Minimal Pairs and Quasi-Minimal Degrees for the Joint Spectra of Structures

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Abstract. Two properties of the Co-spectrum of the Joint spectrum of finitely many abstract structures are presented — a Minimal Pair type theorem and the existence of a Quasi-Minimal degree with respect to the Joint spectrum of the structures.

1 Introduction

Let $\mathfrak{A}$ be a countable abstract structure. The Degree spectrum $\mathrm{DS}(\mathfrak{A})$ of $\mathfrak{A}$ is the set of all enumeration degrees generated by all enumerations of $\mathfrak{A}$. The Co-spectrum of the structure $\mathfrak{A}$ is the set of all enumeration degrees which are lower bounds of the $\mathrm{DS}(\mathfrak{A})$. As a typical example of a spectrum is the cone of the total degrees greater then or equal to some enumeration degree $\mathbf{a}$ and the respective Co-spectrum which is equal to the set all degrees less than or equal to $\mathbf{a}$. There are examples of structures with more complicated degree spectra e.g. [5, 4, 1, 3, 7]. The properties of the Degree spectra are presented in [7] which show that the degree spectra behave with respect to their Co-spectra like the cones of enumeration degrees.

In [8] a generalization of the notions of Degree spectra and Co-spectra for finitely many structures is presented. Let $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ be countable abstract structures. The Joint spectrum of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ is the set $\mathrm{DS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ of all elements of $\mathrm{DS}(\mathfrak{A}_0)$, such that $\mathbf{a}^{(k)} \in \mathrm{DS}(\mathfrak{A}_k)$, for each $k \leq n$.

Here we shall prove two properties of the Co-spectrum of $\mathrm{DS}(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$ — the Minimal Pair type theorem and the existence of a quasi-minimal degree with respect to the Joint spectrum.

The proofs use the technique of regular enumerations introduced in [6], combined with partial generic enumerations used in [7].

2 Preliminaries

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k)$ be a partial structure over the set of all natural numbers $\mathbb{N}$, where each R_i is a subset of $\mathbb{N}^{r_i}$ and $=$ and $\neq$ are among $R_1, \ldots, R_k$.

An enumeration f of $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.

For every $A \subseteq \mathbb{N}^a$ define $f^{-1}(A) = \{\langle x_1 \dots x_a \rangle : (f(x_1), \dots, f(x_a)) \in A\}$. Denote by $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \dots \oplus f^{-1}(R_k)$.

For any sets of natural numbers A and B the set A is enumeration reducible to B ($A \leq_{\mathrm{e}} B$) if there is an enumeration operator Γ_z such that $A = \Gamma_z(B)$. By $d_{\mathrm{e}}(A)$ we denote the enumeration degree of the set A and by $\mathcal{D}_{\mathrm{e}}$ the set of all enumeration degrees. The set A is total if $A \equiv_{\mathrm{e}} A^+$, where $A^+ = A \oplus (\mathbb{N} \setminus A)$. A degree $\mathbf{a}$ is called total if $\mathbf{a}$ contains the e-degree of a total set. The jump operation "$'$" denotes here the enumeration jump introduced by Cooper [2].

Definition 1. *The Degree spectrum of* $\mathfrak{A}$ is the set

$$\mathrm{DS}(\mathfrak{A}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) : f \text{ is an enumeration of } \mathfrak{A}\} \ .$$

Let $B_0, \dots, B_n$ be arbitrary subsets of $\mathbb{N}$. Define the set $\mathcal{P}(B_0, \dots, B_i)$ as follows:

1. $\mathcal{P}(B_0) = B_0$;
2. If $i < n$, then $\mathcal{P}(B_0, \dots, B_{i+1}) = (\mathcal{P}(B_0, \dots, B_i))' \oplus B_{i+1}$.

In the construction of minimal pair we shall use a modification of the "type omitting" version of Jump Inversion Theorem from [6]. In fact, the result follows from the proof of the Theorem 1.7 in [6].

Theorem 2 ([6]). *Let $\{A^k_r\}_r$, $k = 0, \dots, n$ be a sequence of subsets of $\mathbb{N}$ such that for every r and for all k, $0 \leq k < n$, $A^k_r \not\leq_{\mathrm{e}} \mathcal{P}(B_0, \dots, B_k)$. Then there exists a total set F having the following properties:*

1. *$B_i \leq_{\mathrm{e}} F^{(i)}$, for all $i \leq n$;*
2. *$A^k_r \not\leq_{\mathrm{e}} F^{(k)}$, for all r and all $k < n$.*

Definition 3. A set F of natural numbers is called *quasi-minimal over* B_0 if the following conditions hold:

1. $B_0 <_{\mathrm{e}} F$;
2. For any total set $A \subseteq \mathbb{N}$, if $A \leq_{\mathrm{e}} F$, then $A \leq_{\mathrm{e}} B_0$.

