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PROPERTIES OF CO-SPECTRA OF JOINT SPECTRA OF STRUCTURES

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Two properties of co-spectrum of joint spectrum of finitely many abstract structures are presented: a minimal pair type theorem and an existence of a quasi-minimal degree for the joint spectrum.

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1. INTRODUCTION

Let $\mathfrak{A}$ be an abstract structure. The degree spectrum $DS(\mathfrak{A})$ of $\mathfrak{A}$ is the set of all enumeration degrees generated by all presentations of $\mathfrak{A}$ on the natural numbers. In [6, 2, 5, 4, 9] several results about degree spectra of structures are obtained.

The co-spectrum of the structure $\mathfrak{A}$ is the set of all lower bounds of the degree spectra of $\mathfrak{A}$. Co-spectra are introduced and studied in [9].

In [10] a generalization of the notions of degree spectra and co-spectra for finitely many structures is presented - the so called joint spectrum and co-spectrum. A normal form of the sets which generates the elements of the co-spectrum of the joint spectrum in terms of some positive recursive Σ^+ formulae, introduced first in [1], is obtained.

Here we shall prove two properties of the co-spectrum of joint spectrum of structures - the Minimal pair type theorem and the existence of a quasi-minimal degree for the joint spectrum.

The proofs use the technique of regular enumerations introduced in [8], combined with partial generic enumerations used in [9].

2. PRELIMINARIES

Let $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_k)$ be a partial structure over the set of all natural numbers $\mathbb{N}$, where each R_i is a subset of $\mathbb{N}^{r_i}$ and "=" and "≠" are among $R_1, \dots, R_k$.

An enumeration f *of* $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.

If $A \subseteq \mathbb{N}^a$, define

$$f^{-1}(A) = \{\langle x_1 \dots x_a \rangle : (f(x_1), \dots, f(x_a)) \in A\}.$$

Let $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \dots \oplus f^{-1}(R_k)$.

For any sets of natural numbers A and B the set A is enumeration reducible to B ($A \leq_e B$) if there is an enumeration operator Γ_z such that $A = \Gamma_z(B)$. By $d_e(A)$ we denote the enumeration degree of the set A and by $\mathcal{D}_e$ the set of all enumeration degrees. The set A is total if $A \equiv_e A^+$, where $A^+ = A \oplus (\mathbb{N} \setminus A)$. A degree a is called total if a contains the e-degree of a total set. The jump operation "$'$" denotes here the enumeration jump introduced by Cooper [3].

Given $n+1$ subsets $B_0, \dots, B_n$ of $\mathbb{N}$, $i \leq n$, define the set $\mathcal{P}(B_0, \dots, B_i)$ as follows:

(i) $\mathcal{P}(B_0) = B_0$;

(ii) If $i < n$, then $\mathcal{P}(B_0, \dots, B_{i+1})(\mathcal{P}(B_0, \dots, B_i))' \oplus B_{i+1}$.

3. JOINT SPECTRA OF STRUCTURES

Let $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ be abstract structures on $\mathbb{N}$.

The *joint sSpectrum* of $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ is the set

$$DS(\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n)\{\mathbf{a} : \mathbf{a} \in DS(\mathfrak{A}_0), \mathbf{a}' \in DS(\mathfrak{A}_1), \dots, \mathbf{a}^{(n)} \in DS(\mathfrak{A}_n)\}.$$

For every $k \leq n$, *the k-th jump spectrum of* $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ is the set

$$DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)\{\mathbf{a}^{(k)} : \mathbf{a} \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_n)\}.$$

In [10] we prove that $DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$ is closed upwards, i.e. if $\mathbf{a}^{(k)} \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, $\mathbf{b}$ is a total e-degree and $\mathbf{a}^{(k)} \leq \mathbf{b}$, then $\mathbf{b} \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$.

The k-th co-spectrum of $\mathfrak{A}_0, \dots, \mathfrak{A}_n$, $k \leq n$, is the set of all lower bounds of $DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, i.e.

$$CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)\{\mathbf{b} : \mathbf{b} \in \mathcal{D}_e \& (\forall \mathbf{a} \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n))(\mathbf{b} \leq \mathbf{a})\}.$$

From [10] we know that the k-th Co-spectrum for $k \leq n$ depends only of the first k structures:

$$CS_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_k, \ldots, \mathfrak{A}_n) CS_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_k).$$

In [10] we give a normal form of the sets which generates the elements of the k-th co-spectrum of $DS(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$, i.e. for every $A \subseteq \mathbb{N}$ the following are equivalent:

(1) $d_e(A) \in CS_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$;

(2) For every $f_0, \ldots, f_k$ enumerations of $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$, respectively,

$$A \leq_e \mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \ldots, f_k^{-1}(\mathfrak{A}_k));$$

(3) A is forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$;

(4) A is formally k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$.

In Section 4 we shall recall the definition of the forcing k-definable sets on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$.

The analog of the Minimal pair theorem, which we shall prove in Section 5, is in the following form:

Theorem 3.1. *Let $k \leq n$. There exist enumeration degrees* $\mathbf{f}$ *and* $\mathbf{g}$*, elements of $DS(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$, such that for any enumeration degree* $\mathbf{a}$*:*

$$\mathbf{a} \leq \mathbf{f}^{(k)} \ \&\ \mathbf{a} \leq \mathbf{g}^{(k)} \Longrightarrow \mathbf{a} \in CS_k(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n).$$

The proof uses the technique of the regular enumerations from [8], which we will discuss in Section 6.

Given a set $\mathcal{A}$ of enumeration degrees, denote by $co(\mathcal{A})$ the set of all lower bounds of $\mathcal{A}$. Say that the degree $\mathbf{q}$ is *a quasi-minimal with respect to* $\mathcal{A}$ if the following conditions hold:

(i) $\mathbf{q} \notin co(\mathcal{A})$;

(ii) If $\mathbf{a}$ is a total degree and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$;

(iii) If $\mathbf{a}$ is a total degree and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.

The second property, we are going to prove in Section 7, is the existence of a quasi-minimal degree with respect to $DS(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

Theorem 3.2. *There exists an enumeration degree* $\mathbf{q}$ *such that:*

(i) $\mathbf{q}' \in DS(\mathfrak{A}_1), \ldots, \mathbf{q}^{(n)} \in DS(\mathfrak{A}_n)$, $q \notin CS(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$;

(ii) *If* $\mathbf{a}$ *is a total degree and* $\mathbf{a} \geq \mathbf{q}$*, then* $\mathbf{a} \in DS(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$;

(iii) *If* $\mathbf{a}$ *is a total degree and* $\mathbf{a} \le \mathbf{q}$*, then* $\mathbf{a} \in CS(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

4. FORCING k-DEFINABLE SETS

Suppose that $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ are structures on $\mathbb{N}$. Let $f_0, \ldots, f_n$ be enumerations of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$, respectively.

Denote by $\bar{f} = (f_0, \ldots, f_n)$ and $\mathcal{P}^{\bar{f}}_k \mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \ldots, f_k^{-1}(\mathfrak{A}_k))$ for $k = 0, \ldots, n$.

Let $W_0, \ldots, W_z, \ldots$ be a Goedel enumeration of the r.e. sets and D_v be the finite set having a canonical code v.

For every $i \le n$, e and x in $\mathbb{N}$ define the relations $\bar{f} \models_i F_e(x)$ and $\bar{f} \models_i \neg F_e(x)$ by induction on i:

(i) $\bar{f} \models_0 F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ D_v \subseteq f_0^{-1}(\mathfrak{A}_0))$;

(ii)
$$\begin{aligned} \bar{f} \models_{i+1} F_e(x) \iff & (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(\\ & u = \langle 0, e_u, x_u\rangle \ \&\ \bar{f} \models_i F_{e_u}(x_u) \vee \\ & u = \langle 1, e_u, x_u\rangle \ \&\ \bar{f} \models_i \neg F_{e_u}(x_u) \vee \\ & u = \langle 2, x_u\rangle \ \&\ x_u \in f_{i+1}^{-1}(\mathfrak{A}_{i+1})));\end{aligned}$$

(iii) $\bar{f} \models_i \neg F_e(x) \iff \bar{f} \not\models_i F_e(x)$.

