Partial Degree Spectra of Abstract Structures

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Outline

- Enumerations
- Degree spectra of structures
- Definability on structures
- Partial degree spectra
- Relative stability
**Definition.** Let $\mathcal{A} = (A, \omega; \theta_1, \ldots, \theta_n; P_1, \ldots, P_k)$ be a two sorted countable structure.

An enumeration of $\mathcal{A}$ is $\langle f, \mathcal{B}_f \rangle$, where $f$ is a (partial) surjective mapping of $\omega$ onto $A$, $\mathcal{B}_f = (\omega; \varphi_1, \ldots, \varphi_n, \sigma_1, \ldots, \sigma_k)$ and

- $\text{dom}(f)$ is closed under $\varphi_1, \ldots, \varphi_n$;
- $(\forall \bar{x} \in \text{dom}(f))(\forall \bar{y} \in \omega)[f(\varphi_i(\bar{x}, \bar{y})) = \theta_i(f(\bar{x}), \bar{y})]$;
- $(\forall \bar{x} \in \text{dom}(f))(\forall \bar{y} \in \omega)[\sigma_j(\bar{x}, \bar{y}) \iff P_j(f(\bar{x}), \bar{y})]$.

An enumeration $\langle f, \mathcal{B}_f \rangle$ is total if $\text{dom}(f) = \omega$.

Denote by $\langle \varphi \rangle = \{ \langle y, x_1, \ldots, x_n \rangle \mid \varphi(x_1, \ldots, x_n) = y \}$.

$$\langle \mathcal{B}_f \rangle = \langle \varphi_1 \rangle \oplus \cdots \oplus \langle \varphi_n \rangle \oplus \langle \sigma_1 \rangle \oplus \cdots \oplus \langle \sigma_k \rangle$$
Definition. [Richter] The Degree Spectrum of $\mathcal{A}$ is the set

$$DS(\mathcal{A}) = \{d_e(\langle B_f \rangle) \mid \langle f, B_f \rangle \text{ is a total enumeration of } \mathcal{A}\}.$$ 

If $DS(\mathcal{A})$ has a least e-degree $a$, then $a$ is called the degree of $\mathcal{A}$.

Definition. The Co-Spectrum of $\mathcal{A}$ is the set

$$CS(\mathcal{A}) = \{d_e(X) \mid X \leq_e \langle B_f \rangle, \langle f, B_f \rangle \text{ is a tot. enum. of } \mathcal{A}\}.$$ 

If $CS(\mathcal{A})$ has a greatest e-degree $a$ then $a$ is called the co-degree of $\mathcal{A}$.
Proposition. If a structure $\mathcal{A}$ has a degree $a$ then $a$ is also the co-degree of $\mathcal{A}$.

There are examples of structures with no co-degrees and structures with co-degree but no degree.
Let $\mathcal{A} = (A, \omega; \theta_1, \ldots, \theta_n, P_1, \ldots, P_k)$ and $\langle f, \mathcal{B}_f \rangle$ is an enumeration of $\mathcal{A}$.

A function $\theta : \omega^r \times A^m \rightarrow A$ is admissible in $\langle f, \mathcal{B}_f \rangle$ if there is a function $\varphi$ partial recursive in $\mathcal{B}_f$, $(\langle \varphi \rangle \leq_e \langle \mathcal{B}_f \rangle)$ and:

- $\text{dom}(f)$ is closed under $\varphi$;
- $(\forall \bar{x} \in \text{dom}(f))(\forall \bar{y} \in \omega)[f(\varphi(\bar{x}, \bar{y})) = \theta(f(\bar{x}), \bar{y})]$.

And $\theta : \omega^r \times A^m \rightarrow \omega$ is admissible in $\langle f, \mathcal{B}_f \rangle$ if there is a function $\varphi$ partial recursive in $\mathcal{B}_f$

- $\text{dom}(f)$ is closed under $\varphi$;
- $(\forall \bar{x} \in \text{dom}(f))(\forall \bar{y} \in \omega)[\varphi(\bar{x}, \bar{y}) = \theta(f(\bar{x}), \bar{y})]$.
Definition.

- A function $\theta$ is (search) computable in $\mathcal{A}$ iff $\theta$ is admissible in all total enumerations of $\mathcal{A}$.
- A function $\theta$ is (REDS) partially computable in $\mathcal{A}$ iff $\theta$ is admissible in all (partial) enumerations of $\mathcal{A}$.

- Search computability by Moschovakis (Fraissé, Lacombe, Montague);
- Computability by means of Recursively Enumerable Definitional Schemes (REDS) by Shepherdson (Friedman EDS).
The domains of the computable functions in \( \mathbb{A} \) we call the computably enumerable (c.e.) on \( \mathbb{A} \) sets.

