A Jump Inversion Theorem for the Degree Spectra

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Abstract. A jump inversion theorem for the degree spectra is presented. For a structure $\mathfrak{A}$ which degree spectrum is a subset of the jump spectrum of a structure $\mathfrak{B}$, a structure $\mathfrak{C}$ is constructed as a Marker's extension of $\mathfrak{A}$ such that the jump spectrum of $\mathfrak{C}$ is exactly the degree spectrum of $\mathfrak{A}$ and the degree spectrum of $\mathfrak{C}$ is a subset of the degree spectrum of $\mathfrak{B}$.
Key words: enumeration degrees; degree spectra; Marker's extensions; enumerations.

1 Introduction

The notion of a degree spectrum of a countable structure is introduced by Richter [9] as the set of all Turing degrees generated by all one-to-one enumerations of the structure. It is studied by Ash, Downey, Jockush and Knight [1, 4, 7]. It is a kind of a measure of complexity of the structure. Soskov [11] represented the notion of a degree spectrum of a structure from the point of view of enumeration degrees.

Let $\mathfrak{A}$ be a countable structure. *The degree spectrum* of the structure $\mathfrak{A}$ is the set $\mathrm{DS}(\mathfrak{A})$ of all enumeration degrees generated by all enumerations of $\mathfrak{A}$. The main benefit of considering not only one-to-one but all enumerations of the structure is that the degree spectrum is always closed upwards with respect to total degrees [11], i.e. if $\mathbf{a} \in \mathrm{DS}(\mathfrak{A})$ then each total enumeration degree $\mathbf{b}$ greater than $\mathbf{a}$ is in $\mathrm{DS}(\mathfrak{A})$. If $\mathbf{a}$ is the least element of $\mathrm{DS}(\mathfrak{A})$ then $\mathbf{a}$ is called the *degree of* $\mathfrak{A}$.

The jump spectrum of $\mathfrak{A}$ is the set $\mathrm{DS}_1(\mathfrak{A})$ of all enumeration jumps of the elements of $\mathrm{DS}(\mathfrak{A})$. If $\mathbf{a}$ is the least element of $\mathrm{DS}_1(\mathfrak{A})$ then $\mathbf{a}$ is called *the first jump degree of* $\mathfrak{A}$.

For any countable structures $\mathfrak{A}$ and $\mathfrak{B}$ define the relation

$$\mathfrak{B} \preceq \mathfrak{A} \iff \mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}(\mathfrak{B}) \ .$$

And let $\mathfrak{A} \equiv \mathfrak{B}$ if $\mathfrak{A} \preceq \mathfrak{B}$ and $\mathfrak{B} \preceq \mathfrak{A}$.

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Let $\mathfrak{B}' \preceq \mathfrak{A}$ if $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_1(\mathfrak{B})$ and $\mathfrak{A} \preceq \mathfrak{B}'$ if $\mathrm{DS}_1(\mathfrak{B}) \subseteq \mathrm{DS}(\mathfrak{A})$. We say that $\mathfrak{A} \equiv \mathfrak{B}'$ if $\mathfrak{A} \preceq \mathfrak{B}'$ and $\mathfrak{B}' \preceq \mathfrak{A}$.

Soskov [12] showed that each jump spectrum is a degree spectrum of a structure. So, for every structure $\mathfrak{B}$ there is a structure $\mathfrak{A}$ such that $\mathfrak{B}' \equiv \mathfrak{A}$, i.e. $\mathrm{DS}(\mathfrak{A}) = \mathrm{DS}_1(\mathfrak{B})$.

In this paper we shall show that if $\mathfrak{A}$ and $\mathfrak{B}$ are structures and $\mathfrak{B}' \preceq \mathfrak{A}$ then there exists a structure $\mathfrak{C}$ such that $\mathfrak{B} \preceq \mathfrak{C}$ and $\mathfrak{C}' \equiv \mathfrak{A}$.

The structure $\mathfrak{C}$ we shall construct as a Marker's extension of $\mathfrak{A}$. In [6] two model-theoretic extension operators were introduced based on the ideas of Marker's construction from [8]. These extensions are called Marker's $\exists$ and $\forall$-extensions and are studied in [5, 6]. In our construction we will use also the relativized representation lemma for Σ^0_2 sets proved by Goncharov and Khoussainov [6].

As an application we shall show that if a structure $\mathfrak{A}$ has a degree and $\mathfrak{B}' \preceq \mathfrak{A}$ for some structure $\mathfrak{B}$ then there is a torsion free abelian group $\mathfrak{G}$ of rank 1 such that $\mathfrak{B} \preceq \mathfrak{G}$, $\mathfrak{G}' \equiv \mathfrak{A}$ and $\mathfrak{G}$ has a degree as well.

As a corollary of the main result we receive an analogue of the jump inversion theorem for the joint spectra of finitely many structures considered in [13, 15]. Let $\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n$ be countable structures. *The joint spectrum of* $\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set of all enumeration degrees $\mathbf{a} \in \mathrm{DS}(\mathfrak{A})$ such that $\mathbf{a}' \in \mathrm{DS}(\mathfrak{A}_1), \dots, \mathbf{a}^{(n)} \in \mathrm{DS}(\mathfrak{A}_n)$.

We will prove that if there is a structure $\mathfrak{B}$ such that $\mathfrak{B}' \preceq \mathfrak{A}$ then there exists a structure $\mathfrak{C} \succeq \mathfrak{B}$ such that the joint spectrum of $\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n$ is exactly the jump joint spectrum of $\mathfrak{C}, \mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n$.

