Enumeration Degree Spectra and $\omega$-Degree Spectra of Abstract Structures

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Outline

- Enumeration degrees
- Degree spectra and co-spectra
- Characterization of the co-spectra
- Representing the countable ideals as co-spectra
- Properties of upwards closed sets of degrees
- The minimal pair theorem
- Quasi-minimal degrees
- Jump spectra
- $\omega$-degree spectra
**Definition.** (Friedberg and Rogers, 1959) We say that $\Psi : 2^\omega \rightarrow 2^\omega$ is an **enumeration operator** (or e-operator) iff for some c.e. set $W_i$

$$\Psi(B) = \{ x | (\exists D)[\langle x, D \rangle \in W_i \& D \subseteq B] \}$$

for each $B \subseteq \omega$.

**Definition.** For any sets $A$ and $B$ define $A$ is **enumeration reducible to** $B$, written $A \leq_e B$, by $A = \Psi(B)$ for some e-operator $\Psi$. 
The enumeration jump

**Definition.** Given $A \subseteq \omega$, set $A^+ = A \oplus (\omega \setminus A)$.

**Theorem.** For any $A, B \subseteq \omega$,

1. $A$ is c.e. in $B$ iff $A \leq_e B^+$.
2. $A \leq_T B$ iff $A^+ \leq_e B^+$.

**Definition.** (Cooper, McEvoy) Given $A \subseteq \omega$, let $E_A = \{\langle i, x \rangle | x \in \Psi_i(A)\}$. Set $J_e(A) = E_A^+$.

The enumeration jump $J_e$ is monotone and agrees with the Turing jump $J_T$ in the following sense:

**Theorem.** For any $A \subseteq \omega$, $J_T(A)^+ \equiv_e J_e(A^+)$.

**Definition.** A set $A$ is called *total* iff $A \equiv_e A^+$. 
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**Definition.** A set $A$ is called *total* iff $A \equiv_e A^+$. 
**Definition.** Given a set \( A \), let \( d_e(A) = \{ B \subseteq \omega | A \equiv_e B \} \).

Denote by \( \mathcal{D}_T \) the partial ordering of the Turing degrees and by \( \mathcal{D}_e \) the partial ordering of the enumeration degrees.

**The Rogers embedding.** Define \( \iota : \mathcal{D}_T \to \mathcal{D}_e \) by \( \iota(d_T(A)) = d_e(A^+) \). Then \( \iota \) is a Proper embedding of \( \mathcal{D}_T \) into \( \mathcal{D}_e \).

The enumeration degrees in the range of \( \iota \) are called total.

Let \( d_e(A)' = d_e(J_e(A)) \). The jump is always total and agrees with the Turing jump under the embedding \( \iota \).
**Definition.** Given a set $A$, let $d_e(A) = \{B \subseteq \omega | A \equiv_e B\}$.

Denote by $\mathcal{D}_T$ the partial ordering of the Turing degrees and by $\mathcal{D}_e$ the partial ordering of the enumeration degrees.

**The Rogers embedding.** Define $\iota : \mathcal{D}_T \rightarrow \mathcal{D}_e$ by $\iota(d_T(A)) = d_e(A^+)$. Then $\iota$ is a Proper embedding of $\mathcal{D}_T$ into $\mathcal{D}_e$. The enumeration degrees in the range of $\iota$ are called total.

Let $d_e(A)' = d_e(J_e(A))$. The jump is always total and agrees with the Turing jump under the embedding $\iota$. 
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Let $d_e(A)' = d_e(J_e(A))$. The jump is always total and agrees with the Turing jump under the embedding $\iota$. 
Degree Spectra

Let $\mathcal{A} = (\mathbb{N}; R_1, \ldots, R_k)$ be a denumerable structure. Enumeration of $\mathcal{A}$ is every total surjective mapping of $\mathbb{N}$ onto $\mathbb{N}$.

Given an enumeration $f$ of $\mathcal{A}$ and a subset of $\mathcal{A}$ of $\mathbb{N}^a$, let

$$f^{-1}(A) = \{ \langle x_1, \ldots, x_a \rangle : (f(x_1), \ldots, f(x_a)) \in A \}.$$ 

Set $f^{-1}(\mathcal{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq)$.