If $A \subseteq \mathbb{N}$ and $k \le n$, then

$$A \le_e \mathcal{P}^{\bar{f}}_k \iff (\exists e)(A = \{x : \bar{f} \models_k F_e(x)\}).$$

The forcing conditions, which we shall call *finite parts*, are $n+1$-tuples $\bar{\tau} = (\tau_0, \ldots, \tau_n)$ of finite mappings $\tau_0, \ldots, \tau_n$ of $\mathbb{N}$ in $\mathbb{N}$. We suppose that an effective coding of the finite parts is fixed, and by the least finite part with a fixed property we mean a finite part with a minimal code.

For every $i \le n$, e and x in $\mathbb{N}$ and every finite part $\bar{\tau}$ we define the forcing relations $\bar{\tau} \Vdash_i F_e(x)$ and $\bar{\tau} \Vdash_i \neg F_e(x)$ following the definition of relation "$\models_i$".

Definition 4.1. (i) $\bar{\tau} \Vdash_0 F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ D_v \subseteq \tau_0^{-1}(\mathfrak{A}_0))$;

(ii)
$$\begin{aligned} \bar{\tau} \Vdash_{i+1} F_e(x) \iff & \exists v(\langle v, x\rangle \in W_e \ \& \\ & (\forall u \in D_v)(u = \langle 0, e_u, x_u\rangle \ \&\ \bar{\tau} \Vdash_i F_{e_u}(x_u) \vee \\ & u = \langle 1, e_u, x_u\rangle \ \&\ \bar{\tau} \Vdash_i \neg F_{e_u}(x_u) \vee \\ & u = \langle 2, x_u\rangle \ \&\ x_u \in \tau_{i+1}^{-1}(\mathfrak{A}_{i+1})));\end{aligned}$$

(iii) $\bar{\tau} \Vdash_i \neg F_e(x) \iff (\forall \bar{\rho} \supseteq \bar{\tau})(\bar{\rho} \nVdash_i F_e(x))$.

Given finite parts $\bar{\delta} = (\delta_0, \ldots, \delta_n)$ and $\bar{\tau} = (\tau_0, \ldots, \tau_n)$, let

$$\bar{\delta} \subseteq \bar{\tau} \iff \delta_0 \subseteq \tau_0, \ldots, \delta_n \subseteq \tau_n.$$

For any $i \leq n, e, x \in \mathbb{N}$ denote $X^i_{\langle e,x\rangle} = \{\bar{\rho} : \bar{\rho} \Vdash_i F_e(x)\}$.
If $\bar{f} = (f_0, \ldots, f_n)$ is an enumeration of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$, then

$$\bar{\tau} \subseteq \bar{f} \iff \tau_0 \subseteq f_0, \ldots, \tau_n \subseteq f_n.$$

Definition 4.2. An enumeration $\bar{f}$ of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ is *i-generic* if for every $j < i$, $e, x \in \mathbb{N}$

$$(\forall \bar{\tau} \subseteq \bar{f})(\exists \bar{\rho} \in X^j_{\langle e,x\rangle})(\bar{\tau} \subseteq \bar{\rho}) \Longrightarrow (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \in X^j_{\langle e,x\rangle}).$$

From [10] we know that:

(1) If $\bar{f}$ is a k-generic enumeration, then

$$\bar{f} \models_k F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_k F_e(x)).$$

(2) If f is a $(k+1)$-generic enumeration, then

$$\bar{f} \models_k \neg F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_k \neg F_e(x)).$$

Definition 4.3. The set $A \subseteq \mathbb{N}$ is *forcing k-definable* on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ if there exist a finite part $\bar{\delta}$ and $e \in \mathbb{N}$ such that

$$x \in A \iff (\exists \bar{\tau} \supseteq \bar{\delta})(\bar{\tau} \Vdash_k F_e(x)).$$

Proposition 4.1. *Let $\{X^k_r\}_r$, $k = 0, \ldots, n$, be $(n+1)$-sequences of sets of natural numbers. There exists an $(n+1)$-generic enumeration $\bar{f}$ of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ such that for any $k \leq n$ and for all $r \in \mathbb{N}$, if the set X^k_r is not forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$, then $X^k_r \not\leq_e \mathcal{P}^{\bar{f}}_k$.*

Proof. We shall construct an $(n+1)$-generic enumeration $\bar{f}$ such that for all r and all $k = 0, \ldots, n$, if the set X^k_r is not forcing k-definable, then $X^k_r \not\leq_e \mathcal{P}^{\bar{f}}_k$. Let call the last condition an omitting condition.

The construction of the enumeration $\bar{f}$ will be carried out by steps. On each step j we shall define a finite part $\bar{\delta}^j = (\delta^j_0, \ldots, \delta^j_n)$, so that $\bar{\delta}^j \subseteq \bar{\delta}^{j+1}$, and take $f_i = \cup_j \delta^j_i$ for each $i \leq n$.

On the steps $j = 3q$ we shall ensure that each f_i is a total surjective mapping from $\mathbb{N}$ onto $\mathbb{N}$. On the steps $j = 3q+1$ we shall ensure that $\bar{f}$ is $(n+1)$-generic. On the steps $j = 3q+2$ we shall ensure the omitting condition.

Let $\bar{\delta}^0 = (\emptyset, \ldots, \emptyset)$.

Suppose that $\bar{\delta}^j$ is defined.

Case $j = 3q$. For every i, $0 \le i \le n$, let $x_i = \mu x[x \notin \mathrm{dom}(\delta_i^j)]$ and $y_i = \mu y[y \notin \mathrm{ran}(\delta_i^j)]$. Let $\delta_i^{j+1}(x_i) = y_i$ and $\delta_i^{j+1}(x) \simeq \delta_i^j(x)$ for $x \neq x_i$.

Case $j = 3\langle e, i, x\rangle + 1$, $i \le n$. Check if there exists a finite part $\bar{\rho} \supseteq \bar{\delta}^j$ such that $\bar{\rho} \Vdash_i F_e(x)$. If so, then let $\bar{\delta}^{j+1}$ be the least such ρ. Otherwise, let $\bar{\delta}^{j+1} = \bar{\delta}^j$.

Case $j = 3\langle e, k, r\rangle + 2$, $k \le n$. Consider the set X_r^k. If X_r^k is forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ then let $\bar{\delta}^{j+1} = \bar{\delta}^j$.

Suppose now that X_r^k is not forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ and let

$$C = \{x : (\exists \bar{\tau} \supseteq \bar{\delta}^j)(\bar{\tau} \Vdash_k F_e(x))\}.$$

Clearly, C is forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$. Hence $C \neq X_r^k$. Then there exists an x such that either $x \in X_r^k$ and $x \notin C$ or $x \in C$ and $x \notin X_r^k$. Take $\bar{\delta}^{j+1} = \bar{\delta}^j$ in the first case.

If the second case holds, then there exists $\bar{\tau} \supseteq \bar{\delta}^j$ such that $\bar{\tau} \Vdash_k F_e(x)$. Let $\bar{\delta}^{j+1}$ be the least such τ.

In all other cases let $\bar{\delta}^{j+1} = \bar{\delta}^j$.