In the construction of the quasi-minimal degree we shall use the following fact:

Theorem 4. *There exists a set of natural numbers F having the following properties:*

1. *$B_0 <_{\mathrm{e}} F$;*
2. *For all $1 \leq i \leq n$, $B_i \leq_{\mathrm{e}} F^{(i)}$;*
3. *For any total set A, if $A \leq_{\mathrm{e}} F$, then $A \leq_{\mathrm{e}} B_0$.*

The set F from Theorem 4 is quasi-minimal over B_0. We shall prove this theorem in the last section using the technique of partial regular enumerations.

3 Joint Spectra of Structures

Let $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ be abstract structures on $\mathbb{N}$.

Definition 5. *The Joint spectrum of* $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ is the set

$$\mathrm{DS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \{\mathbf{a} : \mathbf{a} \in \mathrm{DS}(\mathfrak{A}_0), \mathbf{a}' \in \mathrm{DS}(\mathfrak{A}_1), \ldots, \mathbf{a}^{(n)} \in \mathrm{DS}(\mathfrak{A}_n)\} \ .$$

Definition 6. For every $k \leq n$, *the k th Jump spectrum of* $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ is the set

$$\mathrm{DS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n) = \{\mathbf{a}^{(k)} : \mathbf{a} \in \mathrm{DS}(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)\} \ .$$

In [8] is shown that $\mathrm{DS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$ is closed upwards, i.e. if $\mathbf{a}^{(k)} \in \mathrm{DS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$, $\mathbf{b}$ is a total e-degree and $\mathbf{a}^{(k)} \leq \mathbf{b}$, then $\mathbf{b} \in \mathrm{DS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$.

Definition 7. *The kth Co-spectrum of* $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$, $k \leq n$, is the set of all lower bounds of $\mathrm{DS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$, i.e.

$$\mathrm{CS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_{\mathrm{e}} \& (\forall \mathbf{a} \in \mathrm{DS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n))(\mathbf{b} \leq \mathbf{a})\} \ .$$

From [8] we know that the kth Co-spectrum for $k \leq n$ depends only of the first k structures:

$$\mathrm{CS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_k, \ldots, \mathfrak{A}_n) = \mathrm{CS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_k) \ .$$

Let $f_0, \ldots, f_n$ be enumerations of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$. Denote by $\bar{f} = (f_0, \ldots, f_n)$ and $\mathcal{P}_k^{\bar{f}} = \mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \ldots, f_k^{-1}(\mathfrak{A}_k))$, $k = 0, \ldots, n$.

Let $W_0, \ldots, W_z, \ldots$ be a Gödel's enumeration of the c.e. sets and D_v be the finite set having canonical code v.

For every $i \leq n$, e and x in $\mathbb{N}$ define the relations $\bar{f} \models_i F_e(x)$ and $\bar{f} \models_i \neg F_e(x)$ by induction on i:

1. $\bar{f} \models_0 F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ D_v \subseteq f_0^{-1}(\mathfrak{A}_0))$;
2. $\bar{f} \models_{i+1} F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(u = \langle 0, e_u, x_u\rangle \ \&$ $\bar{f} \models_i F_{e_u}(x_u) \ \vee u = \langle 1, e_u, x_u\rangle \ \&\ \bar{f} \models_i \neg F_{e_u}(x_u) \ \vee\ u = \langle 2, x_u\rangle \ \&$ $x_u \in f_{i+1}^{-1}(\mathfrak{A}_{i+1})))$;
3. $\bar{f} \models_i \neg F_e(x) \iff \bar{f} \not\models_i F_e(x)$.

It is easy to check that for any $A \subseteq \mathbb{N}$ and $k \leq n$

$$A \leq_{\mathrm{e}} \mathcal{P}_k^{\bar{f}} \iff (\exists e)(A = \{x : \bar{f} \models_k F_e(x)\}) \ .$$

The forcing conditions which we shall call *finite parts* are $n+1$ tuples $\bar{\tau} = (\tau_0, \ldots, \tau_n)$ of finite mappings $\tau_0, \ldots, \tau_n$ of $\mathbb{N}$ in $\mathbb{N}$.

For any $i \leq n$, e and x in $\mathbb{N}$ and every finite part $\bar{\tau}$ we define the forcing relations $\bar{\tau} \Vdash_i F_e(x)$ and $\bar{\tau} \Vdash_i \neg F_e(x)$ following the definition of relation "$\models_i$".

Definition 8. 1. $\bar{\tau} \Vdash_0 F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ D_v \subseteq \tau_0^{-1}(\mathfrak{A}_0))$;

2. $\bar{\tau} \Vdash_{i+1} F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ \ (\forall u \in D_v)(u = \langle 0, e_u, x_u\rangle \ \&$ $\bar{\tau} \Vdash_i F_{e_u}(x_u) \ \vee u = \langle 1, e_u, x_u\rangle \ \&\ \bar{\tau} \Vdash_i \neg F_{e_u}(x_u) \ \vee\ u = \langle 2, x_u\rangle \ \&$ $x_u \in \tau_{i+1}^{-1}(\mathfrak{A}_{i+1})))$;

3. $\bar{\tau} \Vdash_i \neg F_e(x) \iff (\forall \bar{\rho} \supseteq \bar{\tau})(\bar{\rho} \nVdash_i F_e(x))$.