Definition. (Richter) The Turing Degree Spectrum of $\mathcal{A}$ is the set

$$DS_T(\mathcal{A}) = \{ d_T(f^{-1}(\mathcal{A})) : f \text{ is an one to one enumeration of } \mathcal{A} \}.$$ 

If $a$ is the least element of $DS_T(\mathcal{A})$, then $a$ is called the degree of $\mathcal{A}$.
**Definition.** The e-Degree Spectrum of $\mathcal{A}$ is the set

$$DS(\mathcal{A}) = \{d_e(f^{-1}(\mathcal{A})) : f \text{ is an enumeration of } \mathcal{A}\}.$$ 

If $a$ is the least element of $DS(\mathcal{A})$, then $a$ is called the e-degree of $\mathcal{A}$.

**Proposition.** If $\mathcal{A}$ has e-degree $a$ then $a = d_e(f^{-1}(\mathcal{A}))$ for some one to one enumeration $f$ of $\mathcal{A}$.

**Proposition.** If $a \in DS(\mathcal{A})$, $b$ is a total e-degree and $a \leq_e b$ then $b \in DS(\mathcal{A})$. 
**Definition.** The $e$-Degree Spectrum of $\mathcal{A}$ is the set

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Definition. The e-Degree Spectrum of \( \mathbb{A} \) is the set

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DS(\mathbb{A}) = \{d_e(f^{-1}(\mathbb{A})) : f \text{ is an enumeration of } \mathbb{A}\}.
\]

If \( a \) is the least element of \( DS(\mathbb{A}) \), then \( a \) is called the e-degree of \( \mathbb{A} \).

Proposition. If \( \mathbb{A} \) has e-degree \( a \) then \( a = d_e(f^{-1}(\mathbb{A})) \) for some one to one enumeration \( f \) of \( \mathbb{A} \).

Proposition. If \( a \in DS(\mathbb{A}) \), \( b \) is a total e-degree and \( a \leq_e b \) then \( b \in DS(\mathbb{A}) \).
Definition. The structure $\mathcal{A}$ is called total if for every enumeration $f$ of $\mathcal{A}$ the set $f^{-1}(\mathcal{A})$ is total.

Proposition. If $\mathcal{A}$ is a total structure then $DS(\mathcal{A}) = \iota(DS_T(\mathcal{A}))$.

Given a structure $\mathcal{A} = (\mathbb{N}, R_1, \ldots, R_k)$, for every $j$ denote by $R_j^c$ the complement of $R_j$ and let $\mathcal{A}^+ = (\mathbb{N}, R_1, \ldots, R_k, R_1^c, \ldots, R_k^c)$.

Proposition. The following are true:
1 $\iota(DS_T(\mathcal{A})) = DS(\mathcal{A}^+)$.
2 If $\mathcal{A}$ is total then $DS(\mathcal{A}) = DS(\mathcal{A}^+)$.
Clearly if $\mathcal{A}$ is a total structure then $DS(\mathcal{A})$ consists of total degrees. The vice versa is not always true.

**Example.** Let $K$ be the Kleene’s set and $\mathcal{A} = (\mathbb{N}; G_S, K)$, where $G_S$ is the graph of the successor function. Then $DS(\mathcal{A})$ consists of all total degrees. On the other hand if $f = \lambda x. x$, then $f^{-1}(\mathcal{A})$ is an c.e. set. Hence $\bar{K} \nleq_e f^{-1}(\mathcal{A})$. Clearly $\bar{K} \leq_e (f^{-1}(\mathcal{A}))^+$. So $f^{-1}(\mathcal{A})$ is not total.

Is it true that if $DS(\mathcal{A})$ consists of total degrees then there exists a total structure $\mathcal{B}$ s.t. $DS(\mathcal{A}) = DS(\mathcal{B})$?
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Is it true that if $DS(\mathcal{A})$ consists of total degrees then there exists a total structure $\mathcal{B}$ s.t. $DS(\mathcal{A}) = DS(\mathcal{B})$?
Co-spectra

**Definition.** Let $\mathcal{A}$ be a nonempty set of enumeration degrees the *co-set of* $\mathcal{A}$ is the set $co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$. Namely

$$co(\mathcal{A}) = \{ b : b \in D_e \& (\forall a \in \mathcal{A})(b \leq_e a) \}.$$ 

**Example.** Fix $a \in D_e$ and set $\mathcal{A}_a = \{ b \in D_e : a \leq_e b \}$. Then $co(\mathcal{A}_a) = \{ b \in D_e : b \leq_e a \}$.