The so received enumeration $\bar{f} = \cup_j \bar{\delta}^j$ is $(n+1)$-generic. Let $i \le n$, $e, x \in \mathbb{N}$ and suppose that for every finite part $\bar{\tau} \subseteq \bar{f}$ there is an extention $\bar{\rho} \Vdash_i F_e(x)$. Consider the step $j = 3\langle e, i, x\rangle + 1$. From the construction we have that $\bar{\delta}^{j+1} \Vdash_i F_e(x)$.

To prove that the enumeration $\bar{f}$ satisfies the omitting condition, let the set X_r^k be not forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ and suppose that $X_r^k \le_e \mathcal{P}_k^{\bar{f}}$. Then $X_r^k = \{x : \bar{f} \models_k F_e(x)\}$ for some e. Consider the step $j = 3\langle e, k, r\rangle + 2$. From the construction there is an x such that one of the following two cases holds:

(a) $x \in X_r^k$ and $(\forall \bar{\rho} \supseteq \bar{\delta}^j)(\bar{\rho} \nVdash_k F_e(x))$. So, $\bar{\delta}^j \Vdash_k \neg F_e(x)$.

Since $\bar{f}$ is $(n+1)$-generic, and hence $(k+1)$-generic, $x \in X_r^k$ & $\bar{f} \not\models_k F_e(x)$. A contradiction.

(b) $x \notin X_r^k$ & $\bar{\delta}^{j+1} \Vdash_k F_e(x)$. Since $\bar{f}$ is $(k+1)$-generic, $\bar{f} \models_k F_e(x)$. A contradiction. □

5. MINIMAL PAIR THEOREM

First we need a modification of the "type omitting" version of Jump inversion theorem from [8]. In fact, one can see the result from the proof of Theorem 1.7 in [8]. But in this form it is not explicit formulated there. We shall postpone the proof for Section 6, where the technique of regular enumerations will be discussed.

Theorem 5.1. *Let $B_0, \ldots, B_n$ be arbitrary sets of natural numbers. Let $\{A_r^k\}_r$, $k = 0, \ldots, n$, be $(n+1)$-sequences of subsets of $\mathbb{N}$ such that for every r and for all k, $0 \le k < n$, $A_r^k \not\le_e \mathcal{P}(B_0, \ldots, B_k)$. Then there exists a total set F having the following properties:*

(i) *For all $i \le n$, $B_i \le_e F^{(i)}$;*

(ii) *For all r, for all k, $0 \le k < n$, $A_r^k \not\le_e F^{(k)}$.*

Proof of Theorem 3.1. We shall construct total sets F and G such that $d_e(F) \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, $d_e(G) \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$ and for each $k \leq n$ if a total set X, $X \leq_e F^{(k)}$ and $X \leq_e G^{(k)}$, then $d_e(X) \in CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$. And take the degree $\mathbf{f} = d_e(F)$ and $\mathbf{g} = d_e(G)$.

First we construct enumerations $\bar{f}$ and $\bar{h}$ of $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ such that for any $k \leq n$ if a set $A \subseteq \mathbb{N}$, $A \leq_e \mathcal{P}^{\bar{f}}_k$ and $A \leq_e \mathcal{P}^{\bar{h}}_k$, then A is a forcing k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_n$.

Let $g_0, \dots, g_n$ be arbitrary enumerations of $\mathfrak{A}_0, \dots, \mathfrak{A}_n$. By Theorem 5.1 for $B_0 = g_0^{-1}(\mathfrak{A}_0), \dots, B_n = g_n^{-1}(\mathfrak{A}_n)$ there exists a total set F such that: $g_0^{-1}(\mathfrak{A}_0) \leq_e F, g_1^{-1}(\mathfrak{A}_1) \leq_e F', \dots,\ g_n^{-1}(\mathfrak{A}_n) \leq_e F^{(n)}$. Since $DS(\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$ is closed upwards, then $d_e(F) \in DS(\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$, i.e. $d_e(F) \in DS(\mathfrak{A}_0), d_e(F') \in DS(\mathfrak{A}_1)$, $\dots$, $d_e(F^{(n)}) \in DS(\mathfrak{A}_n)$. Hence, there exist $h_0, h_1, \dots, h_n$ enumerations of $\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n$, respectively, such that $h_0^{-1}(\mathfrak{A}_0) \equiv_e F, h_1^{-1}(\mathfrak{A}_1) \equiv_e F', \dots, h_n^{-1}(\mathfrak{A}_n) \equiv_e F^{(n)}$. Notice that for each $k \leq n$, $F^{(k)} \equiv_e \mathcal{P}^{\bar{h}}_k$.

For each k, $0 \leq k \leq n$, let $\{X^k_r\}_r$ be the sequence of all sets enumeration reducible to $\mathcal{P}^{\bar{h}}_k$.

By Proposition 4.1 there is an $(n+1)$-generic enumeration $\bar{f}$ such that for all r and all $k = 0, \dots, n$ if the set X^k_r is not forcing k-definable then $X^k_r \not\leq_e \mathcal{P}^{\bar{f}}_k$.

Suppose now that the set $A \leq_e \mathcal{P}^{\bar{f}}_k$ and $A \leq \mathcal{P}^{\bar{h}}_k$. Then $A = X^k_r$ for some r. From the omitting condition of $\bar{f}$ it follows that A is forcing k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_n$.

Now we apply Theorem 5.1. Let $B_0 = f_0^{-1}(\mathfrak{A}_0), \dots, B_n f_n^{-1}(\mathfrak{A}_n)$ and $B_{n+1} = N$. For each $k \leq n$ consider the sequence $\{A^k_r\}_r$ of these sets among the sets $\{X^k_r\}_r$, which are not forcing k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_n$. From the choice of the enumeration $\bar{f}$ it follows that each of these sets A^k_r, $A^k_r \not\leq_e \mathcal{P}^{\bar{f}}_k$. Then by Theorem 5.1 there is a total set G such that:

(i) For all $i \leq n$, $f_i^{-1}(\mathfrak{A}_i) \leq_e G^{(i)}$;

(ii) For all r and all $k \leq n$, $A^k_r \not\leq_e G^{(k)}$.

Note that since G is a total set and because of the fact that each spectrum is closed upwards, we have that $d_e(G) \in DS(\mathfrak{A}_0), \dots, d_e(G^{(n)}) \in DS(\mathfrak{A}_n)$, and hence $d_e(G) \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$.

Suppose now that a total set X, $X \leq_e F^{(k)}$ and $X \leq_e G^{(k)}$, $k \leq n$. From $X \leq_e F^{(k)}$ and $F^{(k)} \equiv_e \mathcal{P}^{\bar{h}}_k$ it follows that $X = X^k_r$ for some r. It is clear that $X \leq_e \mathcal{P}^{\bar{f}}_k$. Otherwise, from the choice of G it follows that $X \not\leq_e G^{(k)}$. Hence X is forcing k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_n$. By the normal form of the sets, which enumeration degrees are in $CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, we have that $d_e(X) \in CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$. □

6. REGULAR ENUMERATIONS

We shall remind the notion of regular enumerations from [8]. Let $B_0, \dots, B_n$ be non empty subsets of $\mathbb{N}$.

Finite parts are as usual finite mappings of $\mathbb{N}$ into $\mathbb{N}$. The notion of *i-regular finite parts* is defined by induction on $i \le n$.

The 0-*regular finite parts* are finite parts τ such that $\mathrm{dom}(\tau) = [0, 2q+1]$ and for all odd $z \in \mathrm{dom}(\tau), \tau(z) \in B_0$.