For any $i \leq n, e, x \in \mathbb{N}$ denote by $X^i_{\langle e,x \rangle} = \{\bar{\rho} : \bar{\rho} \Vdash_i F_e(x)\}$.

Definition 9. An enumeration $\bar{f}$ of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ is *i-generic* if for every $j < i$, $e, x \in \mathbb{N}$

$$(\forall \bar{\tau} \subseteq \bar{f})(\exists \bar{\rho} \in X^j_{\langle e,x \rangle})(\bar{\tau} \subseteq \bar{\rho}) \Longrightarrow (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \in X^j_{\langle e,x \rangle}) \ .$$

In [8] the following properties of the k-generic enumertions are shown:

1. If $\bar{f}$ is an k-generic enumeration, then

$$\bar{f} \models_k F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_k F_e(x)) \ .$$

2. If $\bar{f}$ is an $(k+1)$-generic enumeration, then

$$\bar{f} \models_k \neg F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_k \neg F_e(x)) \ .$$

Definition 10. The set $A \subseteq \mathbb{N}$ is *forcing k-definable* on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ if there exist a finite part $\bar{\delta}$ and $e \in \mathbb{N}$ such that

$$x \in A \iff (\exists \bar{\tau} \supseteq \bar{\delta})(\bar{\tau} \Vdash_k F_e(x)) \ .$$

In [8] the following characterization of the sets which generates the elements of the kth Co-spectrum of $\mathrm{DS}(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$ is given:

Theorem 11 ([8]). *For every $A \subseteq \mathbb{N}$, the following are equivalent:*

1. $d_{\mathrm{e}}(A) \in \mathrm{CS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$.
2. $A \leq_{\mathrm{e}} \mathcal{P}^{\bar{f}}_k$, *for all* $\bar{f} = (f_0, \ldots, f_k)$ *enumerations of* $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$.
3. *A is forcing k-definable on* $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$.

Theorem 12. *Let $\{X^k_r\}_r$, $k = 0, \ldots, n$ be $n+1$ sequences of sets of natural numbers. There exists a $(n+1)$-generic enumeration $\bar{f}$ of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ such that for any $k \leq n$ and for all $r \in \mathbb{N}$, if the set X^k_r is not forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$, then $X^k_r \not\leq_{\mathrm{e}} \mathcal{P}^{\bar{f}}_k$.*

4 Minimal Pair Theorem

In [7] a Minimal Pair Theorem for Degree spectrum of a structure $\mathfrak{A}$ is presented. Using the technique of splitting generic enumerations it is proven there that for each constructive ordinal α there exist elements $\mathbf{f}$ and $\mathbf{g}$ of $DS(\mathfrak{A})$ such that for any enumeration degree $\mathbf{a}$ and any $\beta + 1 < \alpha$

$$\mathbf{a} \leq \mathbf{f}^{(\beta)} \ \& \ \mathbf{a} \leq \mathbf{g}^{(\beta)} \Rightarrow \mathbf{a} \in CS_\beta(\mathfrak{A}) \ .$$

We shall prove an analogue of the Minimal Pair Theorem for the Joint spectrum.

Theorem 13. *For all structures $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$, there exist enumeration degrees* $\mathbf{f}$ *and* $\mathbf{g}$ *in* $\mathrm{DS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$, *such that for any enumeration degree* $\mathbf{a}$ *and* $k \leq n$*:*

$$\mathbf{a} \leq \mathbf{f}^{(k)} \ \& \ \mathbf{a} \leq \mathbf{g}^{(k)} \Rightarrow \mathbf{a} \in \mathrm{CS}_k(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \ .$$

Proof. We shall construct two total sets F and G, such that $d_{\mathrm{e}}(F) \in \mathrm{DS}(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$, $d_{\mathrm{e}}(G) \in \mathrm{DS}(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$ and for each $k \leq n$, if a set X, $X \leq_{\mathrm{e}} F^{(k)}$ and $X \leq_{\mathrm{e}} G^{(k)}$, then $d_{\mathrm{e}}(X) \in \mathrm{CS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$. And take $\mathbf{f} = d_{\mathrm{e}}(F)$ and $\mathbf{g} = d_{\mathrm{e}}(G)$.

First we construct enumerations $\bar{f}$ and $\bar{h}$ of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ such that for any $k \leq n$, if a set $A \subseteq \mathbb{N}$, $A \leq_{\mathrm{e}} \mathcal{P}_k^{\bar{f}}$ and $A \leq_{\mathrm{e}} \mathcal{P}_k^{\bar{h}}$, then A is a forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$.