Next application is a similar result for another relativized version of the notion of a degree spectrum of a structure with respect to finitely many abstract structures studied in [14]. It is shown [13–15] that both generalized notions of degree spectra have all general properties of the degree spectra of a structure such as minimal pair theorem and the existence of quasi-minimal degrees.

The relative spectrum of the structure $\mathfrak{A}$ *with respect to* $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set of all enumeration degrees generated by those enumerations of $\mathfrak{A}$ which "assume" that each $\mathfrak{A}_i$ is relatively Σ^0_{i+1} on $\mathfrak{A}$ for $i = 1, \dots k$. We will show that if there is a structure $\mathfrak{B}$ such that $\mathfrak{B}' \preceq \mathfrak{A}$ then there exists a structure $\mathfrak{C} \succeq \mathfrak{B}$ such that the relative spectrum of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ coincide with the jump relative spectrum of $\mathfrak{C}$ with respect to $\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n$.

2 Preliminaries

2.1 Enumeration Degrees

For any sets of natural numbers A and B the set A is enumeration reducible to B ($A \leq_e B$) if there is an enumeration operator Γ_z such that $A = \Gamma_z(B)$. In other words:

$$A \leq_e B \iff (\exists z)(\forall x)(x \in A \iff (\exists v)(\langle v, x\rangle \in W_z \;\&\; D_v \subseteq B))$$

where D_v is the finite set with the canonical code v and $\{W_z\}_{z<\omega}$ is a Gödel enumeration of the c.e. sets.

The relation $\leq_{\mathrm{e}}$ is reflexive and transitive and induces an equivalence relation $\equiv_{\mathrm{e}}$ on all sets of natural numbers. The respective equivalence classes are called enumeration degrees.

By $d_{\mathrm{e}}(A)$ we denote the enumeration degree of the set A and by $\mathcal{D}_{\mathrm{e}}$ the set of all enumeration degrees. Let $A^+ = A \oplus (\mathbb{N} \setminus A)$. The set A is total if $A \equiv_{\mathrm{e}} A^+$. An enumeration degree $\mathbf{a}$ is total if $\mathbf{a}$ contains the e-degree of a total set. The jump operation "$'$" denotes here the enumeration jump introduced by Cooper [3].

Definition 1. Let $L_A = \{\langle x, z\rangle \mid x \in \Gamma_z(A)\}$.

The *e-jump* A' of A is the set $(L_A)^+$.

In fact, the set A is Σ^0_2 relatively the set B ($A \in \Sigma^0_2(B)$) if and only if $A \leq_{\mathrm{e}} (B^+)'$. This follows from the observation that $K_B^+ \equiv_{\mathrm{e}} (B^+)'$ where $K_B = \{\langle e, x\rangle \mid x \in W_e^B\}$.

$$A \in \Sigma^0_2(B) \iff A \text{ is c.e. in } K_B \iff A \leq_{\mathrm{e}} K_B^+ \iff A \leq_{\mathrm{e}} (B^+)' \ .$$

So, if the set B is total then $B \equiv_{\mathrm{e}} B^+$ and hence $A \in \Sigma^0_2(B) \iff A \leq_{\mathrm{e}} B'$.

2.2 Degree Spectra

Let $\mathfrak{A} = (A; R_1, \dots, R_s)$ be a countable structure such that $=$ is among the predicates $R_1, \dots, R_s$.

An enumeration f of $\mathfrak{A}$ is a total mapping of $\mathbb{N}$ onto A.

For $B \subseteq A^a$ define $f^{-1}(B) = \{\langle x_1, \dots, x_a\rangle \mid (f(x_1), \dots, f(x_a)) \in B\}$.

For each predicate R of $\mathfrak{A}$ of arity r the pullback R^f of R is defined by $R^f(x_1, \dots, x_r) \iff R(f(x_1), \dots, f(x_r))$. Let

$$f^{-1}(R) = \{\langle x_1, \dots, x_r, 0\rangle \mid R^f(x_1, \dots, x_r)\} \cup \\ \{\langle x_1, \dots, x_r, 1\rangle \mid \neg R^f(x_1, \dots, x_r)\} \ .$$

Denote by $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \ldots \oplus f^{-1}(R_s)$.

Definition 2. *The degree spectrum of* $\mathfrak{A}$ is the set

$$\mathrm{DS}(\mathfrak{A}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\} \ .$$

Our definition of degree spectra is equivalent to Soskov's from [11]. The structure $\mathfrak{A}$ is total if the predicates $R_1, \dots, R_s$ are totally defined on A. We consider here only total structures. It is easy to see that if the structure $\mathfrak{A}$ is total then all elements of the degree spectra of $\mathfrak{A}$ are total enumeration degrees. Let ι be the Roger's embedding of the Turing degrees into the enumeration degrees. Then

$$\mathrm{DS}(\mathfrak{A}) = \{\iota(d_{\mathrm{T}}(f^{-1}(\mathfrak{A}))) \mid f \text{ is an enumeration of } \mathfrak{A}\} \ .$$

Richter [9] and Knight [7] defined the degree spectra by taking into account only the bijective enumerations, while we allow as in [11] arbitrary surjective enumerations.

Proposition 3. *[11] Let f be an arbitrary enumeration of $\mathfrak{A}$. There exists a bijective enumeration g of $\mathfrak{A}$ such that $g^{-1}(\mathfrak{A}) \leq_e f^{-1}(\mathfrak{A})$.*

The above proposition shows that almost all of the known results about Turing degree spectra remain valid also for enumeration degree spectra.

Proposition 4. *[11] Let g be an enumeration of $\mathfrak{A}$. Suppose that F is a total set and $g^{-1}(\mathfrak{A}) \leq_e F$. There exists an enumeration f of $\mathfrak{A}$ such that $f^{-1}(\mathfrak{A}) \equiv_e F$.*

From the last proposition it follows that the degree spectrumDS($\mathfrak{A}$) is closed upwards with respect to the totalenumeration degrees.