Let τ be a 0-regular finite part. If $\mathrm{dom}(\tau) = [0, 2q+1]$, then the 0-rank of τ $|\tau|_0 q+1$ – the number of the odd elements of $\mathrm{dom}(\tau)$. Let B_0^τ be the set of the odd elements of $\mathrm{dom}(\tau)$. If ρ is a 0-regular extention of τ, we shall denote this fact by $\tau \subseteq_0 \rho$. It is clear that if $\tau \subseteq_0 \rho$ and $|\tau|_0 |\rho|_0$, then $\tau = \rho$. Let

$$\tau \Vdash_0 F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \& \ (\forall u \in D_v)(\tau((u)_0) \simeq (u)_1)),$$

$$\tau \Vdash_0 \neg F_e(x) \iff \forall(\rho)(\tau \subseteq_0 \rho \Rightarrow \rho \nVdash_0 F_e(x)).$$

Suppose that for some $i < n$ we have defined the i-regular finite parts and for every i-regular τ – the i-rank $|\tau|_i$ of τ, the set B_i^τ and the relations $\tau \Vdash_i F_e(x)$ and $\tau \Vdash_i \neg F_e(x)$. Suppose also that if τ and ρ are i-regular, $\tau \subseteq \rho$ (we write $\tau \subseteq_i \rho$) and $|\tau|_i = |\rho|_i$, then $\tau = \rho$.

Denote by $X^i_{\langle e,x\rangle} = \{\rho : \rho \text{ is } i\text{-regular} \ \& \ \rho \Vdash_i F_e(x)\}$.

For any i-regular finite part τ and any set X of i-regular finite parts, denote by $\mu_i(\tau, X) = \mu\rho[\tau \subseteq_i \rho \ \& \ \rho \in X]$ if any, and $\mu_i(\tau, X) = \mu\rho[\tau \subseteq_i \rho]$, otherwise.

Definition 6.1. Let τ be a finite part and $m \ge 0$. The finite part δ is called an *i-regular m omitting extension* of τ if $\delta \supseteq_i \tau$, $\mathrm{dom}(\delta) = [0, q-1]$ and there exist natural numbers $q_0 < \cdots < q_m < q_{m+1} = q$ such that:

(a) $\delta \restriction q_0 = \tau$;

(b) For all $p \le m$, $\delta \restriction q_{p+1} \mu_i(\delta \restriction (q_p + 1), X^i_{\langle p, q_p\rangle})$.

Denote by K^δ_τ the sequence $q_0, \ldots, q_m$. If δ and ρ are two i-regular m omitting extensions of τ and $\delta \subseteq \rho$, then $\delta = \rho$.

Let $\mathcal{R}_i$ denote the set of all i-regular finite parts. Given an index j, by S^i_j we shall denote the intersection $\mathcal{R}_i \cap \Gamma_j(\mathcal{P}(B_0, \ldots, B_i))$, where Γ_j is the j-th enumeration operator.

Let τ be a finite part defined on $[0, q-1]$ and $r \ge 0$. Then τ is *$(i+1)$-regular* with *$(i+1)$-rank* $r+1$ if there exist natural numbers

$$0 < n_0 < l_0 < b_0 < n_1 < l_1 < b_1 < \cdots < n_r < l_r < b_r < n_{r+1} = q$$

such that $\tau \restriction n_0$ is an i-regular finite part with i-rank equal to 1 and for all j, $0 \le j \le r$, the following conditions are satisfied:

(a) $\tau \restriction l_j \simeq \mu_i(\tau \restriction (n_j + 1), S^i_j)$;

(b) $\tau \restriction b_j$ is an i-regular j omitting extension of $\tau \restriction l_j$;

(c) $\tau(b_j) \in B_{i+1}$;

(d) $\tau\restriction n_{j+1}$ is an i-regular extension of $\tau\restriction(b_j+1)$ with i-rank equal to $|\tau\restriction b_j|_i+1$.

Let $B^\tau_{i+1}=\{b_0,\ldots,b_r\}$. By K^τ_{i+1} we shall denote the sequence $K^{\tau\restriction b_r}_{\tau\restriction l_r}$.

Let for every $(i+1)$-regular finite part τ

$$\tau \Vdash_{i+1} F_e(x) \iff \exists v(\langle v,x\rangle \in W_e \;\&\; (\forall u\in D_v)((u=\langle e_u,x_u,0\rangle \;\&\; \tau\Vdash_i F_{e_u}(x_u))\vee (u=\langle e_u,x_u,1\rangle \;\&\; \tau\Vdash_i \neg F_{e_u}(x_u)))).$$

$$\tau\Vdash_{i+1}\neg F_e(x) \iff (\forall\rho)(\tau\subseteq_{i+1}\rho\Rightarrow \rho\nVdash_{i+1} F_e(x)).$$

Definition 6.2. Let f be a total mapping of $\mathbb{N}$ in $\mathbb{N}$. Then f is a *regular enumeration* if the following two conditions hold:

(i) For every finite part $\delta\subseteq f$, there exists an n-regular extension τ of δ such that $\tau\subseteq f$.

(ii) If $i\le n$ and $z\in B_i$, then there exists an i-regular extension $\tau\subseteq f$ such that $z\in\tau(B^\tau_i)$.

Let f be a total mapping on $\mathbb{N}$. We define for every $i\le n,e,x$ the relation $f\models_i F_e(x)$ by induction on i:

Definition 6.3.
(i) $f\models_0 F_e(x) \iff \exists v(\langle v,x\rangle\in W_e \;\&\; (\forall u\in D_v)(f((u)_0)\simeq(u)_1))$;
(ii) $f\models_{i+1} F_e(x) \iff \exists v(\langle v,x\rangle\in W_e \;\&\; (\forall u\in D_v)((u=\langle e_u,x_u,0\rangle \;\&\; f\models_i F_{e_u}(x_u))\vee(u=\langle e_u,x_u,1\rangle \;\&\; f\not\models_i F_{e_u}(x_u))))$.

Set $f\models_i\neg F_e(x)\iff f\not\models_i F_e(x)$.
In [8] it is proven that for every regular enumeration f:

1. $B_0\le_e f$.
2. If $i<n$, then $B_{i+1}\le_e f\oplus\mathcal{P}(B_0,\ldots,B_i)'$, and $\mathcal{P}(B_0,\ldots,B_i)<_e f^{(i)}$, for $i\le n$.
3. If $A\subset\mathbb{N}$, then
$$A\le_e f^{(i)}\iff(\exists e)A=\{x: f\models_i F_e(x)\}.$$
4. For all $i\le n$ (for negation $i<n$),
$$f\models_i(\neg)F_e(x)\iff(\exists\tau\subseteq f)(\tau \text{ is } i\text{-regular} \;\&\; \tau\Vdash_i(\neg)F_e(x)).$$

Notice that if f is a regular enumeration, then $B_i\le_e f^{(i)}$, $i\le n$.

Given a finite mapping τ defined on $[0,q-1]$, by $\tau*z$ we shall denote the extension ρ of τ defined on $[0,q]$ and such that $\rho(q)\simeq z$. We shall use the following Lemma, proved in [8].

Lemma 6.1. [8] *Let $A_0, \dots, A_{n-1}$ be subsets of $\mathbb{N}$ such that $A_i \not\leq_e \mathcal{P}(B_0, \dots, B_i)$. Let τ be an n-regular finite part, defined on $[0, q-1]$. Suppose that $|\tau|_n = r+1$, $y \in \mathbb{N}$, $z_0 \in B_0, \dots, z_n \in B_n$ and $s \leq r+1$. Then there is an n-regular extension ρ of τ such that:*

(i) $|\rho|_n = r+2$;

(ii) $\rho(q) \simeq y$, $z_0 \in \rho(B_0^\rho), \dots, z_n \in \rho(B_n^\rho)$;

(iii) *if $i < n$ and $K_{i+1}^\rho = q_0^i, \dots, q_s^i, \dots, q_{m_i}^i$, then*

(a) $\rho(q_s^i) \in A_i \Rightarrow \rho \Vdash_i \neg F_s(q_s^i)$;

(b) $\rho(q_s^i) \notin A_i \Rightarrow \rho \Vdash_i F_s(q_s^i)$.