Let $g_0, \ldots, g_n$ be arbitrary enumerations of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$. By Theorem 2 for $B_0 = g_0^{-1}(\mathfrak{A}_0), \ldots, B_n = g_n^{-1}(\mathfrak{A}_n)$ there exists a total set F, such that: $g_0^{-1}(\mathfrak{A}_0) \leq_{\mathrm{e}} F$, $g_1^{-1}(\mathfrak{A}_1) \leq_{\mathrm{e}} F'$,..., $g_n^{-1}(\mathfrak{A}_n) \leq_{\mathrm{e}} F^{(n)}$. Since $\mathrm{DS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ is closed upwards, then $d_{\mathrm{e}}(F) \in \mathrm{DS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$, i.e. $d_{\mathrm{e}}(F) \in \mathrm{DS}(\mathfrak{A}_0), d_{\mathrm{e}}(F') \in \mathrm{DS}(\mathfrak{A}_1)$, ..., $d_{\mathrm{e}}(F^{(n)}) \in \mathrm{DS}(\mathfrak{A}_n)$.

Hence, there exist enumerations $h_0, h_1, \ldots, h_n$ of $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$, respectively, such that $h_0^{-1}(\mathfrak{A}_0) \equiv_{\mathrm{e}} F, h_1^{-1}(\mathfrak{A}_1) \equiv_{\mathrm{e}} F', \ldots, h_n^{-1}(\mathfrak{A}_n) \equiv_{\mathrm{e}} F^{(n)}$. Notice, that for each $k \leq n$, $F^{(k)} \equiv_{\mathrm{e}} \mathcal{P}_k^{\bar{h}}$.

For each $k \leq n$, let $\{X_r^k\}_r$ be the sequence of all sets enumeration reducible to $\mathcal{P}_k^{\bar{h}}$.

By Theorem 12 there is an $(n+1)$-generic enumeration $\bar{f}$ such that for all r, and all $k = 0, \ldots, n$ if the set X_r^k is not forcing k-definable then $X_r^k \not\leq_{\mathrm{e}} \mathcal{P}_k^{\bar{f}}$.

Suppose now that the set $A \leq_{\mathrm{e}} \mathcal{P}_k^{\bar{f}}$ and $A \leq_{\mathrm{e}} \mathcal{P}_k^{\bar{h}}$. Then $A = X_r^k$ for some r. From the omitting condition of $\bar{f}$ it follows that A is forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$.

Now we apply again the Theorem 2. Let $B_0 = f_0^{-1}(\mathfrak{A}_0), \ldots, B_n = f_n^{-1}(\mathfrak{A}_n)$ and $B_{n+1} = N$. For each $k \leq n$ consider the sequence $\{A_r^k\}_r$ of these sets among the sets $\{X_r^k\}_r$, which are not forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$. From the choice of the enumeration $\bar{f}$ it follows that each of these sets A_r^k, $A_r^k \not\leq_{\mathrm{e}} \mathcal{P}_k^{\bar{f}}$. Then by Theorem 2 there is a total set G, such that

1. For all $k \leq n$, $f_k^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} G^{(k)}$;
2. For all r and all $k \leq n$, $A_r^k \not\leq_{\mathrm{e}} G^{(k)}$.

Note, that since G is a total set, and because of the fact that each spectrum is closed upwards, we have that $d_{\mathrm{e}}(G) \in \mathrm{DS}(\mathfrak{A}_0), \ldots, d_{\mathrm{e}}(G^{(n)}) \in \mathrm{DS}(\mathfrak{A}_n)$, and hence $d_{\mathrm{e}}(G) \in \mathrm{DS}(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$.

Suppose now, that a set X, $X \leq_{\mathrm{e}} F^{(k)}$ and $X \leq_{\mathrm{e}} G^{(k)}$, $k \leq n$. From $X \leq_{\mathrm{e}} F^{(k)}$ and $F^{(k)} \equiv_{\mathrm{e}} \mathcal{P}_k^{\bar{h}}$, it follows that $X = X_r^k$ for some r. It is clear that $X \leq_{\mathrm{e}} \mathcal{P}_k^{\bar{f}}$. Otherwise from the choice of G it follows that $X \not\leq_{\mathrm{e}} G^{(k)}$. Hence X is forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$. By the normal form of the sets which enumeration degrees are in $\mathrm{CS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$, we have that $d_{\mathrm{e}}(X) \in \mathrm{CS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$.

5 Quasi-Minimal Degree

Given a set $\mathcal{A}$ of enumeration degrees denote by $co(\mathcal{A})$ the set of all lower bounds of $\mathcal{A}$. Say that the degree $\mathbf{q}$ is *quasi-minimal with respect to* $\mathcal{A}$ if the following conditions hold ([7]):

1. $\mathbf{q} \notin co(\mathcal{A})$.
2. If $\mathbf{a}$ is a total degree and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$.
3. If $\mathbf{a}$ is a total degree and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.

In [7] it is shown that there is a quasi-minimal degree $\mathbf{q}_0$ with respect to $\mathrm{DS}(\mathfrak{A}_0)$, i.e. $\mathbf{q}_0 \notin \mathrm{CS}(\mathfrak{A}_0)$ and for every total degree $\mathbf{a}$: if $\mathbf{a} \geq \mathbf{q}_0$, then $\mathbf{a} \in \mathrm{DS}(\mathfrak{A}_0)$ and if $\mathbf{a} \leq \mathbf{q}_0$, then $\mathbf{a} \in \mathrm{CS}(\mathfrak{A}_0)$.