Let \( L \) be the language of \( \mathbb{A} \). We add a unary predicate symbol \( T_0 \) to \( L \) to represent a predicate which is true everywhere.

**Proposition.** A set \( X \subseteq \omega^r \times A^m \) is c.e. on \( \mathbb{A} \) iff there is a recursive function \( \gamma : \omega^{r+1} \rightarrow \omega \), such that for any \( n \), \( E^{\gamma(n,y)}(\tilde{X}, \tilde{W}) \) is an elementary \( \Sigma_1 \) formula in \( L \) and there exist parameters \( t_1, \ldots, t_l \) of \( A \) such that:

\[
(\tilde{y}, \tilde{x}) \in X \iff (\exists n \in \omega)[\mathbb{A} \models E^{\gamma(n,y)}(\tilde{X}/\tilde{x}, \tilde{W}/\tilde{t})].
\]

These sets are exactly the relative intrinsically sets on \( \mathbb{A} \).
The Partially Computably enumerable Sets on $\mathcal{A}$

The domains of the partially computable functions in $\mathcal{A}$ we call partially c.e. on $\mathcal{A}$ sets.

**Proposition.** A set $X \subseteq \omega^r \times A^m$ is p.c.e. in $\mathcal{A}$ if there is a recursive function $\gamma : \omega^{r+1} \to \omega$, such that for any $n$, $P^{\gamma(n,\bar{y})}(\bar{X}, \bar{W})$ is a finite conjunctions of atoms or negated atoms in $L$ and there exist parameters $t_1, \ldots, t_l$ of $A$ such that:

$$(\bar{y}, \bar{x}) \in X \iff (\exists n \in \omega) [\mathcal{A} \models P^{\gamma(n,\bar{y})}(\bar{X}/\bar{x}, \bar{W}/\bar{t})].$$
Example of a structure with no co-degree

Consider $\mathcal{A} = (\mathbb{N}, \omega; \Psi; P)$, where $\Psi : \mathbb{N} \to \mathbb{N}$ and $\Psi(\langle n, x \rangle) = \langle n, x + 1 \rangle$ and the predicate $P \subseteq \mathbb{N}$:

$$
P(x) = \begin{cases} 
0 & \exists t(x = \langle 0, t \rangle), \\
0 & \exists n \exists t(x = \langle n + 1, t \rangle \& t \in \emptyset^{(n+1)}), \\
\bot & \text{otherwise.}
\end{cases}
$$

For every $X \subseteq \omega$: $X$ is c.e. in $\mathcal{A}$ iff $\exists n(X \leq_e \emptyset^{(n)})$.
Consider the sequence $\emptyset <_e \emptyset' <_e \cdots <_e \emptyset^{(n)} <_e \cdots$. There is no set $W$ so that:

$$(\forall X \subseteq \omega)(X \leq_e W \iff \exists n(X \leq_e \emptyset^{(n)})).$$

And hence $\mathcal{A}$ has no co-degree.
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$$\forall X \subseteq \omega)(X \leq_e W \iff \exists n(X \leq_e \emptyset^{(n)})).$$

And hence $\mathcal{A}$ has no co-degree.
Proposition. Let $\mathcal{A} = (A, \omega; R, =_A)$, where $A$ is countable set and $R \subseteq A$ is a linear order. Then $d_e(\emptyset)$ is a co-degree of $\mathcal{A}$.

For every $X \subseteq \omega$, if $X$ is c.e. in $\mathcal{A}$ then there is a recursive function $\gamma$ and there exist parameters $t_1, \ldots, t_l$ of $A$ such that:

$$y \in X \iff (\exists n \in \omega)[\mathcal{A} \models E^{(n,y)}_{\gamma}(\bar{W}/\bar{t})].$$

And then $X \leq_e \emptyset$.
Hence $d_e(\emptyset)$ is a co-degree of $\mathcal{A}$.

Corollary. [Richter] If $\mathcal{A}$ is a countable linear ordering with a degree, then this degree is $0_e = d_e(\emptyset)$.
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**Corollary.** [Richter] If $\mathcal{A}$ is a countable linear ordering with a degree, then this degree is $0_e = d_e(\emptyset)$. 

Alexandra A. Soskova  Partial Degree Spectra
An ordinal $\xi$ is constructive if the structure $\xi = (\xi, \omega; \in, =)$ is isomorphic to a computable well ordering.