Now we turn to the proof of Theorem 5.1. Set $B_{n+1} = \mathbb{N}$ and $\mathcal{P}(B_0, \dots, B_{n+1}) = \mathcal{P}(B_0, \dots, B_n)' \oplus B_{n+1}$. By a regular enumeration f we mean a regular one with respect to $B_0, \dots, B_n, B_{n+1}$.

Proof of Theorem 5.1.

Let $\{A_r^k\}_r$, $k \leq n$, be seqences of subsets of $\mathbb{N}$ such that $A_r^k \not\leq_e \mathcal{P}(B_0, \dots, B_k)$.

We shall construct a regular enumeration f such that f "omits" the sets A_r^k for all r, $k \leq n$, i.e. $A_r^k \not\leq_e f^{(k)}$.

The construction of f will be carried out by steps. At each step s we shall construct an $(n+1)$-regular finite part δ_s, so that $|\delta_s|_{n+1} \geq s+1$ and $\delta_s \subseteq_{n+1} \delta_{s+1}$. On the even steps we shall ensure the genericity of f, i.e. conditions (a) and (d) from the definition of i-regular finite part, and on the odd steps we shall ensure the omitting conditions, the conditions (b), (c).

Let $\mathcal{R}_{n+1}$ be the set of all $(n+1)$-regular finite parts and $S_j^{n+1} = \mathcal{R}_{n+1} \cap \Gamma_j(\mathcal{P}(B_0, \dots, B_{n+1}))$. Let $\sigma_0, \dots, \sigma_{n+1}$ be recursive in $\mathcal{P}(B_0, \dots, B_{n+1})$ enumerations of the sets $B_0, \dots, B_{n+1}$, respectively.

Let δ_0 be an arbitrary $(n+1)$-regular finite part with $(n+1)$-rank equal to 1. Suppose that δ_s is defined.

Case $s = 2m$. Check whether there exists a $\rho \in S_m^{n+1}$ such that $\delta_s \subset \rho$. If so, let δ_{s+1} be the least such ρ. Otherwise, let δ_{s+1} be the least $(n+1)$-regular extension of δ_s with $(n+1)$-rank equal to $|\delta_s|_{n+1} + 1$.

Case $s = 2m+1$. Let $|\delta_s|_{n+1} = r+1 \geq s+1$. Let $m\langle p, e\rangle$. We may assume that $e \leq m$ and then $e < r+1$. Let $\sigma_0(m) \simeq z_0, \dots, \sigma_{n+1}(m) \simeq z_{n+1}$. Set $\tau_0 \simeq \mu_n(\delta_s * z_{n+1}, S_{r+1}^n)$. Let $l_{r+1} = \mathrm{lh}(\tau_0)$ and $q_0^n = l_{r+1}$. For $j < e$, let $\tau_{j+1} = \mu_n(\tau_j * 0, X_{\langle j, q_j^n\rangle}^n)$ and $q_{j+1}^n = \mathrm{lh}(\tau_{j+1})$. So, τ_e and q_e^n are defined. Let

$$C = \{x : (\exists \tau \supseteq \tau_e)(\tau \in \mathcal{R}_n \ \&\ \tau(q_e^n) \simeq x \ \&\ \tau \Vdash_n F_e(q_e^n))\}.$$

The set $C \leq_e \mathcal{P}(B_0, \dots, B_{n+1})$ and $A_p^n \not\leq_e \mathcal{P}(B_0, \dots, B_{n+1})$. Then there is an a such that

$$a \in C \ \&\ a \notin A_p^n \vee a \notin C \ \&\ a \in A_p^n. \tag{6.1}$$

Let a_0 be the least a satisfying (6.1).

Next we extend the finite $\tau_e * a_0$ to a finite part τ, so that τ is an n-regular $r+1$ omitting extension of τ_0. Set $b_{r+1} = \mathrm{lh}(\tau)$. Now consider the sets $A_p^0, \ldots, A_p^{n-1}$. By Lemma 6.1 we can construct an n-regular extension ρ of τ such that:

(i) $|\rho|_n = |\tau|_n + 1$;

(ii) $\rho(b_{r+1}) \simeq z_{n+1}$ and $z_0 \in \rho(B_0^\rho), \ldots, z_n \in \rho(B_n^\rho)$;

(iii) if $k < n$ and $K_{k+1}^\rho = q_0^k, \ldots, q_e^k, \ldots, q_{m_k}^k$, then

(a) $\rho(q_e^k) \in A_p^k \Rightarrow \rho \Vdash_k \neg F_e(q_e^k)$;

(b) $\rho(q_e^k) \notin A_p^k \Rightarrow \rho \Vdash_k F_e(q_e^p)$.

Set $\delta_{s+1} = \rho$.

Let $f = \bigcup \delta_s$. From the construction it follows that f is a regular enumeration. For every e, x, $\{\tau : \tau \in \mathcal{R}_{n+1} \;\&\; \tau \Vdash_{n+1} F_e(x)\}$ is e-reducible to $\mathcal{P}(B_0, \ldots, B_{n+1})$. From here, by the even stages of the construction, it follows that for all e, x,

$$f \models_{n+1} (\neg)F_e(x) \iff (\exists \tau \subseteq f)(\tau \in \mathcal{R}_{n+1} \;\&\; \tau \Vdash_{n+1} (\neg)F_e(x)).$$

Since f is regular, we have that if $k \leq n$, then for all e and x,

$$f \models_k (\neg)F_e(x) \iff (\exists \tau \subseteq f)(\tau \in \mathcal{R}_k \;\&\; \tau \Vdash_k (\neg)F_e(x)).$$

Now suppose that for some $k \leq n$ and p, $A_p^k \leq_e f^{(k)}$. Then the set $C_p^k = \{x : f(x) \in A_p^k\}$ is also e-reducible to $f^{(k)}$. Fix an e such that for all x,

$$f(x) \in A_p^k \iff x \in C_p^k \iff f \models_k F_e(x). \tag{6.2}$$

Consider the step $s = 2\langle p, e\rangle + 1$. By the construction, there exists a $q_e^k \in \mathrm{dom}(\delta_{s+1})$ such that

$$(f(q_e^k) \in A_p^k \Rightarrow f \models_k \neg F_e(q_e^k)) \;\&\; (f(q_e^k) \notin A_p^k \Rightarrow f \models_k F_e(q_e^k)).$$

Clearly, $\delta_{s+1}(q_e^k) \simeq f(q_e^k)$. Now assume that $f(q_e^k) \in A_p^k$. Then $\delta_{s+1} \Vdash_k \neg F_e(q_e^k)$. Hence $f \models_k \neg F_e(q_e^k)$, which is impossible. It remains that $f(q_e^k) \notin A_p^k$. In this case $\delta_{s+1} \Vdash_k F_e(q_e^k)$ and hence $f \models_k F_e(q_e^k)$. The last again contradicts (6.2). So $A_p^k \not\leq_e f^{(k)}$. $\square$

7. QUASI-MINIMAL DEGREE

Definition 7.1. Let $B_0 \subseteq \mathbb{N}$. A set F of natural numbers is called *quasi-minimal over* B_0 if the following conditions hold:

(i) $B_0 <_e F$;

(ii) For any total set $A \subseteq \mathbb{N}$, if $A \leq_e F$, then $A \leq_e B_0$.

The following theorem we shall prove in the next section using the technique of partial regular enumerations.