The jump spectrum of $\mathfrak{A}$ is the set $\mathrm{DS}_1(\mathfrak{A}) = \{\mathbf{a}' \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A})\}$.

Since by [12] every jump spectrum is a degree spectrum of a structure it follows that $\mathrm{DS}_1(\mathfrak{A})$ is also closed upwards with respect to total enumeration degrees. One can see this fact directly using the jump inversion theorem from [10].

3 Marker's Extensions

Marker [8] presented a method of constructing for any $n \geq 1$ a $\aleph_0$-categorical almost strongly minimal theory which is not Σ_n-axiomatizable. Further Goncharov and Khoussainov [6] adapted the construction to the general case in order to find for any $n \geq 1$ examples of $\aleph_1$-categorical computable models as well as $\aleph_0$-categorical computable models whose theories are Turing equivalent to $\emptyset^{(n)}$. We shall give the definition of Marker's $\exists$ and $\forall$ extensions following [6].

Let $\mathfrak{A} = (A; R_1, \ldots, R_s, =)$ be a countable total structure and for each i the predicate R_i has arity r_i.

Marker's $\exists$-extension of R_i, denoted by $R_i^{\exists}$, is defined as follows. Consider a set X_i with new elements such that $X_i = \{x^i_{\langle a_1,\ldots,a_{r_i}\rangle} \mid R_i(a_1, \ldots, a_{r_i})\}$. The set X_i we shall call a $\exists$-fellow for R_i. We suppose that all sets A, X_1,..., X_s are pairwise disjoint.

The predicate $R_i^{\exists}$ is a predicate of arity r_i+1 such that $R_i^{\exists}(a_1, \ldots, a_{r_i}, x) \iff a_1, \ldots, a_{r_i} \in A \;\&\; x \in X_i \;\&\; x = x^i_{\langle a_1,\ldots,a_{r_i}\rangle}$ (and so $R_i(a_1, \ldots, a_{r_i})$).

From the definition of $R_i^{\exists}$ it follows that if $a_1, \ldots, a_{r_i} \in A$ then $(\exists x \in X_i) R_i^{\exists}(a_1, \ldots, a_{r_i}, x) \iff R_i(a_1, \ldots, a_{r_i})$.

Definition 5. The structure $\mathfrak{A}^{\exists}$ is defined as follows:

$$(A \cup \bigcup_{i=1}^{s} X_i, R_1^{\exists}, \ldots, R_s^{\exists}, \bar{X}_1, \ldots, \bar{X}_s, =),$$

where each $R_i^{\exists}$ is a Marker's $\exists$-extension of R_i with $\exists$-fellow X_i and $\bar{X}_i$ is a unary predicate true over the elements of the $\exists$-fellow for R_i.

Marker's $\forall$-extension of R_i, denoted by $R_i^{\forall}$, is defined as follows. Consider an infinite set Y_i of new elements such that $Y_i = \{y^i_{\langle a_1,\ldots,a_{r_i}\rangle} \mid \neg R_i(a_1, \ldots, a_{r_i})\}$. The set Y_i we shall call a $\forall$-fellow for R_i.

The predicate $R_i^\forall$ is a predicate of arity r_i+1 such that

1. If $R_i^\forall(a_1,\ldots,a_{r_i},y)$ then $a_1,\ldots,a_{r_i}\in A$ and $y\in Y_i$;
2. If $a_1,\ldots,a_{r_i}\in A$ & $y\in Y_i$ then $\neg R_i^\forall(a_1,\ldots,a_{r_i},y) \iff y=y^i_{\langle a_1,\ldots,a_{r_i}\rangle}$.

Note that from the definition of $R_i^\forall$ it follows that if $a_1,\ldots,a_{r_i}\in A$ then $(\forall y\in Y_i)R_i^\forall(a_1,\ldots,a_{r_i},y) \iff R_i(a_1,\ldots,a_{r_i})$.

Definition 6. The structure $\mathfrak{A}^\forall$ is defined as follows:

$$(A\cup\bigcup_{i=1}^{s}Y_i, R_1^\forall,\ldots,R_s^\forall,\bar{Y}_1,\ldots,\bar{Y}_s,=),$$

where each $R_i^\forall$ is a Marker's $\forall$-extension of R_i with $\forall$-fellow Y_i and $\bar{Y}_i$ is a unary predicate true over the elements of the $\forall$-fellow for R_i. The $\forall$-fellows of the distinct predicates and the set A are pairwise disjoint.

Definition 7. The structure $\mathfrak{A}^{\exists\forall}$ is obtained from $\mathfrak{A}$ as $(\mathfrak{A}^\exists)^\forall$, i.e.

$$(A\cup\bigcup_{i=1}^{s}X_i\cup\bigcup_{i=1}^{s}Y_i, R_1^{\exists\forall},\ldots,R_s^{\exists\forall},\bar{X}_1,\ldots,\bar{X}_s,\bar{Y}_1,\ldots,\bar{Y}_s,=),$$

where X_i is a $\exists$-fellow for R_i and Y_i is a $\forall$-fellow for $R_i^\exists$.

The structure $\mathfrak{A}^{\exists\forall}$ has the following properties:

Proposition 8. *Let $a_1,\ldots,a_{r_i}\in A$. Then:*

1. $R_i(a_1,\ldots,a_{r_i}) \iff (\exists x\in X_i)(\forall y\in Y_i)R_i^{\exists\forall}(a_1,\ldots,a_{r_i},x,y)$;

2. For each $y\in Y_i$ there exists a unique sequence $a_1,\ldots,a_{r_i}\in A$ and $x\in X_i$ such that $\neg R_i^{\exists\forall}(a_1,\ldots,a_{r_i},x,y)$;

3. For each $x\in X_i$ there exists a unique sequence $a_1,\ldots,a_{r_i}\in A$ such that for all $y\in Y_i$ it holds that $R_i^{\exists\forall}(a_1,\ldots,a_{r_i},x,y)$.