We are going to prove the existence of a quasi-minimal degree with respect to $\mathrm{DS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

Theorem 14. *For all structures $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$ there exists an enumeration degree $\mathbf{q}$ such that:*

1. $\mathbf{q}' \in \mathrm{DS}(\mathfrak{A}_1), \ldots, \mathbf{q}^{(n)} \in \mathrm{DS}(\mathfrak{A}_n)$, $\mathbf{q} \notin \mathrm{CS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$;
2. *If $\mathbf{a}$ is a total degree and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathrm{DS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$;*
3. *If $\mathbf{a}$ is a total degree and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in \mathrm{CS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.*

Proof. Let $\mathbf{q}_0$ be a quasi-minimal degree $\mathbf{q}_0$ with respect to $\mathrm{DS}(\mathfrak{A}_0)$ from [7].

Let $B_0 \subseteq \mathbb{N}$, such that $d_e(B_0) = \mathbf{q}_0$, and $f_1, \ldots, f_n$ be fixed total enumerations of $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. Set $B_1 = f_1^{-1}(\mathfrak{A}_1), \ldots, B_n = f_n^{-1}(\mathfrak{A}_n)$. By Theorem 4 there is quasi-minimal over B_0 set F, such that $B_0 <_e F$, $B_i \leq_e F^{(i)}$, for each $1 \leq i \leq n$, and moreover for any total set A, if $A \leq_e F$, then $A \leq_e B_0$. We will show that $\mathbf{q} = d_e(F)$ is quasi-minimal with respect to $\mathrm{DS}(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$.

Since $\mathbf{q}_0$ is quasi-minimal with respect to $\mathrm{DS}(\mathfrak{A}_0)$, $\mathbf{q}_0 \notin \mathrm{CS}(\mathfrak{A}_0)$. But $\mathbf{q}_0 < \mathbf{q}$ and thus $\mathbf{q} \notin \mathrm{CS}(\mathfrak{A}_0)$. Hence $\mathbf{q} \notin \mathrm{CS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

For each $1 \leq i \leq n$, the set $F^{(i)}$ is total and $f_i^{-1}(\mathfrak{A}_i) \leq_e F^{(i)}$. Since any degree spectrum is closed upwards it follows that $d_e(F^{(i)}) \in \mathrm{DS}(\mathfrak{A}_i)$, i.e. $\mathbf{q}^{(i)} \in \mathrm{DS}(\mathfrak{A}_i)$.

Consider a total set X, such that $X \geq_e F$. Then $d_e(X) \geq \mathbf{q}_0$. From the fact that $\mathbf{q}_0$ is quasi-minimal with respect to $\mathrm{DS}(\mathfrak{A}_0)$ it follows that $d_e(X) \in \mathrm{DS}(\mathfrak{A}_0)$. Moreover for each $1 \leq i \leq n$, $X^{(i)} \geq_e F^{(i)} \geq_e f_i^{-1}(\mathfrak{A}_i)$, and $X^{(i)}$ is a total set. Then for each $i \leq n$, $d_e(X^{(i)}) \in \mathrm{DS}(\mathfrak{A}_i)$, and hence $d_e(X) \in \mathrm{DS}(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$.

Suppose that X is a total set and $X \leq_e F$. Then, from the choice of F, since X is total, $X \leq_e B_0$. Apply again the quasi-minimality of $\mathbf{q}_0$ and then $d_e(X) \in \mathrm{CS}(\mathfrak{A}_0)$. But $\mathrm{CS}(\mathfrak{A}_0, \ldots, \mathfrak{A}_n) = \mathrm{CS}(\mathfrak{A}_0)$ and therefore $d_e(X) \in \mathrm{CS}(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$. □

In the rest of the paper we shall present the proof of Theorem 4.

6 Partial Regular Enumerations

Let $B_0, \ldots, B_n$ be fixed sets of natural numbers. Combining the technique of the (total) regular enumerations from [6] with the partial generic enumerations, introduced in [7], we shall construct a partial regular enumeration f, which graph will be quasi-minimal over the set B_0 and such that $B_i \leq_e f^{(i)}$, for $0 \leq i \leq n$. In [7] a partial generic enumeration of B_0 is constructed, which is quasi-minimal over B_0. In addition, the enumeration f we are going to obtain, will code the sets $B_1, \ldots, B_n$ in its jumps ($B_i \leq_e f^{(i)}$).

Definition 15. *A partial enumeration f of B_0* is a partial surjective mapping from $\mathbb{N}$ onto $\mathbb{N}$ with the following properties:

1. For all odd x, if $f(x)$ is defined, then $f(x) \in B_0$;
2. For all $y \in B_0$, there is an odd x, such that $f(x) \simeq y$.