Theorem 7.1. *Let $B_0, \ldots, B_n, n \geq 1$, be arbitrary sets of natural numbers. There exists a set F having the following properties*:

(i) $B_0 <_e F$;

(ii) *For all* $1 \leq i \leq n$, $B_i \leq_e F^{(i)}$;

(iii) *For any total set A, if $A \leq_e F$, then $A \leq_e B_0$.*

In fact, the set F from Theorem 7.1 is a quasi-minimal over B_0.

Let the structures $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ be fixed.

Proof of Theorem 3.2. By [9], there is a quasi-minimal degree $\mathbf{q}_0$ with respect to $DS(\mathfrak{A}_0)$, i.e.:

(i) $\mathbf{q}_0 \notin CS(\mathfrak{A}_0)$;

(ii) If $\mathbf{a}$ is a total degree and $\mathbf{a} \geq \mathbf{q}_0$, then $\mathbf{a} \in DS(\mathfrak{A}_0)$;

(iii) If $\mathbf{a}$ is a total degree and $\mathbf{a} \leq \mathbf{q}_0$, then $\mathbf{a} \in CS(\mathfrak{A}_0)$.

Let $B_0 \subseteq \mathbb{N}$ such that $d_e(B_0) = \mathbf{q}_0$, and $f_1, \ldots, f_n$ be fixed total enumerations of $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. Denote $B_1 = f_1^{-1}(\mathfrak{A}_1), \ldots, B_n = f_n^{-1}(\mathfrak{A}_n)$. By Theorem 7.1, there is a quasi-minimal over B_0 set F such that:

(i) $B_0 <_e F$;

(ii) For all $1 \leq i \leq n$, $f_i^{-1}(\mathfrak{A}_i) \leq_e F^{(i)}$;

(iii) For any total set A, if $A \leq_e F$, then $A \leq_e B_0$.

We will show that $\mathbf{q} = d_e(F)$ is a quasi-minimal with respect to $DS(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$, i.e.:

(i) $\mathbf{q}' \in DS(\mathfrak{A}_1), \ldots, \mathbf{q}^{(n)} \in DS(\mathfrak{A}_n)$, $\mathbf{q} \notin CS(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$;

(ii) If $\mathbf{a}$ is a total degree and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in DS(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$;

(iii) If $\mathbf{a}$ is a total degree and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in CS(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

In order to prove (i), suppose that $\mathbf{q} \in CS(\mathfrak{A}_0)$. By Theorem 7.1, $\mathbf{q}_0 < \mathbf{q}$ and thus $\mathbf{q}_0 \in CS(\mathfrak{A}_0)$. A contradiction with the fact that $\mathbf{q}_0$ is quasi-minimal with respect to $DS(\mathfrak{A}_0)$. Then $\mathbf{q} \notin CS(\mathfrak{A}_0)$ and hence $\mathbf{q} \notin CS(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

For each i, $1 \leq i \leq n$, the set $F^{(i)}$ is total and $f_i^{-1}(\mathfrak{A}_i) \leq_e F^{(i)}$. Since any degree spectrum is closed upwards, it follows that $d_e(F^{(i)}) \in DS(\mathfrak{A}_i)$, i.e. $\mathbf{q}^{(i)} \in DS(\mathfrak{A}_i)$.

For (ii) consider a total set X such that $X \geq_e F$. Then $d_e(X) \geq \mathbf{q}_0$. From the fact that $\mathbf{q}_0$ is quasi-minimal with respect to $DS(\mathfrak{A}_0)$ it follows that $d_e(X) \in DS(\mathfrak{A}_0)$. Moreover, for each $1 \leq i \leq n$, $X^{(i)} \geq_e F^{(i)} \geq_e f_i^{-1}(\mathfrak{A}_i)$, and $X^{(i)}$ is a total set. Then for each $i \leq n$, $d_e(X^{(i)}) \in DS(\mathfrak{A}_i)$, and hence $d_e(X) \in DS(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$.

For (iii) suppose that X is a total set and $X \leq_e F$. Then, from the choice of F, $X \leq_e B_0$. Because $\mathbf{q}_0$ is quasi-minimal with respect to $DS(\mathfrak{A}_0)$, it follows that $d_e(X) \in CS(\mathfrak{A}_0)$. But $CS(\mathfrak{A}_0, \ldots, \mathfrak{A}_n) = CS(\mathfrak{A}_0)$ and therefore $d_e(X) \in CS(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$. □

8. PARTIAL REGULAR ENUMERATIONS

Let $B_0 \subseteq \mathbb{N}$.

Definition 8.1. *A partial enumeration f of B_0* is a partial surjective mapping from $\mathbb{N}$ onto $\mathbb{N}$ with the following properties:

(i) For all odd x, if $f(x)$ is defined, then $f(x) \in B_0$;

(ii) For all $y \in B_0$, there is an odd x such that $f(x) \simeq y$.

It is clear that if f is a partial enumeration of B_0, then $B_0 \leq_e f$ since

$$y \in B_0 \iff (\exists n)(f(2n+1) \simeq y).$$

Let $\perp \notin \mathbb{N}$.

Definition 8.2. A *partial finite part* τ is a finite mapping of $\mathbb{N}$ into $\mathbb{N} \cup \{\perp\}$ such that $(\forall x)(x \in \mathrm{dom}(\tau)\ \&\ x \text{ is odd} \Rightarrow (\tau(x) = \perp \vee \tau(x) \in B_0))$.

If τ is a partial finite part and f is a partial enumeration of B_0, say that

$$\tau \subseteq f \iff (\forall x \in \mathrm{dom}(\tau))((\tau(x) = \perp \Rightarrow f(x) \text{ is not defined }) \ \& \\ (\tau(x) \neq \perp \Rightarrow \tau(x) \simeq f(x)).$$

Let $B_0, \ldots, B_n$ be fixed sets of natural numbers. Combining the technique of the regular enumerations with the partial (generic) enumerations on the 0-level for B_0, we shall construct a partial regular enumeration f, which will be quasi-minimal over the set B_0 and such that $B_i \leq_e f^{(i)}$ for $i \leq n$.

A 0-*regular partial finite part* is a partial finite part τ such that $\mathrm{dom}(\tau) = [0, 2q+1]$ and for all odd $z \in \mathrm{dom}(\tau)$, $\tau(z) \in B_0$ or $\tau(z) = \perp$.

Let B_0^τ be the set of all odd elements z of $\mathrm{dom}(\tau)$ such that $\tau(z) \in B_0$. The 0-rank of τ, $|\tau|_0 = q+1$, we call the number of the odd elements of $\mathrm{dom}(\tau)$. If ρ is a 0-regular partial extention of τ, we shall denote this fact again by $\tau \subseteq_0 \rho$. It is clear that if $\tau \subseteq_0 \rho$ and $|\tau|_0 |\rho|_0$, then $\tau = \rho$. Let

$$\tau \Vdash_0 F_e(x) \iff \exists v(\langle v, x\rangle \in W_e\ \&\ (\forall u \in D_v)(u = \langle s, t\rangle,\ \&\ \tau(s) \simeq t\ \&\ t \neq \perp.)),$$

$$\tau \Vdash_0 \neg F_e(x) \iff \forall(\rho)(\tau \subseteq_0 \rho \Rightarrow \rho \nVdash_0 F_e(x)).$$

The definition of $(i+1)$-regular partial finite part τ, the set B^{τ}_{i+1}, the $(i+1)$-rank of τ and the relations $\tau \Vdash_{i+1} F_e(x)$ and $\tau \Vdash_{i+1} \neg F_e(x)$ are defined in the same way as in Section 6, the only difference is that instead of i-regular finite parts we use i-regular partial finite parts. Notice that again if τ is an i-regular partial finite part, then τ is a j-regular partial finite part for each $j < i$.

