Joint Spectra and Relative Spectra of structures

Alexandra A. Soskova

Faculty of Mathematics and Computer Science
Sofia University

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Degree spectra

Definition

Let $\mathfrak{A}$ be a countable structure. The *spectrum* of $\mathfrak{A}$ is the set of Turing degrees

$$\mathrm{Sp}(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \text{ computes the diagram of an isomorphic copy of } \mathfrak{A}\}.$$

Enumeration of a structure

Let $\mathfrak{A} = (A; R_1, \dots, R_k)$ be a countable abstract structure.

- An enumeration f of $\mathfrak{A}$ is a bijection from $\mathbb{N}$ onto A.
- Let for any $X \subseteq A^a$
  $f^{-1}(X) = \{\langle x_1 \dots x_a \rangle : (f(x_1), \dots, f(x_a)) \in X\}$.
- $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \dots \oplus f^{-1}(R_k)$.

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The spectrum of $\mathfrak{A}$ is the set $\mathrm{Sp}(\mathfrak{A}) = \{\mathbf{a} \mid (\exists f)(d_T(f^{-1}(\mathfrak{A})) \leq_T \mathbf{a})\}$.

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The spectrum of $\mathfrak{A}$ is the set $\mathrm{Sp}(\mathfrak{A}) = \{\mathbf{a} \mid (\exists f)(d_T(f^{-1}(\mathfrak{A})) \leq_T \mathbf{a})\}$.

The k-th jump spectrum of $\mathfrak{A}$ is the set $\mathrm{Sp}_k(\mathfrak{A}) = \{\mathbf{a}^{(k)} \mid \mathbf{a} \in \mathrm{Sp}(\mathfrak{A})\}$.

Joint Spectra

Let $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ be arbitrary countable abstract structures.

Definition

The Joint spectrum of $\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set

$$\begin{aligned}\mathrm{JSp}(\mathfrak{A}_0, \mathfrak{A}_1, \dots, &\mathfrak{A}_n) = \\ &\{\mathbf{a} : \mathbf{a} \in \mathrm{Sp}(\mathfrak{A}_0), \mathbf{a}' \in \mathrm{Sp}(\mathfrak{A}_1), \dots, \mathbf{a}^{(\mathbf{n})} \in \mathrm{Sp}(\mathfrak{A}_n)\}.\end{aligned}$$

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Proposition

The joint spectrum of $\vec{\mathfrak{A}} = \{\mathfrak{A}_k\}_{k \leq n}$ *is the set*
$\mathrm{JSp}(\vec{\mathfrak{A}}) = \{d_T(B) \mid (\exists \{f_k\}_{k \leq n})(\forall k \leq n)(f_k^{-1}(\mathfrak{A}_k)$ *is c.e. in* $B^{(k)})\}$.

Enumeration reducibility

1. A set X is *c.e. in* a set Y if X can be enumerated by a computable in Y function.
2. A set X is enumeration reducible to a set Y if and only if there is an effective procedure to transform an enumeration of Y to an enumeration of X.

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Denote by Y^+ the set $Y \oplus \overline{Y}$.

Proposition

X is c.e. in Y if and only if $X \leq_e Y^+$.

Co-spectra of structures

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_k\}_{k \leq n}$ be a finite sequence of structures. Let $A = \bigcup_k A_k$.

Definition

The k-th co-spectrum of $\vec{\mathfrak{A}}$ is the set

$$\mathrm{CoJSp}_k(\vec{\mathfrak{A}}) = \left\{ \mathbf{a} \in \mathcal{D}_e \mid \forall \mathbf{x} \in \mathrm{JSp}_k(\vec{\mathfrak{A}})(\mathbf{a} \leq_e \mathbf{x}) \right\}.$$

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For a set X, $d_e(X) \in \mathrm{CoJSp}_k(\vec{\mathfrak{A}})$ is equivalent to X is definable by a computable sequence of infinitary computable Σ^+_{k+1} formulae with predicates only from the first k structures, such that the predicates for the j-th appear for the first time at level $j+1$ positively, but considered as a many-sorted formulae.

Relative Spectra of Structures

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_k\}_{k \leq n}$ be a finite sequence of countable structures. Denote by $A = \bigcup_k A_k$.

Definition

The relative spectrum of $\vec{\mathfrak{A}}$ is

$$\mathrm{RSp}(\vec{\mathfrak{A}}) = \{d_T(B) \mid (\exists f \text{ enumeration of } A)(\forall k \leq n)(f^{-1}(\mathfrak{A}_k) \text{ is c.e. in } B^{(k)})\}$$

where $f^{-1}(\mathfrak{A}_k) = f^{-1}(A_k) \oplus f^{-1}(R_1^k) \oplus \cdots \oplus f^{-1}(R_{m_k}^k)$.