It is clear that if f is a partial enumeration of B_0, then $B_0 \leq_e f$.

Let $\perp \notin \mathbb{N}$.

Definition 16. A *partial finite part* τ is a finite mapping of $\mathbb{N}$ into $\mathbb{N} \cup \{\perp\}$, such that $(\forall x)(x \in \mathrm{dom}(\tau)\ \&\ x \text{ is odd} \Rightarrow (\tau(x) = \perp \vee \tau(x) \in B_0))$.

If τ is a partial finite part and f is a partial enumeration of B_0, say that

$$\tau \subseteq f \iff (\forall x \in \mathrm{dom}(\tau))((\tau(x) = \perp \Rightarrow f(x) \text{ is not defined })\ \&$$
$$(\tau(x) \neq \perp \Rightarrow \tau(x) \simeq f(x)) \ .$$

A 0-*regular partial finite part* is a partial finite part τ such that $\mathrm{dom}(\tau) = [0, 2q+1]$ and for all odd $z \in \mathrm{dom}(\tau)$, $\tau(z) \in B_0$ or $\tau(z) = \perp$. The 0-rank of τ, $|\tau|_0 = q+1$ we call the number of the odd elements of $\mathrm{dom}(\tau)$. If ρ is a 0-regular partial extention of τ we shall denote this by $\tau \subseteq_0 \rho$. It is clear that if $\tau \subseteq_0 \rho$ and $|\tau|_0 = |\rho|_0$, then $\tau = \rho$. Let

$$\tau \Vdash_0 F_e(x) \iff \exists v(\langle v, x\rangle \in W_e\ \&\ (\forall u \in D_v)(u = \langle s, t\rangle,\ \&\ \tau(s) \simeq t\ \&\ t \neq \perp))$$

$$\tau \Vdash_0 \neg F_e(x) \iff (\forall \rho)(\tau \subseteq_0 \rho \Rightarrow \rho \nVdash_0 F_e(x)) \ .$$

The $(i+1)$-regular partial finite part τ, the $(i+1)$-rank $|\tau|_{i+1}$ of τ and the relations $\tau \Vdash_{i+1} F_e(x)$ and $\tau \Vdash_{i+1} \neg F_e(x)$ are defined by induction on i, in the same way as in [6]. The only difference is that instead of i-regular finite parts we use i-regular partial finite parts. Denote by $\mathcal{R}_i$ the set of all i-regular partial finite parts.

For any i-regular finite part τ and any set X of i-regular finite parts, denote by $\mu_i(\tau, X) = \mu\rho[\tau \subseteq \rho\ \&\ \rho \in \mathcal{R}_i\ \&\ \rho \in X]$ if any, and $\mu_i(\tau, X) = \mu\rho[\tau \subseteq \rho\ \&\ \rho \in \mathcal{R}_i]$, otherwise.

Denote by $X^i_{\langle e,x\rangle} = \{\rho : \rho \text{ is } i\text{-regular}\ \&\ \rho \Vdash_i F_e(x)\}$.

Let τ be a finite part and $m \geq 0$. The finite part δ is called an *i-regular m omitting extension* of τ if $\delta \supseteq \tau$, $\delta \in \mathcal{R}_i$, $\mathrm{dom}(\delta) = [0, q-1]$ and there exist natural numbers $q_0 < \ldots < q_m < q_{m+1} = q$ such that:

1. $\delta \restriction q_0 = \tau$.
2. For all $p \leq m$, $\delta \restriction q_{p+1} = \mu_i(\delta \restriction (q_p + 1), X^i_{\langle p, q_p\rangle})$.

If δ and ρ are two i-regular m omitting extensions of τ and $\delta \subseteq \rho$ then $\delta = \rho$. Given an index j, by S^i_j we shall denote the intersection $\mathcal{R}_i \cap \Gamma_j(\mathcal{P}(B_0, \dots, B_i))$, where Γ_j is the jth enumeration operator.

Let τ be a finite part defined on $[0, q-1]$ and $r \geq 0$. Then τ is $(i+1)$-*regular* with $(i+1)$-*rank* $r+1$ if there exist natural numbers

$$0 < n_0 < l_0 < b_0 < n_1 < l_1 < b_1 \ldots < n_r < l_r < b_r < n_{r+1} = q$$

such that $\tau \upharpoonright n_0$ is an i-regular finite part with i-rank equal to 1 and for all j, $0 \leq j \leq r$, the following conditions are satisfied:

(a) $\tau \upharpoonright l_j \simeq \mu_i(\tau \upharpoonright (n_j + 1), S^i_j)$;
(b) $\tau \upharpoonright b_j$ is an i-regular j omitting extension of $\tau \upharpoonright l_j$;
(c) $\tau(b_j) \in B_{i+1}$;
(d) $\tau \upharpoonright n_{j+1}$ is an i-regular extension of $\tau \upharpoonright (b_j + 1)$ with i-rank equal to $|\tau \upharpoonright b_j|_i + 1$.