Definition 8.3. *A partial regular enumeration* is a partial mapping f from $\mathbb{N}$ onto $\mathbb{N}$ such that the following two conditions hold:

(i) For every partial finite part $\delta \subseteq f$, there exists an n-regular partial extension τ of δ such that $\tau \subseteq f$.

(ii) If $i \leq n$ and $z \in B_i$, then there exists an i-regular partial finite part $\tau \subseteq f$ such that $z \in \tau(B^{\tau}_i)$.

If f is a partial regular enumeration and $i \leq n$, then for every $\delta \subseteq f$, $\mathrm{dom}(\delta) \subseteq [0, q-1]$, there exists an i-regular partial $\tau \subseteq f$ such that $\delta \subseteq \tau$, and for every $x \in [0, q-1]$ if $f(x)$ is not defined, then $\tau(x) = \bot$. Moreover, there exist i-regular partial finite parts of f of arbitrary large rank.

The relation $f \models_i F_e(x)$ is the same as in Definition 6.3. By induction on i one could check that for any $A \subseteq \mathbb{N}$, $A \leq_e f^{(i)}$ iff there exists e such that for all x,

$$x \in A \iff f \models_i F_e(x).$$

Lemma 8.1. *Suppose that f is a partial regular enumeration. Then*:

(1) *For all $i \leq n$, $f \models_i F_e(x) \iff (\exists \tau \subseteq f)(\tau$ is i-regular & $\tau \Vdash_i F_e(x))$.*

(2) *For all $i < n$, $f \models_i \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau$ is i-regular & $\tau \Vdash_i \neg F_e(x))$.*

The proof follows from the definitions by induction on i as in the total case.

Let $\mathcal{R}_i$ be the set of all i-regular partial finite parts. It is clear that $\mathcal{R}_i \leq_e \mathcal{P}_i$, where $\mathcal{P}_i = \mathcal{P}(B_0, \dots, B_n)$.

Definition 8.4. A partial enumeration f is *i-generic* if for any $j < i$ and for every enumeration reducible to $\mathcal{P}_j$ set S of j-regular partial finite parts the following condition holds:

$$(\exists \tau \subseteq f)(\tau \in S \vee (\forall \rho \supseteq_j \tau)(\rho \notin S)).$$

Proposition 8.1. *Every partial regular enumeration is $(i+1)$-generic enumeration for every $i < n$.*

Proof. Let S be a set of i-regular partial finite parts such that $S \leq_e \mathcal{P}_i$. Then there exists an e such that $S = \mathcal{R}_i \cap \Gamma_e(\mathcal{P}_i)$. Consider an $(i+1)$-regular partial finite part $\tau \subseteq f$ with $(i+1)$-rank greater than e. From the definition of $(i+1)$-regular partial finite part it follows that there is an i-regular partial finite part $\sigma \subseteq_i \tau$, and hence $\sigma \subseteq f$ such that $\sigma \in S$ or $(\forall \rho \supseteq_i \sigma)(\rho \notin S)$. □

Proposition 8.2. *Suppose that f is a partial regular enumeration. Then*:

(1) *For each $i \leq n$, $B_i \leq_e f^{(i)}$.*

(2) *If $i < n$, then $f \not\leq_e \mathcal{P}_i$.*

Proof. We know that $B_0 \leq_e f$. Let $i < n$. Suppose that for each $j \leq i$, $B_j \leq_e f^{(j)}$. Then $\mathcal{P}_i \leq_e f^{(i)}$.

Since f is partial regular, for every partial finite part δ of f there exists an $(i+1)$-regular partial finite part $\tau \subseteq f$ such that $\delta \subseteq \tau$, where if $f(x)$ is not defined and $x \in \mathrm{dom}(\tau)$, then $\tau(x) = \perp$. For each q denote by $f\lceil_q$ the partial finite part τ with $\mathrm{dom}(\tau) = [0, q-1]$, $\tau \subseteq f$, and for each $x < q$ if $f(x)$ is not defined, then $\tau(x) = \perp$.

Let

$$0 < n_0 < l_0 < b_0 < n_1 < l_1 < b_1 < \dots < n_r < l_r < b_r < n_{r+1} < \dots$$

be the numbers satisfying the conditions (a)–(d) from the definition of the $(i+1)$-regular partial finite part τ_r. Clearly, if $B^f_{i+1} = \{b_0, b_1 \dots\}$, then $f(B^f_{i+1}) = B_{i+1}$. We shall show that there exists an effective in $f^{(i+1)}$ procedure which lists $n_0, l_0, b_0, \dots$ in an increasing order.

Using the oracle f', we can generate consecutively the partial finite parts $f\lceil q$ for $q = 1, 2 \dots$. Notice that $f\lceil n_0$ is i-regular and $|f\lceil n_0|_i = 1$, and it is the first element of this sequence which belongs to $\mathcal{R}_i$. Clearly, $n_0 = \mathrm{lh}(f\lceil n_0)$.

Suppose that $n_0, l_0, b_0, \dots, n_r$ have already been listed. Since $f\lceil l_r \simeq \mu_i(f\lceil(n_r+1), S^i_r)$, we can find effectively in $f^{(i+1)}$ the partial finite part $f\lceil l_r$. Then $l_r = \mathrm{lh}(f\lceil l_r)$. Next $f\lceil b_r$ is an i-regular partial r omitting extension of $f\lceil l_r$. So, there exist natural numbers $l_r = q_0 < \dots < q_r < q_{r+1} = b_r$. Using the oracle $f^{(i+1)}$, we can find consecutively the numbers $q_0, \dots, q_r, q_{r+1} = b_r$. By definition, $f\lceil n_{r+1}$ is an i-regular partial extension of $f\lceil(b_r+1)$ having i-rank equal to $|f\lceil b_r|_i + 1$. Using f', we can generate consecutively the partial finite parts $f\lceil(b_r+1+q)$, $q = 0, 1, \dots$ Then $f\lceil n_{r+1}$ is the first element of this sequence which belongs to $\mathcal{R}_i$.

Then B^f_{i+1} is effective in $f^{(i+1)}$ and $B_{i+1} \leq_e f^{(i+1)}$.

To prove (2), assume that $f \leq_e \mathcal{P}_i$. Then the set

$$S = \{\tau : \tau \in \mathcal{R}_i \;\&\; (\exists x, y_1 \neq y_2 \in \mathbb{N})(\tau(x) \simeq y_1 \;\&\; f(x) \simeq y_2)\},$$

$S \leq_e \mathcal{P}_i$. Using the fact that f is $(i+1)$-generic, there is an i-regular partial finite part $\tau \subseteq f$ such that either $\tau \in S$ or $(\forall \rho \supseteq_i \tau)(\rho \notin S)$. It is obvious that both of these cases are impossible. A contradiction. □

Lemma 8.2. *Let $i \leq n$ and τ be an i-regular partial finite part with domain $[0, q-1]$.*

(1) *For every $y \in \mathbb{N}, z_0 \in B_0, \dots, z_i \in B_i$, we can find effectively in $\mathcal{P}'_{i-1}$ an i-regular partial extension ρ of τ such that $|\rho|_i = |\tau|_i + 1$ and $\rho(q) \simeq y, z_0 \in \rho(B^\rho_0), \dots, z_i \in \rho(B^\rho_i)$.*

(2) *For every sequence* $\vec{a} = a_0, \ldots, a_m$ *of natural numbers, one can find effectively in* $\mathcal{P}'_i$ *an* i*-regular* m *omitting partial extension* δ *of* τ *such that* $\delta(K^{\delta}_{\tau}) = \vec{a}$.

Proof. The proof is as in the total case [8]. By induction on i, (1) and (2) are proven simultaneously. □

Proof of Theorem 7.1. By Proposition 8.2, it is sufficient to show that there exists a partial regular enumeration f which is quasi-minimal over B_0.