**Definition.** Given a structure $\mathfrak{A}$, set $CS(\mathfrak{A}) = co(DS(\mathfrak{A}))$. If $a$ is the greatest element of $CS(\mathfrak{A})$ then call $a$ the *co-degree* of $\mathfrak{A}$.

If $\mathfrak{A}$ has a degree $a$ then $a$ is also the co-degree of $\mathfrak{A}$. The vice versa is not always true.
**Definition.** Let \( A \) be a nonempty set of enumeration degrees the co-set of \( A \) is the set \( co(A) \) of all lower bounds of \( A \). Namely

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**Example.** Fix \( a \in D_e \) and set \( A_a = \{ b \in D_e : a \leq_e b \} \). Then

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co(A_a) = \{ b \in D_e : b \leq_e a \}.
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**Definition.** Given a structure \( \mathcal{A} \), set \( CS(\mathcal{A}) = co(DS(\mathcal{A})) \). If \( a \) is the greatest element of \( CS(\mathcal{A}) \) then call \( a \) the co-degree of \( \mathcal{A} \).

If \( \mathcal{A} \) has a degree \( a \) then \( a \) is also the co-degree of \( \mathcal{A} \). The vice versa is not always true.
The admissible sets

**Definition.** A set $A$ of natural numbers is admissible in $\mathcal{A}$ if for every enumeration $f$ of $\mathcal{A}$, $A \leq_e f^{-1}(\mathcal{A})$.

Clearly $a \in CS(\mathcal{A})$ iff $a = d_e(A)$ for some admissible in $\mathcal{A}$ set $A$.

Every finite mapping of $\mathbb{N}$ into $\mathbb{N}$ is called *finite part*. For every finite part $\tau$ and natural numbers $e, x$, let

$$
\tau \models F_e(x) \iff x \in \psi_e(\tau^{-1}(\mathcal{A})) \text{ and } \\
\tau \models \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nvdash F_e(x)).
$$

**Definition.** An enumeration $f$ is *generic* if for every $e, x \in \mathbb{N}$, there exists a $\tau \subseteq f$ s.t. $\tau \models F_e(x) \lor \tau \models \neg F_e(x)$. 

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The admissible sets

**Definition.** A set $A$ of natural numbers is admissible in $\mathcal{U}$ if for every enumeration $f$ of $\mathcal{U}$, $A \leq_e f^{-1}(\mathcal{U})$.

Clearly $a \in CS(\mathcal{U})$ iff $a = d_e(A)$ for some admissible in $\mathcal{U}$ set $A$. Every finite mapping of $\mathbb{N}$ into $\mathbb{N}$ is called *finite part*. For every finite part $\tau$ and natural numbers $e, x$, let

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**Definition.** A set $A$ of natural numbers is *forcing definable in the structure* $\mathcal{A}$ iff there exist finite part $\delta$ and natural number $e$ s.t.

$$A = \{x | (\exists \tau \supseteq \delta)(\tau \models F_e(x))\}.$$ 

**Theorem.** Let $A \subseteq \mathbb{N}$ and $d_e(B) \in DS(\mathcal{A})$. Then the following are equivalent:

1. $A$ is admissible in $\mathcal{A}$.
2. $A \leq_e f^{-1}(\mathcal{A})$ for all generic enumerations $f$ of $\mathcal{A}$ s.t. $(f^{-1}(\mathcal{A}))' \equiv_e B'$.
3. $A$ is forcing definable.
**Definition.** A set $A$ of natural numbers is *forcing definable in the structure* $\mathcal{A}$ iff there exist finite part $\delta$ and natural number $e$ s.t.

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2. $A \leq_e f^{-1}(\mathcal{A})$ for all generic enumerations $f$ of $\mathcal{A}$ s.t. $(f^{-1}(\mathcal{A}))' \equiv_e B'$.
3. $A$ is forcing definable.
**Example.** (Richter 1981) Let \( \mathcal{A} = (\mathbb{N}; <) \) be a linear ordering. Then \( DS(\mathcal{A}) \) contains a minimal pair of degrees and hence \( CS(\mathcal{A}) = \{ 0_e \} \). Clearly \( 0_e \) is the co-degree of \( \mathcal{A} \). Therefore if \( \mathcal{A} \) has a degree \( a \), then \( a = 0_e \).