If τ is an i-regular partial finite part, then τ is a j-regular partial finite part for each $j < i$ and $|\tau|_j > |\tau|_i$.

Definition 17. *A partial regular enumeration* is a partial enumeration, such that:

1. For every partial finite part $\delta \subseteq f$, there exists an n-regular partial extension τ of δ such that $\tau \subseteq f$.
2. If $i \leq n$ and $z \in B_i$, then there exists an i-regular partial finite part $\tau \subseteq f$, such that $z \in \mathrm{dom}(\tau)$.

If f is a partial regular enumeration, $\delta \subseteq f$ and $i \leq n$, then there exists an i-regular partial finite part τ of an arbitrary large rank such that $\delta \subseteq \tau$ and $\tau \subseteq f$.

Denote by $\mathcal{P}_i = \mathcal{P}(B_0, \dots, B_i)$. It is clear that $\mathcal{R}_i \leq_{\mathrm{e}} \mathcal{P}_i$.

Definition 18. A partial enumeration f is i-*generic* if for any $j < i$ and for every enumeration reducible to $\mathcal{P}_j$ set S of j-regular partial finite parts the following condition holds:

$$(\exists \tau \subseteq f)(\tau \in S \vee (\forall \rho \supseteq \tau)(\rho \in \mathcal{R}_i \Rightarrow \rho \notin S)) \ .$$

Proposition 19. *Every partial regular enumeration is $(i+1)$-generic enumeration, for every $i < n$.*

Proposition 20. *Suppose that f is a partial regular enumeration. Then*

1. *For each $i \leq n$, $B_i \leq_{\mathrm{e}} f^{(i)}$.*
2. *If $i < n$, then $f \not\leq_{\mathrm{e}} \mathcal{P}_i$.*

Definition 21. If f is a partial enumeration define:

$$f \models_0 F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \& \ (\forall u \in D_v)(f((u)_0) \simeq (u)_1)) \ .$$

Proof of Theorem 4. By Proposition 20 it is sufficient to show that there exists a partial regular enumeration f which is quasi-minimal over B_0.

We shall construct f as a union of n-regular partial finite parts δ_s such that for all s, $\delta_s \subseteq \delta_{s+1}$ and $|\delta_s|_n = s+1$. Suppose that for $i \leq n$, σ_i is a recursively in B_i enumeration of B_i.

Let δ_0 be a 0-regular partial finite part such that $|\delta_0|_n = 1$. Suppose that δ_s is defined. Set $z_0 = \sigma_0(s), \ldots, z_n = \sigma_n(s)$. We can construct effectively in $\mathcal{P}'_{n-1}$ a n-regular partial finite part $\rho \supseteq \delta_s$ such that $|\rho|_n = |\delta_s|_n + 1$, $\rho(\mathrm{lh}(\delta_s)) = s$ and $z_0 = \rho(x_0)$ for some $x_0 \in B_0, \ldots, z_n = \rho(x_n)$ for some $x_n \in B_n$. Set $\delta_{s+1} = \rho$.

The obtained enumeration f is surjective on $\mathbb{N}$ and it is a union of n-regular partial finite parts. From the construction is obvious that for every $z \in B_i$ there is an i-regular partial finite part τ of f, such that $z \in \mathrm{dom}(\tau)$. Hence f is a partial regular enumeration. By Proposition 19 f is $(i+1)$-generic for each $i < n$.

Then by Proposition 20, for $i \leq n$, $B_i \leq f^{(i)}$. Moreover f is a partial 1-generic enumeration and hence $B_0 <_{\mathrm{e}} f$.

To prove that f is quasi-minimal over B_0, it is sufficient to show that if ψ is a total function and $\psi \leq_{\mathrm{e}} f$, then $\psi \leq_{\mathrm{e}} B_0$. It is clear that for any total set $A \subseteq \mathbb{N}$ one can construct a total function ψ, $\psi \equiv_{\mathrm{e}} A$. Let ψ be a total function and $\psi = \Gamma_e(f)$. Then

$$(\forall x, y \in \mathbb{N})(f \models_0 F_e(\langle x, y\rangle) \iff \psi(x) \simeq y) \ .$$

Consider the set

$$S_0 = \{\rho : \rho \in \mathcal{R}_0 \ \& \ (\exists x, y_1 \neq y_2 \in \mathbb{N})(\rho \Vdash_0 F_e(\langle x, y_1\rangle) \ \& \ \rho \Vdash_0 F_e(\langle x, y_2\rangle))\} \ .$$

Since $S_0 \leq_{\mathrm{e}} B_0$, we have that there exists a 0-regular partial finite part $\tau_0 \subseteq f$ such that either $\tau_0 \in S_0$ or $(\forall \rho \supseteq_0 \tau_0)(\rho \notin S_0)$. Assume that $\tau_0 \in S_0$. Then there exist $x, y_1 \neq y_2$ such that $f \models_0 F_e(\langle x, y_1\rangle)$ and $f \models_0 F_e(\langle x, y_2\rangle)$. Then $\psi(x) \simeq y_1$ and $\psi(x) \simeq y_2$ which is impossible. So, $(\forall \rho \supseteq_0 \tau_0)(\rho \notin S_0)$.