**Proposition.** Let $\xi$ be a countable ordinal. Then the structure $\xi = (\xi, \omega; \in, =)$ has a degree if and only if $\xi$ is a constructive ordinal.

**Corollary.** If $\xi$ is a countable $\xi \geq \omega_1^{CK}$ then $\xi$ has a co-degree and no degree.
Definition. The Partial Degree Spectrum of $\mathcal{A}$ is the set 

$$PDS(\mathcal{A}) = \{d_e(\langle \mathcal{B}_f \rangle) \mid \langle f, \mathcal{B}_f \rangle \text{ is a partial enumeration of } \mathcal{A}\}.$$ 

The least element of $\mathcal{A}$ (if it exists) is called a partial degree of $\mathcal{A}$.

Definition. The Partial Co-Spectrum of $\mathcal{A}$ is the set 

$$PCS(\mathcal{A}) = \{d_e(X) \mid X \leq_e \langle \mathcal{B}_f \rangle, \langle f, \mathcal{B}_f \rangle \text{ is an enumeration of } \mathcal{A}\}.$$ 

If $PCS(\mathcal{A})$ has a greatest e-degree $a$ then $a$ is called a partial co-degree of $\mathcal{A}$.

Proposition. If $a$ is a partial degree of $\mathcal{A}$ then $a$ is a partial co-degree of $\mathcal{A}$. 
If a is a degree of $\mathfrak{A}$ and b is a partial degree of $\mathfrak{A}$ then $b \leq a$. There are structures (e.g. that from Example 1) with no partial degree.

**Definition.** A set $W \subseteq \mathbb{N}$ is *total* if $(\omega \setminus W) \leq_e W$. An e-degree is *total* if it contains a total set.

**Proposition.** Let $\mathfrak{A}$ be a total countable structure with a partial co-degree $a$. Then $a$ is a total e-degree.

Consider a set $W \in a$. Then $W$ is p.c.e. in $\mathfrak{A}$, i.e. there is a recursive function $\gamma$ and parameters $t_1, \ldots, t_l$ of $A$ such that:

$$y \in W \iff (\exists n \in \omega)[\mathfrak{A} \models P^{(n,y)}(\bar{Z}/\bar{t})].$$

The set $\{\hat{L} \mid L(\bar{Z}/\bar{t}) = 0\}$ is total and e-equivalent to $W$. 
**Theorem.** If the structure $\mathcal{A}$ has a p. co-degree which is a total e-degree then $\mathcal{A}$ has a p. degree too.

Let $a$ be e p.co-degree of $\mathcal{A}$ and $W \in a$ be a total set. We construct a standard enumeration $\langle f, \mathcal{B}_f \rangle$ of $\mathcal{A}$ such that $\langle \mathcal{B}_f \rangle \leq_e W$.

**Fact:** Since $W$ is a total set then $W$ is e-equivalent to its characteristic function. Hence for each $r$ there is a p.r in $W$ universal function $\Phi_r$ for the p.r. in $W$ functions of $r$ arguments.

If $W$ is not total, then we can construct an enumeration $\langle f, \mathcal{B}_f \rangle$ of $\mathcal{A}$, $W \equiv_e \langle \mathcal{B}_f \rangle$, but the functions in $\mathcal{B}_f$ are not single valued outside the domain of $f$. 

Alexandra A. Soskova  Partial Degree Spectra
**Corollary.** Every total structure $\mathcal{A}$ with a partial co-degree has a partial degree.

**Proposition.** Let $\mathcal{A} = (A, \omega; R_1, \ldots, R_k)$, where all the predicates $R_j \subseteq A^{m_j}$. Then $\mathcal{A}$ has a partial co-degree $0_e$.

**Corollary.** Every countable linear ordering has a partial degree $0_e$. And hence if $\xi$ is not constructive ordinal, then the structure $(\xi, \omega; \in, =)$ has a partial degree $0_e$ and has no degree.
Let $\mathcal{A} = (\mathbb{N}, \omega; \theta_1, \ldots, \theta_n; P_1, \ldots, P_k)$.

**Definition.** The structure $\mathcal{A}$ is *relatively stable* if for every total enumeration $\langle f, \mathcal{V}_f \rangle$ of $\mathcal{A}$ the mapping $f$ is partially recursive in $\mathcal{V}_f$.

**Definition.** The structure $\mathcal{A}$ is *algorithmic complete* if all the p.r. functions on $\mathbb{N}$ are computable in $\mathcal{A}$ considered as functions on $\mathbb{N}$ and on $\omega$.