**Definition.** Let \( n \geq 0 \). The \( n \)-th jump spectrum of a structure \( \mathcal{A} \) is defined by \( DS_n(\mathcal{A}) = \{ a^{(n)} \mid a \in DS(\mathcal{A}) \} \). Set \( CS_n(\mathcal{A}) = co(DS_n(\mathcal{A})) \).

**Example.** (Knight 1986) Consider again a linear ordering \( \mathcal{A} \). Then \( CS_1(\mathcal{A}) \) consists of all \( \Sigma^0_2 \) sets. The first jump co-degree of \( \mathcal{A} \) is \( 0'_e \).

**Example.** (Slaman 1998, Whener 1998) There exists an \( \mathcal{A} \) s.t. \( DS(\mathcal{A}) = \{ a : a \text{ is total and } 0_e < a \} \).

Clearly the structure \( \mathcal{A} \) has co-degree \( 0_e \) but has not a degree.
Some examples

Example. (Richter 1981) Let $\mathcal{A} = (\mathbb{N}; <)$ be a linear ordering. Then $DS(\mathcal{A})$ contains a minimal pair of degrees and hence $CS(\mathcal{A}) = \{0_e\}$. Clearly $0_e$ is the co-degree of $\mathcal{A}$. Therefore if $\mathcal{A}$ has a degree $a$, then $a = 0_e$.

Definition. Let $n \geq 0$. The $n$-th jump spectrum of a structure $\mathcal{A}$ is defined by $DS_n(\mathcal{A}) = \{a^{(n)} \mid a \in DS(\mathcal{A})\}$. Set $CS_n(\mathcal{A}) = co(DS_n(\mathcal{A}))$.

Example. (Knight 1986) Consider again a linear ordering $\mathcal{A}$. Then $CS_1(\mathcal{A})$ consists of all $\Sigma^0_2$ sets. The first jump co-degree of $\mathcal{A}$ is $0_e'$.

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Example. (Richter 1981) Let $\mathcal{A} = (\mathbb{N}; <)$ be a linear ordering. Then $DS(\mathcal{A})$ contains a minimal pair of degrees and hence $CS(\mathcal{A}) = \{0_e\}$. Clearly $0_e$ is the co-degree of $\mathcal{A}$. Therefore if $\mathcal{A}$ has a degree $a$, then $a = 0_e$.

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Clearly the structure $\mathcal{A}$ has co-degree $0_e$ but has not a degree.
Example. (based on Coles, Dawney, Slaman - 1998) Let $G$ be a torsion free Abelian group of rank 1, i.e. $G$ is a subgroup of $Q$. There exists an enumeration degree $s_G$ such that

- $DS(G) = \{ b : b \text{ is total and } s_G \leq_e b \}$.
- The co-degree of $G$ is $s_G$.
- $G$ has a degree iff $s_G$ is total
- If $1 \leq n$, then $s_G^{(n)}$ is the $n$-th jump degree of $G$.

For every $d \in D_e$ there exists a $G$, s.t. $s_G = d$. Hence every principle ideal of enumeration degrees is $CS(G)$ for some $G$. 
Example. Let $B_0, \ldots, B_n, \ldots$ be a sequence of sets of natural numbers. Set $\mathcal{A} = (\mathbb{N}; f; \sigma)$,

$$f(\langle i, n \rangle) = \langle i + 1, n \rangle;$$
$$\sigma = \{\langle i, n \rangle : n = 2k + 1 \lor n = 2k \land i \in B_k\}.$$ 

Then $CS(\mathcal{A}) = I(d_e(B_0), \ldots, d_e(B_n), \ldots)$.
Definition. Consider a subset $A$ of $D_e$. Say that $A$ is *upwards closed* if for every $a \in A$ all total degrees greater than $a$ are contained in $A$.

Let $A$ be an upwards closed set of degrees. Note that if $B \subseteq A$, then $\text{co}(A) \subseteq \text{co}(B)$.