Proof. 1.($\Rightarrow$) Let $R_i(a_1,\ldots,a_{r_i})$. Then there exists $x\in X_i$ such that $R_i^\exists(a_1,\ldots,a_{r_i},x)$. From the definition of Y_i it follows that for any $y\in Y_i$ $R_i^{\exists\forall}(a_1,\ldots,a_{r_i},x,y)$.

($\Leftarrow$) Let $x\in X_i$ and $R_i^{\exists\forall}(a_1,\ldots,a_{r_i},x,y)$ for all $y\in Y_i$. Then $R_i^\exists(a_1,\ldots,a_{r_i},x)$ and hence $R_i(a_1,\ldots,a_{r_i})$.

2. Follows from the definition of Y_i.

3. Let $x\in X_i$ then $x=x^i_{\langle a_1,\ldots,a_{r_i}\rangle}$ and $R_i(a_1,\ldots,a_{r_i})$. Hence $R_i^\exists(a_1,\ldots,a_{r_i},x)$. Then for any $y\in Y_i$ it is not possible that $\neg R_i^{\exists\forall}(a_1,\ldots,a_{r_i},x,y)$.

4 Join of Two Structures

Let $\mathfrak{A}=(A;R_1,\ldots,R_s,=)$ and $\mathfrak{B}=(B;P_1,\ldots,P_t,=)$ be countable structures in the language $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Suppose that $\mathcal{L}_1\cap\mathcal{L}_2=\{=\}$ and $A\cap B=\emptyset$. Let $\mathcal{L}=\mathcal{L}_1\cup\mathcal{L}_2\cup\{\bar{A},\bar{B}\}$, where $\bar{A}$ and $\bar{B}$ are unary predicates.

Definition 9. The join of the structures $\mathfrak{A}$ and $\mathfrak{B}$ is the structure $\mathfrak{A} \oplus \mathfrak{B} = (A \cup B; R_1, \dots, R_s, P_1, \dots, P_t, \bar{A}, \bar{B}, =)$ in the language $\mathcal{L}$, where

(a) the predicate $\bar{A}$ is true only over the elements of A and similarly $\bar{B}$ is true only over the elements of B;

(b) the predicate R_i is defined on the elements of A as in the structure $\mathfrak{A}$ and false on all elements not in A and the predicate P_j is defined similarly.

Lemma 10. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable total structures and $\mathfrak{C} = \mathfrak{A} \oplus \mathfrak{B}$. Then $\mathfrak{A} \preceq \mathfrak{C}$ and $\mathfrak{B} \preceq \mathfrak{C}$.*

Proof. We have to prove that $\mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{A})$ and $\mathrm{DS}(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{B})$.

Let f be an enumeration of $\mathfrak{C}$. Fix $x_0 \in f^{-1}(A)$. Define
$m(0) = x_0$; $m(i+1) = \mu z \in f^{-1}(A)[(\forall k \leq i)(\langle m(k), z\rangle \notin f^{-1}(=))]$.

Set $h = \lambda x.f(m(x))$. Note that $m \leq_{\mathrm{e}} f^{-1}(\mathfrak{C})$ since
$z \in f^{-1}(A) \iff \langle z, 0\rangle \in f^{-1}(\bar{A})$.

Define $h^{-1}(R_i) = \{\langle x_1, \dots, x_{r_i}, e\rangle \mid \langle m(x_1), \dots, m(x_{r_i}), e\rangle \in f^{-1}(R_i)\}$. And $h^{-1}(=) = \{\langle x, y, e\rangle \mid \langle m(x), m(y), e\rangle \in f^{-1}(=)\}$.

Then h is an enumeration of $\mathfrak{A}$ and $h^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} f^{-1}(\mathfrak{C})$. Since $\mathfrak{C}$ is a total structure and $\mathrm{DS}(\mathfrak{A})$ is closed upwards then $\mathrm{d}_e(f^{-1}(\mathfrak{C})) \in \mathrm{DS}(\mathfrak{A})$.

5 Representation of $\Sigma^0_2(D)$ Sets

Let $D \subseteq \mathbb{N}$. A set $M \subseteq \mathbb{N}$ is in $\Sigma^0_2(D)$ if there exists a computable in D predicate Q such that

$$n \in M \iff \exists a \forall b Q(n, a, b) \enspace .$$

Definition 11. [6] If $M \in \Sigma^0_2(D)$ then M is one-to-one representable if there is a computable in D predicate Q with the following properties:

1. $n \in M \iff$ there exists a unique a such that $\forall b Q(n, a, b)$;
2. for every b there is a unique pair $\langle n, a\rangle$ such that $\neg Q(n, a, b)$;
3. for every a there exists a unique n such that $\forall b Q(n, a, b)$.

The predicate Q from the above definition is called an one-to-one representation of M. Goncharov and Khoussainov [6] proved the following lemma:

Lemma 12. *[6] If M is a coinfinite $\Sigma^0_2(D)$ subset of $\mathbb{N}$ which has an infinite computable in D subset S such that $M \setminus S$ is infinite then M has an one-to-one representation.*

Remark 13. We will use this lemma in the next section in our proof of Theorem 14. In order to satisfy the conditions of the lemma we need the following technical explanations.

Let $\mathfrak{A} = (A; R_1, \dots, R_s)$. Suppose that each R_i is true over infinitely many elements and it is false over infinitely many elements also.