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There are structures $\mathfrak{A}_0$ and $\mathfrak{A}_1$ s.t. $\mathrm{CoJSp}_1(\mathfrak{A}_0, \mathfrak{A}_1) \neq \mathrm{CoRSp}_1(\mathfrak{A}_0, \mathfrak{A}_1)$:

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Question

Given a sequence of structures $\vec{\mathfrak{A}}$,

1. *does there exist a structure $\mathfrak{M}$, such that $\mathrm{JSp}(\vec{\mathfrak{A}}) = \mathrm{Sp}(\mathfrak{M})$?*
2. *does there exist a structure $\mathfrak{M}$, such that $\mathrm{RSp}(\vec{\mathfrak{A}}) = \mathrm{Sp}(\mathfrak{M})$?*

A parallel between classical computability theory and effective definability in abstract structures

A close parallel between notions of classical computability theory and of the theory of effective definability in abstract structures:

1. The notion of "c.e. in" corresponds to the notion of Σ_1 definability;
2. The "Σ^0_{n+1} in" sets correspond to the sets definable by means of infinitary computable Σ_{n+1} formulae.

From sets to sequences of sets

Definition

A sequence of sets of natural numbers $\mathcal{X} = \{X_n\}_{n<\omega}$ is *c.e. in* a set $A \subseteq \mathbb{N}$ if for every n, X_n is c.e. in $A^{(n)}$ uniformly in n.

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Theorem (Selman)

$X \leq_e A$ if an only if for every B, if A is c.e. in B then X is c.e. in B.

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Definition

(i) Given a set X of natural numbers and a sequence $\mathcal{Y}$ of sets of natural numbers, let $X \leq_n \mathcal{Y}$ if for all sets B, $\mathcal{Y}$ is c.e. in B implies X is Σ^0_{n+1} in B;

(ii) Given sequences $\mathcal{X}$ and $\mathcal{Y}$ of sets of natural numbers, say that $\mathcal{X}$ is ω-enumeration reducible to $\mathcal{Y}$ ($\mathcal{X} \leq_\omega \mathcal{Y}$) if for all sets B, $\mathcal{Y}$ is c.e. in B implies $\mathcal{X}$ is c.e. in B.

Sequences of sets

Ash presents a characterization of "$\leq_n$" and "$\leq_\omega$" using computable infinitary propositional sentences. Soskov and Kovachev give another characterizations in terms of enumeration computability.

Definition

The *jump sequence* $\mathcal{P}(\mathcal{X}) = \{\mathcal{P}_n(\mathcal{X})\}_{n<\omega}$ of $\mathcal{X}$ is defined by induction:

(i) $\mathcal{P}_0(\mathcal{X}) = X_0$;

(ii) $\mathcal{P}_{n+1}(\mathcal{X}) = \mathcal{P}_n(\mathcal{X})' \oplus X_{n+1}$.

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Theorem (Soskov)

1. *$X \leq_n \mathcal{Y}$ if and only if $X \leq_e \mathcal{P}_n(\mathcal{Y})$.*
2. *$\mathcal{X} \leq_\omega \mathcal{Y}$ if and only if for every n, $X_n \leq_e \mathcal{P}_n(\mathcal{Y})$ uniformly in n.*

Sequences of structures

Now consider a sequence of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, where $\mathfrak{A}_n = (A_n; R_1^n, R_2^n, \ldots R_{m_n}^n)$. Let $A = \bigcup_n A_n$.

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Definition

For $R \subseteq A$ we say that $R \leq_n \vec{\mathfrak{A}}$ if for every enumeration f of $\vec{\mathfrak{A}}$, $f^{-1}(R) \leq_n f^{-1}(\vec{\mathfrak{A}})$.

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Definition

A sequence $\{Y_n\}$ of subsets of A is (relatively intrinsically) ω-enumeration reducible to $\vec{\mathfrak{A}}$ if for every enumeration f of $\vec{\mathfrak{A}}$, $\{f^{-1}(Y_n)\} \leq_\omega f^{-1}(\vec{\mathfrak{A}})$.

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More generally let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ be a sequence of countable structures.