**Proposition.** *(Selman)* Let $A_t = \{a : a \in A \& a \text{ is total}\}$. Then $\text{co}(A) = \text{co}(A_t)$.

**Proposition.** Let $b$ be an arbitrary enumeration degree and $n > 0$. Set $A_{b,n} = \{a : a \in A \& b \leq_e a^{(n)}\}$. Then $\text{co}(A) = \text{co}(A_{b,n})$. 
Specific Properties of Degree Spectra

**Theorem.** Let $\mathcal{A}$ be a structure, $1 \leq n$ and $c \in DS_n(\mathcal{A})$. Then

$$CS(\mathcal{A}) = co(\{b \in DS(\mathcal{A}) : b^{(n)} = c\}).$$

**Example.** (Upwards closed set for which the Theorem is not true)

Let $B \nleq_e A$ and $A \nleq_e B'$. Let

$$D = \{a : d_e(A) \leq_e a\} \cup \{a : d_e(B) \leq_e a\}.$$

Set $A = \{a : a \in D \land a' = d_e(B')\}$.

- $d_e(B)$ is the least element of $A$ and hence $d_e(B) \in co(A)$.
- $d_e(B) \nleq_e d_e(A)$ and hence $d_e(B) \notin co(D)$. 
Theorem. Let $\mathcal{A}$ be a structure, $1 \leq n$ and $c \in DS_n(\mathcal{A})$. Then

$$CS(\mathcal{A}) = \text{co}(\{b \in DS(\mathcal{A}) : b^{(n)} = c\}).$$

Example. (Upwards closed set for which the Theorem is not true)

Let $B \not\leq_e A$ and $A \not\leq_e B'$. Let

$$\mathcal{D} = \{a : d_e(A) \leq_e a\} \cup \{a : d_e(B) \leq_e a\}.$$ 

Set $A = \{a : a \in \mathcal{D} \& a' = d_e(B)'^{\prime}\}$.

- $d_e(B)$ is the least element of $A$ and hence $d_e(B) \in \text{co}(A)$.
- $d_e(B) \not\leq d_e(A)$ and hence $d_e(B) \not\in \text{co}(\mathcal{D})$. 
The minimal pair theorem

**Theorem.** Let \( c \in DS_2(\mathcal{A}) \). There exist \( f, g \in DS(\mathcal{A}) \) s.t. \( f, g \) are total, \( f'' = g'' = c \) and \( CS(\mathcal{A}) = co(\{f, g\}) \).

Notice that for every enumeration degree \( a \) there exists a structure \( \mathcal{A}_a \) s.t. \( DS(\mathcal{A}_a) = \{x \in \mathcal{D}_T | a < e, x\} \). Hence

**Corollary.** (Rozinas) For every \( b \in \mathcal{D}_e \) there exist total \( f, g \) below \( b'' \) which are a minimal pair over \( b \).

Not every upwards closed set of enumeration degrees has a minimal pair:
An upwards closed set with no minimal pair
**Definition.** Let $\mathcal{A}$ be a set of enumeration degrees. The degree $q$ is quasi-minimal with respect to $\mathcal{A}$ if:

- $q \not\in \text{co}(\mathcal{A})$.
- If $a$ is total and $a \geq q$, then $a \in \mathcal{A}$.
- If $a$ is total and $a \leq q$, then $a \in \text{co}(\mathcal{A})$.

**Theorem.** If $q$ is quasi-minimal with respect to $\mathcal{A}$, then $q$ is an upper bound of $\text{co}(\mathcal{A})$.

**Theorem.** For every structure $\mathcal{A}$ there exists a quasi-minimal with respect to $DS(\mathcal{A})$ degree.
Corollary. (Slaman and Sorbi) Let $I$ be a countable ideal of enumeration degrees. There exist an enumeration degree $q$ s.t.

1. If $a \in I$ then $a <_e q$.
2. If $a$ is total and $a \leq_e q$ then $a \in I$.

Definition. Let $B \subseteq A$ be sets of degrees. Then $B$ is a base of $A$ if

$$(\forall a \in A)(\exists b \in B)(b \leq a).$$

Theorem. Let $A$ be an upwards closed set of degrees possessing a quasi-minimal degree. Suppose that there exists a countable base $B$ of $A$ such that all elements of $B$ are total. Then $A$ has a least element.