We can add to the domain A of the structure $\mathfrak{A}$ two new elements say "T" and "F". Define the predicate R^*_i as follows:

1. Let $r_i \geq 2$. Then $R_i^*(a_1, \ldots, a_{r_i})$ is defined as $R_i(a_1, \ldots, a_{r_i})$ if F and T are not among the arguments $\{a_1, \ldots, a_{r_i}\}$. If $T \in \{a_1 \ldots a_{r_i}\}$ then $R_i^*(a_1, \ldots, a_{r_i})$ and if $F \in \{a_1, \ldots, a_{r_i}\}$ and $T \notin \{a_1, \ldots, a_{r_i}\}$ then $\neg R_i^*(a_1, \ldots, a_{r_i})$.

2. Let the predicate R_i be unary. Then we define the binary predicate R_i^* as follows: $R_i^*(a,a) \iff R_i(a)$ if $a \notin \{T, F\}$. If $T \in \{a,b\}$ then $R_i^*(a,b)$ is true and if $F \in \{a,b\}$ and $T \notin \{a,b\}$ then $\neg R_i^*(a,b)$.

Let $\mathfrak{A}^*$ be the obtained structure with domain $A \cup \{T, F\}$ and predicates R_i^* for $i = 1, \ldots, s$. Then one can easily see using Proposition 4 and Proposition 3 that $\mathrm{DS}(\mathfrak{A}) = \mathrm{DS}(\mathfrak{A}^*)$. Indeed, note that if an enumeration of the structure $\mathfrak{A}$ is bijective then the pullback of the equality is computable. Let f be an enumeration of $\mathfrak{A}$ and $\mathrm{d}_e(f^{-1}(\mathfrak{A})) \in \mathrm{DS}(\mathfrak{A})$. By Proposition 3 there is a bijective enumeration g of $\mathfrak{A}$ such that $g^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})$. Then there is a bijective enumeration h of $\mathfrak{A}^*$ such that $h^{-1}(\mathfrak{A}^*) \equiv_{\mathrm{e}} g^{-1}(\mathfrak{A})$. Moreover in $h^{-1}(\mathfrak{A}^*)$ each $h^{-1}(R_i^*)$ is infinite and posses a computable subset S such that $h^{-1}(R_i^*) \setminus S$ is infinite. The set S is formed by all tuples containing the number $h^{-1}(T)$. Since $h^{-1}(\mathfrak{A}^*) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})$ and $\mathfrak{A}$ is total then $\mathrm{d}_e(f^{-1}(\mathfrak{A})) \in \mathrm{DS}(\mathfrak{A}^*)$ by Proposition 4. The proof of $\mathrm{DS}(\mathfrak{A}^*) \subseteq \mathrm{DS}(\mathfrak{A})$ is similar.

6 Jump Inversion Theorem for the Degree Spectra

Theorem 14. *Let $\mathfrak{A}$ and B be total structures such that $\mathfrak{B}' \preceq \mathfrak{A}$. Then there exists a structure $\mathfrak{C}$ such that $\mathfrak{B} \preceq \mathfrak{C}$ and $\mathfrak{C}' \equiv \mathfrak{A}$.*

Proof (Sketch). Without loss of generality we may suppose that the structures $\mathfrak{B}$ and $\mathfrak{A}^{\exists\forall}$ are disjoint. Let $\mathfrak{C} = \mathfrak{B} \oplus \mathfrak{A}^{\exists\forall}$. By Lemma 10 $\mathfrak{B} \preceq \mathfrak{C}$. We shall prove that $\mathfrak{C}' \equiv \mathfrak{A}$, i.e. $\mathrm{DS}(\mathfrak{A}) = \mathrm{DS}_1(\mathfrak{C})$.

1. $\Longrightarrow$ $[\mathrm{DS}_1(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{A})]$.

Let $\mathbf{c} \in \mathrm{DS}_1(\mathfrak{C})$ and let h be an enumeration of $\mathfrak{C}$ such that $\mathbf{c} = \mathrm{d}_e(h^{-1}(\mathfrak{C}))'$. We shall construct an enumeration f of $\mathfrak{A}$ such that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} h^{-1}(\mathfrak{C})'$. Since $h^{-1}(\mathfrak{C})'$ is a total set, by Proposition 4 it will follow that $\mathbf{c} \in \mathrm{DS}(\mathfrak{A})$.

Fix $x_0 \in h^{-1}(A)$. Define

$m(0) = x_0$; $m(i+1) = \mu z \in h^{-1}(A)[(\forall k \leq i)(\langle m(k), z\rangle \notin h^{-1}(=))]$.

Set $f = \lambda a.h(m(a))$. We have $m \leq_{\mathrm{e}} h^{-1}(\mathfrak{A}^{\exists\forall})$ since $z \in h^{-1}(A) \iff (\forall i \leq s)(\langle z, 1\rangle \in h^{-1}(\bar{X}_i) \cap h^{-1}(\bar{Y}_i) \cap h^{-1}(\bar{B}))$. Define:

$$R_i^{\exists\forall,h} = \{\langle a_1, \ldots, a_{r_i}, x, y, e\rangle \mid \langle m(a_1), \ldots, m(a_{r_i}), x, y, e\rangle \in h^{-1}(R_i^{\exists\forall})\ \&\ \langle x, 0\rangle \in h^{-1}(\bar{X}_i)\ \&\ \langle y, 0\rangle \in h^{-1}(\bar{Y}_i)\}\ .$$

$$R_i^f(a_1, \ldots, a_{r_i}) \iff (\exists x)(\forall y)(\ \langle a_1, \ldots, a_{r_i}, x, y, 0\rangle \in R_i^{\exists\forall,h}\ \&\ \langle x, 0\rangle \in h^{-1}(\bar{X}_i)\ \&\ \langle y, 0\rangle \in h^{-1}(\bar{Y}_i))\ .$$

Then it is clear that f is an enumeration of $\mathfrak{A}$ and $f^{-1}(\mathfrak{A}) \in \Sigma_2^0(h^{-1}(\mathfrak{A}^{\exists\forall}))$. Then $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} h^{-1}(\mathfrak{A}^{\exists\forall})' \leq_{\mathrm{e}} h^{-1}(\mathfrak{C})'$ by the monotonicity of the e-jump.