Let

$$\begin{aligned} S_1 = \{\rho : \rho \in \mathcal{R}_0 \ \& \ &(\exists \tau \supseteq_0 \tau_0)(\exists \delta_1 \supseteq_0 \tau)(\exists \delta_2 \supseteq_0 \tau)(\exists x, y_1 \neq y_2)(\tau \subseteq_0 \rho \ \& \\ &\delta_1 \Vdash_0 F_e(\langle x, y_1\rangle) \ \& \ \delta_2 \Vdash_0 F_e(\langle x, y_2\rangle) \ \& \ \mathrm{dom}(\rho) = \mathrm{dom}(\delta_1) \cup \mathrm{dom}(\delta_2) \\ &\& (\forall x)(x \in \mathrm{dom}(\rho) \setminus \mathrm{dom}(\tau) \Rightarrow \rho(x) \simeq \bot))\} \ . \end{aligned}$$

We have that $S_1 \leq_{\mathrm{e}} B_0$ and hence there exists a 0-regular partial finite part $\tau_1 \subseteq f$ such that either $\tau_1 \in S_1$ or $(\forall \rho \supseteq_0 \tau_1)(\rho \notin S_1)$.

Assume that $\tau_1 \in S_1$. Then there exists a 0-regular partial finite part τ such that $\tau_0 \subseteq_0 \tau \subseteq_0 \tau_1$ and for some $\delta_1 \supseteq_0 \tau$, $\delta_2 \supseteq_0 \tau$ and $x_0, y_1 \neq y_2 \in \mathbb{N}$ we have

$$\begin{aligned} &\delta_1 \Vdash_0 F_e(\langle x_0, y_1\rangle) \ \& \ \delta_2 \Vdash_0 F_e(\langle x_0, y_2\rangle) \ \& \ \mathrm{dom}(\tau_1) = \mathrm{dom}(\delta_1) \cup \mathrm{dom}(\delta_2) \ \& \\ &\& \ (\forall x)(x \in \mathrm{dom}(\tau_1) \setminus \mathrm{dom}(\tau) \Rightarrow \tau_1(x) \simeq \bot) \ . \end{aligned}$$

Let $\psi(x_0) \simeq y$. Then $f \models_0 F_e(\langle x_0, y\rangle)$. Hence there exists a $\rho \supseteq_0 \tau_1$ such that $\rho \Vdash_0 F_e(\langle x_0, y\rangle)$. Let $y \neq y_1$. Define the partial finite part ρ_0 as follows:

$$\rho_0(x) \simeq \begin{cases} \delta_1(x) \text{ if } x \in \mathrm{dom}(\delta_1), \\ \rho(x) \ \text{ if } x \in \mathrm{dom}(\rho) \setminus \mathrm{dom}(\delta_1). \end{cases}$$

Then $\tau_0 \subseteq_0 \rho_0$, $\delta_1 \subseteq_0 \rho_0$ and notice that for all $x \in \mathrm{dom}(\rho)$ if $\rho(x) \not\simeq \bot$, then $\rho(x) \simeq \rho_0(x)$. Hence $\rho_0 \Vdash_0 F_e(\langle x_0, y_1\rangle)$ and $\rho_0 \Vdash_0 F_e(\langle x_0, y\rangle)$. So, $\rho_0 \in S_0$. A contradiction.

Thus, $(\forall \rho)(\rho \supseteq_0 \tau_1 \Rightarrow \rho \notin S_1)$.

Let $\tau = \tau_1 \cup \tau_0$. Notice that $\tau \subseteq f$. We shall show that

$$\psi(x) \simeq y \iff (\exists \delta \supseteq_0 \tau)(\delta \Vdash_0 F_e(\langle x, y\rangle)) \ .$$

And hence $\psi \leq_{\mathrm{e}} B_0$.

If $\psi(x) \simeq y$, then $f \models_0 F_e(x)$, and since f is regular, $(\exists \rho \subseteq f)(\rho \Vdash_0 F_e(x))$ and ρ is 0-regular. Then take $\delta = \tau \cup \rho$.

Assume that $\delta_1 \supseteq_0 \tau$, $\delta_1 \Vdash_0 F_e(\langle x, y_1\rangle)$. Suppose that $\psi(x) \simeq y_2$ and $y_1 \neq y_2$. Then there exists a $\delta_2 \supseteq_0 \tau$ such that $\delta_2 \Vdash_0 F_e(\langle x, y_2\rangle)$. Set

$$\rho(x) \simeq \begin{cases} \tau(x) & \text{if } x \in \mathrm{dom}(\tau), \\ \bot & \text{if } x \in (\mathrm{dom}(\delta_1) \cup \mathrm{dom}(\delta_2)) \setminus \mathrm{dom}(\tau). \end{cases}$$

Clearly $\rho \supseteq_0 \tau_1$ and $\rho \in S_1$. A contradiction. □

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