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The Relative spectrum of $\vec{\mathfrak{A}}$ is

$$\mathrm{RSp}(\vec{\mathfrak{A}}) = \{d_T(B) \mid \begin{array}{l}(\exists f \text{ enumeration of } A)\\ (\forall n)(f^{-1}(\mathfrak{A}_n) \text{ is c.e. in } B^{(n)} \text{ uniformly in } n)\},\end{array}$$

Omega enumeration co-spectra

Definition

The ω-enumeration relative Co-spectrum of $\vec{\mathfrak{A}}$ is the set

$$\mathrm{OCoSp}(\vec{\mathfrak{A}}) = \left\{ \mathbf{a} \in \mathcal{D}_\omega \mid \forall \mathbf{x} \in \mathrm{RSp}(\vec{\mathfrak{A}})(\mathbf{a} \leq_\omega \mathbf{x}) \right\}.$$

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For any enumeration f of A denote by $f^{-1}(\vec{\mathfrak{A}}) = \{f^{-1}(\mathfrak{A}_n)\}_{n<\omega}$.

Proposition

For every sequence of sets of natural numbers $\mathcal{X} = \{X_n\}_{n<\omega}$:

1. $d_\omega(\mathcal{X}) \in \mathrm{OCoSp}(\vec{\mathfrak{A}})$ *iff*
2. $\mathcal{X} \leq_\omega \{\mathcal{P}_k(f^{-1}(\vec{\mathfrak{A}}))\}_{k<\omega}$, *for every enumeration f of A iff*
3. *each X_n is definable by a computable sequence of Σ^+_{n+1} formulae with parameters uniformly in n.*

Questions

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Given a sequence of structures $\vec{\mathfrak{A}}$, does there exist a structure $\mathfrak{M}$, such that for every sequence $\mathcal{X}$ of subsets of $A = \bigcup_n A_n$, $\mathcal{X} \leq_\omega \vec{\mathfrak{A}}$ if and only if $\mathcal{X}$ c.e. in $\mathfrak{M}$?
Here $\mathcal{X}$ c.e. in $\mathfrak{M}$ if for each enumeration f of $\mathfrak{M}$, $f^{-1}(X_n)$ is c.e. in $f^{-1}(\mathfrak{M})^{(n)}$ uniformly in n.

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The n-th Marker's extension $\mathfrak{M}_n(R)$ of R

Let $X_0, X_1, \dots X_n$ be infinite disjoint countable - companions to $\mathfrak{M}_n(R)$.
Fix bijections: $h_0 : R \to X_0$
$h_1 : (A^m \times X_0) \setminus G_{h_0} \to X_1 \dots$
$h_n : (A^m \times X_0 \times X_1 \dots \times X_{n-1}) \setminus G_{h_{n-1}} \to X_n$

Let $M_n = G_{h_n}$ and $\mathfrak{M}_n(R) = (A \cup X_0 \cup \dots \cup X_n; X_0, X_1, \dots X_n, M_n)$.

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$\exists x_0 \in X_0 \forall x_1 \in X_1 \exists x_2 \in X_2[(\bar{a}, x_0, x_1, x_2) \in G_{h_2}] \iff \dots$

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$\exists x_0 \in X_0 \forall x_1 \in X_1 \exists x_2 \in X_2[(\bar{a}, x_0, x_1, x_2) \in G_{h_2}] \iff \dots$

$\exists x_0 \in X_0 \forall x_1 \in X_1 \dots \exists x_n \in X_n[M_n(\bar{a}, x_0, \dots x_n)]$.

Marker's extensions

For $\mathfrak{A} = (A; R_1, R_2, \dots R_m)$ and $\mathfrak{B} = (B; P_1, P_2, \dots P_k)$ let $\mathfrak{A} \cup \mathfrak{B} = (A \cup B; R_1, R_2, \dots R_m, P_1, P_2, \dots P_k)$.

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The positive answers of the questions [Soskov]

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}$, $A = \bigcup_n |\mathfrak{A}_n|$ and $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$ the Marker's extension of $\vec{\mathfrak{A}}$.

Theorem

A sequence $\mathcal{Y}$ of subsets of A is (r.i.) ω-enumeration reducible to $\vec{\mathfrak{A}}$ if and only if $\mathcal{Y}$ is (r.i) c.e. in $\mathfrak{M}(\vec{\mathfrak{A}})$.

Theorem

For every $R \subseteq \mathbb{N}$, $R \leq_n \vec{\mathfrak{A}} \iff R$ is relatively intrinsically Σ_{n+1} in $\mathfrak{M}$.