2. $\Longrightarrow$ $[\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_1(\mathfrak{C})]$.

Let $\mathbf{a} \in \mathrm{DS}(\mathfrak{A})$ and $\bar{f}$ be an enumeration of $\mathfrak{A}$ such that $\mathbf{a} = \mathrm{d}_e(\bar{f}^{-1}(\mathfrak{A}))$. By Proposition 3 there is a bijective enumeration f of $\mathfrak{A}$ such that $f^{-1}(\mathfrak{A}) \leq_e \bar{f}^{-1}(\mathfrak{A})$. We are going to construct an enumeration h of $\mathfrak{C}$ such that $h^{-1}(\mathfrak{C})' \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})$. Then since $\mathfrak{A}$ is a total structure and the $\mathrm{DS}_1(\mathfrak{C})$ is upwards closed with respect to total degrees then $\mathbf{a} \in \mathrm{DS}_1(\mathfrak{C})$.

Since $\mathfrak{B}' \preceq \mathfrak{A}$, i.e. $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_1(\mathfrak{B})$ there is an enumeration g of $\mathfrak{B}$ such that $f^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} (g^{-1}(\mathfrak{B}))'$. Denote by $D = g^{-1}(\mathfrak{B})$ and note that D is a total set since the structure $\mathfrak{B}$ is total. So for each predicate R_i of the structure $\mathfrak{A}$ we have that $f^{-1}(R_i) \leq_{\mathrm{e}} D'$. Then $f^{-1}(R_i) \in \Sigma^0_2(D)$. Denote by $M_i = f^{-1}(R_i)$. If the positive part or the negative part of $f^{-1}(R_i)$ is finite then $f^{-1}(R_i)$ is computable. Otherwise by Remark 13 we can suppose that M_i satisfies all conditions from Lemma 12. Then by Lemma 12 for each $i \leq s$ there exists a computable in D predicate Q_i which is an one-to-one representation of M_i. Then

— $\bar{n} \in M_i \iff$ there exists a unique a such that $(\forall b) Q_i(\bar{n}, a, b)$;

— for every b let $r(b) = \langle \bar{n}, a\rangle$ be the unique pair such that $\neg Q_i(\bar{n}, a, b)$;

— for every a let $l(a) = \bar{n}$ be the unique $\bar{n}$ such that $\forall b Q_i(\bar{n}, a, b)$.

Denote by $\mathbb{N}_1 = \{\langle 1, n\rangle \mid n \in \mathbb{N}\}$, $\mathbb{N}_2 = \{\langle 2, i, a\rangle \mid i \leq s \;\&\; a \in \mathbb{N}\}$ and $\mathbb{N}_3 = \{\langle 3, i, b\rangle \mid i \leq s \;\&\; b \in \mathbb{N}\}$. Let $\mathbb{N}_0 = \mathbb{N} \setminus (\bigcup_{i=1}^{3} \mathbb{N}_i)$. Consider a computable bijection m of $\mathbb{N}_0$ onto $\mathbb{N}$ and denote by $\langle 0, n\rangle = m(n)$.

The definition of the enumeration h of $\mathfrak{C}$ is the following:

$h(\langle 0, n\rangle) = g(n)$;

$h(\langle 1, n\rangle) = f(n)$;

$h(\langle 2, i, a\rangle) = x^i_{\langle f(n_1), \dots, f(n_{r_i})\rangle}$, if $l(a) = \langle n_1, \dots, n_{r_i}\rangle$;

$h(\langle 3, i, b\rangle) = y^i_{\langle f(n_1), \dots, f(n_{r_i}), h(\langle 2, i, a\rangle)\rangle}$, if $r(b) = \langle\langle n_1, \dots, n_{r_i}\rangle, a\rangle$.

Here $X_i = \{x^i_{\langle a_1, \dots, a_{r_i}\rangle} \mid R_i(a_1, \dots, a_{r_i})\}$ is the $\exists$-fellow for R_i and $Y_i = \{y^i_{\langle a_1, \dots, a_{r_i}, x\rangle} \mid \neg R_i^{\exists}(a_1, \dots, a_{r_i}, x)\}$ is the $\forall$-fellow for $R_i^{\exists}$. Define

$$R_i^{\exists\forall, h}(\langle 1, n_1\rangle, \dots, \langle 1, n_{r_i}\rangle, \langle 2, i, a\rangle, \langle 3, i, b\rangle) \iff Q_i(\langle n_1, \dots, n_{r_i}\rangle, a, b)\ .$$

Let $h^{-1}(A) = \mathbb{N}_1$, $h^{-1}(X_i) = \mathbb{N}_2$, $h^{-1}(Y_i) = \mathbb{N}_3$.