Corollary. A total structure $\mathfrak{A}$ has a degree if and only if $DS(\mathfrak{A})$ has a countable base.
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1. If $a \in I$ then $a \leq_e q$.
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Definition. Let $B \subseteq A$ be sets of degrees. Then $B$ is a base of $A$ if

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Theorem. Let $A$ be an upwards closed set of degrees possessing a quasi-minimal degree. Suppose that there exists a countable base $B$ of $A$ such that all elements of $B$ are total. Then $A$ has a least element.

Corollary. A total structure $\mathcal{A}$ has a degree if and only if $DS(\mathcal{A})$ has a countable base.
An upwards closed set with no quasi-minimal degree

\[ a \quad b \]

Enumeration Degree Spectra and \( \omega \)-Degree Spectra of Abstract Structures
Definition. The $n$-th jump spectrum of a structure $\mathcal{A}$ is the set

$$DS_n(\mathcal{A}) = \{a^{(n)} | a \in DS(\mathcal{A})\}.$$ 

If $a$ is the least element of $DS_n(\mathcal{A})$ then $a$ is called $n$-th jump degree of $\mathcal{A}$.

Proposition. For every $\mathcal{A}$, $DS_1(\mathcal{A}) \subseteq DS(\mathcal{A})$.

Is it true that for every $\mathcal{A}$, $DS_1(\mathcal{A}) \subset DS(\mathcal{A})$? Probably the answer is ”no”.

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Enumeration Degree Spectra and $\omega$-Degree Spectra of Abstract Structures
Every jump spectrum is spectrum of a total structure

Let $\mathcal{A} = (\mathbb{N}; R_1, \ldots, R_n)$. Let $\bar{0} \not\in \mathbb{N}$. Set $\mathbb{N}_0 = \mathbb{N} \cup \{\bar{0}\}$. Let $\langle ., . \rangle$ be a pairing function s.t. none of the elements of $\mathbb{N}_0$ is a pair and $\mathbb{N}^*$ be the least set containing $\mathbb{N}_0$ and closed under $\langle ., . \rangle$.

**Definition.** Moschovakis’ extension of $\mathcal{A}$ is the structure

$$\mathcal{A}^* = (\mathbb{N}^*, R_1, \ldots, R_n, \mathbb{N}_0, G_{\langle ., . \rangle}).$$

**Proposition.** $DS(\mathcal{A}) = DS(\mathcal{A}^*)$

Let $K_{2\mathfrak{I}} = \{\langle \delta, e, x \rangle : (\exists \tau \supseteq \delta)(\tau \models F_e(x))\}$. Set $\mathcal{A}' = (\mathcal{A}^*, K_{2\mathfrak{I}}, \mathbb{N}^* \setminus K_{2\mathfrak{I}})$.

**Theorem.**
1. The structure $\mathcal{A}'$ is total.
2. $DS_1(\mathcal{A}) = DS(\mathcal{A}')$. 
Every jump spectrum is spectrum of a total structure

Let \( \mathcal{A} = (\mathbb{N}; R_1, \ldots, R_n) \).
Let \( \bar{0} \notin \mathbb{N} \). Set \( \mathbb{N}_0 = \mathbb{N} \cup \{\bar{0}\} \). Let \( \langle ., . \rangle \) be a pairing function s.t. none of the elements of \( \mathbb{N}_0 \) is a pair and \( \mathbb{N}^* \) be the least set containing \( \mathbb{N}_0 \) and closed under \( \langle ., . \rangle \).

**Definition.** Moschovakis’ extension of \( \mathcal{A} \) is the structure

\[
\mathcal{A}^* = (\mathbb{N}^*, R_1, \ldots, R_n, \mathbb{N}_0, G_{\langle ., . \rangle}).
\]

**Proposition.** \( DS(\mathcal{A}) = DS(\mathcal{A}^*) \)

Let \( K_{2\mathcal{A}} = \{\langle \delta, e, x \rangle : (\exists \tau \supseteq \delta)(\tau \models F_e(x))\} \).
Set \( \mathcal{A}' = (\mathcal{A}^*, K_{2\mathcal{A}}, \mathbb{N}^* \setminus K_{2\mathcal{A}}) \).