Theorem

1. *There is a structure $\mathfrak{M}_1$ with $\mathrm{JSp}(\vec{\mathfrak{A}}) = \mathrm{Sp}(\mathfrak{M}_1)$.*
2. *There is a structure $\mathfrak{M}_2$ with $\mathrm{RSp}(\vec{\mathfrak{A}}) = \mathrm{Sp}(\mathfrak{M}_2)$.*

Co-spectra of Marker's extensions

Theorem (Soskov)

Fix $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ *and let* $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$.

1. $\mathrm{CoSp}_n(\mathfrak{M}) = \left\{ d_e(Y) \mid (\forall g)(Y \leq_e \mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}}))) \right\}$.
2. $\mathrm{OCoSp}(\mathfrak{M}) = \left\{ d_\omega(\mathcal{Y}) \mid (\forall g)(\mathcal{Y} \leq_\omega g^{-1}(\vec{\mathfrak{A}})) \right\}$.

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Let $\mathcal{R} = \{R_n\}_{n<\omega}$ be a sequence of sets and $\vec{\mathfrak{A}}$ the sequence of structures, constructed by $\mathcal{R}$:

- $\mathfrak{A}_0 = (\mathbb{N}; G_s, R_0)$;
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$\mathcal{P}_n(\mathcal{R}) \leq_e \mathcal{P}_n(g^{-1}(\vec{\mathfrak{A}}))$ uniformly in n for any enumeration g of $\vec{\mathfrak{A}}$.

1. $\mathrm{CoSp}_n(\mathfrak{M}) = \{d_e(Y) \mid Y \leq_e \mathcal{P}_n(\mathcal{R})\}$.
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$\mathcal{D}_T \subset \mathcal{D}_e \subset \mathcal{D}_\omega$

- The Turing degrees are embedded in to the enumeration degrees by: $\iota(d_T(X)) = d_e(X^+)$.
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  Sequences with this property are called *almost zero*.

Embedding the ω-enumeration degrees into the Muchnik degrees generated by spectra of structures

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Consider again the structure $\vec{\mathfrak{A}}$ obtained from a sequence of sets $\mathcal{R}$. $\mathfrak{A}_0 = (\mathbb{N}; G_s, R_0)$ and for all $n \geq 1$, $\mathfrak{A}_n = (\mathbb{N}; R_n)$.

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- By Main Theorem there is a structure $\mathfrak{M}_\mathcal{R}$ such that $\mathrm{Sp}(\mathfrak{M}_\mathcal{R}) = \left\{ d_T(B) \mid (\exists g)(g^{-1}(\vec{\mathfrak{A}}) \text{ is c.e. in } B) \right\}$.

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- $\mathrm{Sp}(\mathfrak{M}_\mathcal{R}) = \{d_T(B) \mid \mathcal{R} \text{ is c.e. in } B\}$.

$\mathcal{R} \leq_\omega \mathcal{Q} \iff$
$\{d_T(B) \mid \mathcal{R} \text{ is c.e. in } B\} \supseteq \{d_T(B) \mid \mathcal{Q} \text{ is c.e. in } B\} \iff$
$\mathrm{Sp}(\mathfrak{M}_\mathcal{R}) \supseteq \mathrm{Sp}(\mathfrak{M}_\mathcal{Q})$.
Let $\mu(d_\omega(\mathcal{R})) = \mathrm{Sp}(\mathfrak{M}_\mathcal{R})$.

Spectrum with all non low_n degrees for each n

Theorem

For every sequence $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ *there exists a structure* $\mathfrak{M}$ *such that* $\mathrm{Sp}(\mathfrak{M}) = \mathrm{JSp}(\vec{\mathfrak{A}})$.

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$\mathrm{Sp}(\mathfrak{M}) \subseteq \mathrm{Sp}(\mathfrak{A}_0)$, $\mathrm{Sp}_1(\mathfrak{M}) \subseteq \mathrm{Sp}(\mathfrak{A}_1)$, $\ldots$, $\mathrm{Sp}_n(\mathfrak{M}) \subseteq \mathrm{Sp}(\mathfrak{A}_n) \ldots$

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Apply this to the sequence $\vec{\mathfrak{A}}$, where $\mathfrak{A}_n$ is obtained by Wehner's construction relativized to $\mathbf{0}^{(n)}$.

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Theorem (Soskov)

There is a structure $\mathfrak{M}$ with $\mathrm{Sp}(\mathfrak{M}) = \{\mathbf{b} \mid \forall n(\mathbf{b}^{(n)} > \mathbf{0}^{(n)})\}$.

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