We shall construct f as a union of n-regular partial finite parts δ_s such that for all s, $\delta_s \subseteq_n \delta_{s+1}$ and $|\delta_s|_n = s+1$. Suppose that for $i \leq n$ σ_i is a recursively in B_i enumeration of B_i.

Let δ_0 be a 0-regular partial finite part such that $|\delta_0|_n = 1$. Suppose that δ_s is defined. Set $z_0 = \sigma_0(s), \ldots, z_n \sigma_n(s)$. Using Lemma 8.2, we can construct effectively in $\mathcal{P}'_{n-1}$ an n-regular partial finite part $\rho \supseteq_n \delta_s$ such that $|\rho|_n |\delta_s|_n + 1$, $\rho(\mathrm{lh}(\delta_s)) = s$ and $z_0 \in \rho(B^{\rho}_0), \ldots, z_n \in \rho(B^{\rho}_n)$. Set $\delta_{s+1} = \rho$.

The obtained enumeration f is surjective on $\mathbb{N}$ and it is a union of n-regular partial finite parts. From the construction is obvious that for every $z \in B_i$ there is an i-regular partial finite part τ of f such that $z \in B^{\tau}_i$. Hence f is a partial regular enumeration. By Proposition 8.1, f is $(i+1)$-generic for each $i < n$.

Then by Proposition 8.2, for $i \leq n$, $B_i \leq f^{(i)}$. Moreover, f is a partial 1-generic enumeration and hence $B_0 <_e f$.

To prove that f is quasi-minimal over B_0, it is sufficient to show that if ψ is a total function and $\psi \leq_e f$, then $\psi \leq_e B_0$. It is clear that for any total set $A \subseteq \mathbb{N}$ one can construct a total function ψ, $\psi \equiv_e A$. Let ψ be a total function and $\psi = \Gamma_e(f)$. Then

$$(\forall x, y \in \mathbb{N})(f \models_0 F_e(\langle x, y\rangle) \iff \psi(x) \simeq y).$$

Consider the set

$$S_0 = \{\rho : \rho \in \mathcal{R}_0 \ \&\ (\exists x, y_1 \neq y_2 \in \mathbb{N})(\rho \Vdash_0 F_e(\langle x, y_1\rangle) \ \&\ \rho \Vdash_0 F_e(\langle x, y_2\rangle))\}.$$

Since $S_0 \leq_e B_0$, we have that there exists a 0-regular partial finite part $\tau_0 \subseteq f$ such that either $\tau_0 \in S_0$ or $(\forall \rho \supseteq_0 \tau_0)(\rho \notin S_0)$. Assume that $\tau_0 \in S_0$. Then there exist $x, y_1 \neq y_2$ such that $f \models_0 F_e(\langle x, y_2\rangle)$ and $f \models_0 F_e(\langle x, y_2\rangle)$. Then $\psi(x) \simeq y_1$ and $\psi(x) \simeq y_2$, which is impossible. So, $(\forall \rho \supseteq_0 \tau_0)(\rho \notin S_0)$.

Let

$$\begin{aligned} S_1 = \{\rho : &\rho \in \mathcal{R}_0 \ \&\ (\exists \tau \supseteq_0 \tau_0)(\exists \delta_1 \supseteq_0 \tau)(\exists \delta_2 \supseteq_0 \tau) \\ &(\exists x, y_1 \neq y_2 \in \mathbb{N})(\tau \subseteq_0 \rho \ \&\ \delta_1 \Vdash_0 F_e(\langle x, y_1\rangle) \ \&\ \delta_2 \Vdash_0 F_e(\langle x, y_2\rangle) \ \&\\ &\mathrm{dom}(\rho) = \mathrm{dom}(\delta_1) \cup \mathrm{dom}(\delta_2) \ \&\\ &(\forall x)(x \in \mathrm{dom}(\rho) \setminus \mathrm{dom}(\tau) \Rightarrow \rho(x) \simeq \bot))\}. \end{aligned}$$

We have that $S_1 \leq_e B_0$ and hence there exists a 0-regular partial finite part $\tau_1 \subseteq f$ such that either $\tau_1 \in S_1$ or $(\forall \rho \supseteq_0 \tau_1)(\rho \notin S_1)$.

Assume that $\tau_1 \in S_1$. Then there exists a 0-regular partial finite part τ such that $\tau_0 \subseteq_0 \tau \subseteq_0 \tau_1$ and for some $\delta_1 \supseteq_0 \tau$, $\delta_2 \supseteq_0 \tau$ and $x_0, y_1 \neq y_2 \in \mathbb{N}$ we have

$$\delta_1 \Vdash_0 F_e(\langle x_0, y_1\rangle) \;\&\; \delta_2 \Vdash_0 F_e(\langle x_0, y_2\rangle) \;\&\; \mathrm{dom}(\tau_1) = \mathrm{dom}(\delta_1) \cup \mathrm{dom}(\delta_2) \;\&\; \\ \& \; (\forall x)(x \in \mathrm{dom}(\tau_1) \setminus \mathrm{dom}(\tau) \Rightarrow \tau_1(x) \simeq \bot).$$

Let $\psi(x_0) \simeq y$. Then $f \models_0 F_e(\langle x_0, y\rangle)$. Hence there exists a $\rho \supseteq_0 \tau_1$ such that $\rho \Vdash_0 F_e(\langle x_0, y\rangle)$. Let $y \neq y_1$. Define the partial finite part ρ_0 as follows:

$$\rho_0(x) \simeq \begin{cases} \delta_1(x) & \text{if } x \in \mathrm{dom}(\delta_1), \\ \rho(x) & \text{if } x \in \mathrm{dom}(\rho) \setminus \mathrm{dom}(\delta_1). \end{cases}$$

Then $\tau_0 \subseteq_0 \rho_0$, $\delta_1 \subseteq_0 \rho_0$ and notice that for all $x \in \mathrm{dom}(\rho)$ if $\rho(x) \not\simeq \bot$, then $\rho(x) \simeq \rho_0(x)$. Hence $\rho_0 \Vdash_0 F_e(\langle x_0, y_1\rangle)$ and $\rho_0 \Vdash_0 F_e(\langle x_0, y\rangle)$. So, $\rho_0 \in S_0$. A contradiction.

Thus, $(\forall \rho)(\rho \supseteq_0 \tau_1 \Rightarrow \rho \notin S_1)$.

Let $\tau = \tau_1 \cup \tau_0$. Notice that $\tau \subseteq f$. We shall show that

$$\psi(x) \simeq y \iff (\exists \delta \supseteq_0 \tau)(\delta \Vdash_0 F_e(\langle x, y\rangle)).$$

And hence $\psi \leq_e B_0$.

If $\psi(x) \simeq y$, then $f \models_0 F_e(x)$, and by Lemma 8.1 $(\exists \rho \subseteq f)(\rho \Vdash_0 F_e(x))$ and ρ is 0-regular. Then take $\delta = \tau \cup \rho$.

Assume that $\delta_1 \supseteq_0 \tau$, $\delta_1 \Vdash_0 F_e(\langle x, y_1\rangle)$. Suppose that $\psi(x) \simeq y_2$ and $y_1 \neq y_2$. Then there exists a $\delta_2 \supseteq_0 \tau$ such that $\delta_2 \Vdash_0 F_e(\langle x, y_2\rangle)$. Set

$$\rho(x) \simeq \begin{cases} \tau(x) & \text{if } x \in \mathrm{dom}(\tau), \\ \bot & \text{if } x \in (\mathrm{dom}(\delta_1) \cup \mathrm{dom}(\delta_2)) \setminus \mathrm{dom}(\tau). \end{cases}$$

Clearly, $\rho \supseteq_0 \tau_1$ and $\rho \in S_1$. A contradiction. □

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