It follows that

$$\begin{aligned} R_i(f(n_1) \dots f(n_{r_i})) &\iff \langle n_1, \dots, n_{r_i}, 0\rangle \in f^{-1}(R_i) \\ &\iff (\exists a)(\forall b) Q_i(\langle n_1, \dots, n_{r_i}\rangle, a, b) \\ &\iff (\exists a)(\forall b) R_i^{\exists\forall, h}(\langle 1, n_1\rangle, \dots, \langle 1, n_{r_i}\rangle, \langle 2, i, a\rangle, \langle 3, i, b\rangle) \\ &\iff (\exists x)(\forall y) R_i^{\exists\forall}(f(n_1) \dots f(n_{r_i}), x, y) \;\&x \in X_i \& y \in Y_i. \end{aligned}$$

From the definition of h it follows that h is an enumeration of $\mathfrak{A}^{\exists\forall}$. It is clear that $h^{-1}(\mathfrak{A}^{\exists\forall}) \leq_{\mathrm{e}} D$.

Let $\mathfrak{B} = (B; P_1, \dots, P_t, =)$ then for each $j \leq t$

$h^{-1}(P_j) = \{\langle\langle 0, n_1\rangle, \dots, \langle 0, n_{p_j}\rangle, e\rangle \mid \langle n_1, \dots, n_{p_j}, e\rangle \in g^{-1}(P_j)\}$ and $h^{-1}(B) = \mathbb{N}_0$. It is obvious that $h^{-1}(\mathfrak{B}) \leq_{\mathrm{e}} D$.

The pullback of the equality is defined naturally over the elements which are pullbacks of elements of A as $f^{-1}(=)$ and over the elements which are pullbacks of elements of B as $g^{-1}(=)$. Over the elements which are the pullbacks of X_i and

Y_i is a normal equality, since the special form of the Marker's $\exists$ and $\forall$ extensions. So, $h^{-1}(=) \leq_{\mathrm{e}} D$ since $f^{-1}(=)$ is computable.

Thus h is an enumeration of $\mathfrak{C} = \mathfrak{B} \oplus \mathfrak{A}^{\exists\forall}$. Moreover $h^{-1}(\mathfrak{C}) \leq_{\mathrm{e}} D$. Hence $h^{-1}(\mathfrak{C})' \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})$ as $D' \equiv_{\mathrm{e}} f^{-1}(\mathfrak{A})$.

7 Some Applications

The degree of the structure $\mathfrak{A}$, if it exists, is the least element of the degree spectrum of $\mathfrak{A}$. The results of Richter [9] show that there exist structures, e.g. linear orders, which do not have degrees. Richter proved that if the degree spectrum of a linear order has a degree then it is $\mathbf{0}$.

If the jump spectrum $\mathrm{DS}_1(\mathfrak{A})$ has a least element then it is called *the first jump degree of* $\mathfrak{A}$. For example Knight [7] shows that if a linear order has a first jump degree then it is $\mathbf{0}'$. There are examples of structures [1, 4] which have a first jump degree but do not posses a degree. In [2, 11] it is shown that every torsion free abelian group $\mathfrak{G}$ of rank 1, i.e. $\mathfrak{G}$ is a subgroup of the group of the rational numbers Q, has a first jump degree.

Let $\mathfrak{G}$ be a nontrivial subgroup of the additive group of the rational numbers. Fix $a \neq 0$ an element of $\mathfrak{G}$. For every prime number p set

$$h_p(a) = \begin{cases} k & \text{if } k \text{ is the greatest number such that } p^k | a \text{ in } \mathfrak{G}, \\ \infty & \text{if } p^k | a \text{ in } \mathfrak{G} \text{ for all } k. \end{cases}$$

Let $p_0, p_1, \ldots$ be the standard enumeration of the prime numbers and set

$$S_a(\mathfrak{G}) = \{\langle i, j \rangle : j \leq h_{p_i}(a)\}.$$

If a and b are non-zero elements of $\mathfrak{G}$ then $S_a(\mathfrak{G}) \equiv_e S_b(\mathfrak{G})$. Let $\mathbf{d}_{\mathfrak{G}} = \mathrm{d}_e(S_a(\mathfrak{G}))$, where a is some non-zero element of $\mathfrak{G}$.

In [11] it is proved that for every total enumeration degree $\mathbf{d}$, there exists a bijective enumeration f of $\mathfrak{G}$ such that $f^{-1}(\mathfrak{G}) \in \mathbf{d}$ if and only if $\mathbf{d}_{\mathfrak{G}} \leq \mathbf{d}$. Since for every enumeration f we have that $f^{-1}(\mathfrak{G})$ is a total set and $\mathbf{d}_{\mathfrak{G}} \leq d_e(f^{-1}(\mathfrak{G}))$, $\mathrm{DS}(\mathfrak{G}) = \{\mathbf{a} : \mathbf{a} \text{ is total } \& \ \mathbf{a} \geq \mathbf{d}_{\mathfrak{G}}\}$.

It turns out that for any total structures $\mathfrak{A}$ and $\mathfrak{C}$ such that $\mathfrak{C}' \equiv \mathfrak{A}$ if $\mathfrak{C}$ has a degree $\mathbf{a}$ then $\mathbf{a}'$ is the first jump degree of $\mathfrak{C}$ and clearly $\mathbf{a}'$ is the degree of $\mathfrak{A}$ since $\mathrm{DS}(\mathfrak{A}) = \mathrm{DS}_1(\mathfrak{C})$.

Proposition 15. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be total structures such that $\mathfrak{B}' \preceq \mathfrak{A}$. Then if the structure $\mathfrak{A}$ has a degree then there exists a torsion free abelian group $\mathfrak{G}$ of rank 1 which has a degree such that $\mathfrak{B} \preceq \mathfrak{G}$ and $\mathfrak{G}' \equiv \mathfrak{A}$.*

Proof. Let $\mathfrak{C} = \mathfrak{B} \oplus \mathfrak{A}^{\exists\forall}$ be the structure constructed in Theorem 14 such that $\mathfrak{B} \preceq \mathfrak{C}$ and $\mathfrak{C}' \equiv \mathfrak{A}$.