**Proposition.** *The following conditions are equivalent:*

- $\mathcal{A}$ is relatively stable;
- the converting function $\alpha : \mathbb{N} \to \omega$, $\lambda n.\alpha(n) = n$ is computable;
- $\mathcal{A}$ is algorithmic complete.
Theorem. \( \mathfrak{A} \) is algorithmic complete if there exists a recursive function \( \gamma(n, x) \) and parameters \( t_1, \ldots, t_l \in \mathbb{N} \) such that

\[
(\forall x \in \mathbb{N})(\forall y \in \omega)(x = y \iff (\exists n \in \omega)(\mathfrak{A} \models E^{\gamma(n,y)}(\bar{Z}/\bar{t}, X/x))).
\]

Proposition. The structure \( \mathfrak{A} = (\mathbb{N}, \omega; S, =_{\mathbb{N}}) \), where \( S : \mathbb{N} \to \mathbb{N} \) is the successor function on \( \mathbb{N} \) is algorithmic complete.

If \( E^y = T(F^y(Z), X) \) then \( \mathfrak{A} \models E^y(Z/0, X/x) \iff x = y \).
**Definition.** The structure $\mathcal{A}$ is *super relatively stable* if for every enumeration $\langle f, \mathcal{B}_f \rangle$ of $\mathcal{A}$ the mapping $f$ has a p.r. in $\mathcal{B}_f$ function $g \supseteq f$, i.e. for every $n$ if $f(n)$ is defined then $g(n)$ is defined and $f(n) = g(n)$.

Let $\langle f, \mathcal{B}_f \rangle$ be an enumeration of $\mathcal{A}$. Then for every function $\varphi$ with the property $\varphi(x) = \alpha(f(x))$ for $x \in \text{dom}(\alpha)$, $\varphi \supseteq f$. 
Proposition. The following conditions are equivalent:

- $\mathcal{A}$ is super relatively stable;
- The converting function $\alpha : \mathbb{N} \to \omega$, $\lambda n.\alpha(n) = n$ is partially computable in $\mathcal{A}$;
- Every c.e subset of $\omega^{r+m}$, considered as a subset of $\omega^{r} \times \mathbb{N}^{m}$, is c.e. in $\mathcal{A}$.
- There exists a recursive function $\gamma(n, x)$ and parameters $t_1, \ldots, t_l \in \mathbb{N}$ such that

\[
(\forall x \in \mathbb{N})(\forall y \in \omega)(x = y \Leftrightarrow (\exists n \in \omega)(\mathcal{A} \models P^{(n,y)}(\overline{Z}/\overline{t}, X/x))).
\]
**Definition.** The structure $\mathcal{A}$ is *partially algorithmic complete* if all the p.r. functions on $\mathbb{N}$ are partially computable in $\mathcal{A}$ considered as functions on $\mathbb{N}$ and on $\omega$.

**Definition.** A structure $\mathcal{A}$ is finitely generated if there are finitely many elements $t_1, \ldots, t_l$ and variables $W_1, \ldots, W_l$, such that

$$\mathcal{A} = \{ \lambda(\bar{W}/\bar{t}) \mid \lambda \text{ is a term on } \bar{W} \}.$$

**Proposition.** *If a structure $\mathcal{A}$ is partially algorithmic complete then it is finitely generated and hence the computable functions in $\mathcal{A}$ and the partially computable functions coincide.*

**Theorem.** *A structure $\mathcal{A}$ is partially algorithmic complete if and only if $\mathcal{A}$ is super relatively stable and finitely generated.*
Example of algorithmic complete structures

Consider the structure $\mathfrak{A} = (\mathbb{N}, \omega; P; Z)$, where $P : \mathbb{N} \to \mathbb{N}$, $P(x) = x - 1$ for $x > 0$ and $P(0) = 0$, and $Z(x) = 0$ if $x = 0$, and $Z(x) = 1$ if $x > 0$.

It is clear that $\mathfrak{A}$ is not finitely generated. Thus it is not partially algorithmic complete.

Let $L = (F, T)$ be the language of $\mathfrak{A}$ and $x \in \mathbb{N}$, $y \in \omega$.

$$x = y \iff \mathfrak{A} \models \neg T(X/x) \land \cdots \land \neg T(F^{y-1}(X/x)) \land T(F^y(x/x)).$$

Since it is super relative stable and hence relatively stable. Then it is algorithmic complete.

An example of partially algorithmic complete structure is $\mathfrak{A} = (\mathbb{N}, \omega; S, P; Z)$, where

- $S(x) = x + 1$,
- $P(x) = x - 1$ for $x > 0$ and not defined if $x = 0$,
- $Z(x) = 0$ if $x = 0$ and not defined if $x > 0$. 
Thank you!