Suppose now that $\mathbf{a}$ is the degree of $\mathfrak{A}$. Then there is a total degree $\mathbf{c} \in \mathrm{DS}(\mathfrak{C})$ such that $\mathbf{c}' = \mathbf{a}$. Then by [11] since $\mathbf{c}$ is a total degree there exists a subgroup $\mathfrak{G}$ of Q such that $\mathbf{d}_{\mathfrak{G}} = \mathbf{c}$. So, $\mathrm{DS}(\mathfrak{G}) = \{\mathbf{e} : \mathbf{e} \text{ is total and } \mathbf{e} \geq \mathbf{d}_{\mathfrak{G}}\}$. And hence $\mathrm{DS}_1(\mathfrak{G}) = \{\mathbf{e}' : \mathbf{e} \text{ is total } \& \ \mathbf{e}' \geq \mathbf{a}\}$. It is clear that $\mathrm{DS}_1(\mathfrak{G}) \subseteq \mathrm{DS}(\mathfrak{A})$. If

$\mathbf{d} \in \mathrm{DS}(\mathfrak{A})$ then $\mathbf{d} \geq \mathbf{a}$. Since the structure $\mathfrak{A}$ is total $\mathbf{d}$ is total. By the jump inversion theorem from [10] there is a total enumeration degree $\mathbf{e}$ such that $\mathbf{e}' = \mathbf{d}$ and $\mathbf{e} \geq \mathbf{c}$. Then $\mathbf{e}' \in \mathrm{DS}_1(\mathfrak{G})$ and thus $\mathbf{d} \in \mathrm{DS}_1(\mathfrak{G})$. Hence $\mathrm{DS}(\mathfrak{A}) = \mathrm{DS}_1(\mathfrak{G})$. Clearly $\mathrm{DS}(\mathfrak{G}) \subseteq \mathrm{DS}(\mathfrak{B})$ since $\mathbf{d}_{\mathfrak{G}} = \mathbf{c} \in \mathrm{DS}(\mathfrak{G}) \subseteq \mathrm{DS}(\mathfrak{B})$.

The next application concerns a generalization of the notion of degree spectra considered in [13, 15]. Let $\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n$ be countable structures.

Definition 16. *The joint spectrum of* $\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set

$$\mathrm{DS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{\mathbf{a} \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A}), \mathbf{a}' \in \mathrm{DS}(\mathfrak{A}_1), \dots, \mathbf{a}^{(n)} \in \mathrm{DS}(\mathfrak{A}_n)\} \ .$$

The next proposition follows directly from Theorem 14.

Proposition 17. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be total structures such that $\mathfrak{B}' \preceq \mathfrak{A}$. Then there exists a structure $\mathfrak{C} \succeq \mathfrak{B}$ such that* $\mathrm{DS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \mathrm{DS}_1(\mathfrak{C}, \mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.

We can show a similar result for the relativized spectra from [14].

Definition 18. An enumeration f of $\mathfrak{A}$ is *n-acceptable with respect to the structures $\mathfrak{A}_1$, ..., $\mathfrak{A}_n$*, if $f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} (f^{-1}(\mathfrak{A}))^{(i)}$ for each $i \leq n$.

The relative spectrum of the structure $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set

$$\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f \text{ is a } n\text{-acceptable enumeration of } \mathfrak{A}\} \ .$$

Proposition 19. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be total structures such that $\mathfrak{B}' \preceq \mathfrak{A}$. Then there exists a structure $\mathfrak{C} \succeq \mathfrak{B}$ such that* $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \mathrm{RS}_1(\mathfrak{C}, \mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.

Proof (sketch). Let $\mathfrak{C} = \mathfrak{B} \oplus \mathfrak{A}^{\exists\forall}$. Suppose that h is a $(n+1)$-acceptable enumeration of $\mathfrak{C}$ and $\mathrm{d}_e(h^{-1}(\mathfrak{C}))' \in \mathrm{RS}_1(\mathfrak{C}, \mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$. Let $F = h^{-1}(\mathfrak{C})$. Consider a computable in F function m with range $h^{-1}(A)$. Let $s \neq t \in A$. Define an enumeration of $\mathfrak{A}$:

$$f(x) \simeq \begin{cases} h(m(x/2)) & \text{if } x \text{ is even,} \\ s & \text{if } x = 2z+1 \text{ and } z \in F', \\ t & \text{if } x = 2z+1 \text{ and } z \notin F'. \end{cases}$$

Then $f^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F'$ and $f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} h^{-1}(\mathfrak{A}_i) \oplus F' \leq_{\mathrm{e}} h^{-1}(\mathfrak{C})^{(i+1)} \oplus F' \equiv_{\mathrm{e}} F^{(i+1)} \equiv_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(i)}$ for every $i \leq n$. So, $\mathrm{d}_e(h^{-1}(\mathfrak{C}))' \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.

Let f be a n-acceptable enumeration of $\mathfrak{A}$ such that $\mathrm{d}_e(f^{-1}(\mathfrak{A})) \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$. Then as in Theorem 14 one can construct an enumeration h of $\mathfrak{C}$ such that $h^{-1}(\mathfrak{C})' \equiv_{\mathrm{e}} f^{-1}(\mathfrak{A})$ and additionally $h^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A}_i)$ for each $i \leq n$. Then $h^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(i)} \leq_{\mathrm{e}} h^{-1}(\mathfrak{C})^{(i+1)}$. Then $\mathrm{d}_e(f^{-1}(\mathfrak{A})) \in \mathrm{RS}_1(\mathfrak{C}, \mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.

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