**Theorem.**

1. The structure \( \mathcal{A}' \) is total.
2. \( DS_1(\mathcal{A}) = DS(\mathcal{A}') \).
The Jump Inversion Theorem

Consider two structures \( \mathcal{A} \) and \( \mathcal{B} \). Suppose that

\[
DS(\mathcal{B})_t = \{ a | a \in DS(\mathcal{B}) \text{ and } a \text{ is total} \} \subseteq DS_1(\mathcal{A}).
\]

**Theorem.** There exists a structure \( \mathcal{C} \) s.t. \( DS(\mathcal{C}) \subseteq DS(\mathcal{A}) \) and \( DS_1(\mathcal{C}) = DS(\mathcal{B})_t \).

**Corollary.** Let \( DS(\mathcal{B}) \subseteq DS_1(\mathcal{A}) \). Then there exists a structure \( \mathcal{C} \) s.t. \( DS(\mathcal{C}) \subseteq DS(\mathcal{A}) \) and \( DS(\mathcal{B}) = DS_1(\mathcal{C}) \).

**Corollary.** Suppose that \( DS(\mathcal{B}) \) consists of total degrees greater than or equal to \( 0' \). Then there exists a total structure \( \mathcal{C}' \) such that \( DS(\mathcal{B}) = DS(\mathcal{C}') \).
**Theorem.** Let $n \geq 1$. Suppose that $DS(\mathcal{B}) \subseteq DS_n(\mathcal{A})$. There exists a structure $\mathcal{C}$ s.t. $DS_n(\mathcal{C}) = DS(\mathcal{B})$.

**Corollary.** Suppose that $DS(\mathcal{B})$ consists of total degrees greater than or equal to $0^{(n)}$. Then there exists a total structure $\mathcal{C}$ s.t. $DS_n(\mathcal{C}) = DS(\mathcal{B})$. 
**Example.** Let $n \geq 0$. There exists a total structure $\mathcal{C}$ s.t. $\mathcal{C}$ has a $n+1$-th jump degree $0^{(n+1)}$ but has no $k$-th jump degree for $k \leq n$.

*It is sufficient to construct a structure $\mathcal{B}$ satisfying:*

1. $DS(\mathcal{B})$ has not least element.
2. $0^{(n+1)}$ is the least element of $DS_1(\mathcal{B})$.
3. All elements of $DS(\mathcal{B})$ are total and above $0^{(n)}$.

Consider a set $B$ satisfying:

1. $B$ is quasi-minimal above $0^{(n)}$.
2. $B' \equiv_e 0^{(n+1)}$.

Let $G$ be a subgroup of the additive group of the rationales s.t. $S_G \equiv_e B$. Recall that $DS(G) = \{a | d_e(S_G) \leq_e a \text{ and } a \text{ is total} \}$ and $d_e(S_G)'$ is the least element of $DS_1(G)$. 
Example. Let $n \geq 0$. There exists a total structure $\mathcal{C}$ s.t. $\mathcal{C}$ has a $n + 1$-th jump degree $0^{(n+1)}$ but has no $k$-th jump degree for $k \leq n$.

It is sufficient to construct a structure $\mathfrak{B}$ satisfying:

1. $DS(\mathfrak{B})$ has not least element.
2. $0^{(n+1)}$ is the least element of $DS_1(\mathfrak{B})$.
3. All elements of $DS(\mathfrak{B})$ are total and above $0^{n}$.

Consider a set $B$ satisfying:

1. $B$ is quasi-minimal above $0^{n}$.
2. $B' \equiv_e 0^{(n+1)}$.

Let $G$ be a subgroup of the additive group of the rationales s.t. $S_G \equiv_e B$. Recall that $DS(G) = \{a | d_e(S_G) \leq_e a \text{ and } a \text{ is total}\}$ and $d_e(S_G)'$ is the least element of $DS_1(G)$.
Let $n \geq 0$. There exists a total structure $\mathcal{C}$ such that
$DS_n(\mathcal{C}) = \{a|0^{(n)} <_e a\}$.
It is sufficient to construct a structure $\mathcal{B}$ such that the elements of
$DS(\mathcal{B})$ are exactly the total e-degrees greater than $0^{(n)}$.
This is done by Whener's construction using a special family of sets:

**Theorem.** Let $n \geq 0$. There exists a family $\mathcal{F}$ of sets of natural
number s.t. for every $X$ strictly above $0^{(n)}$ there exists a recursive
in $X$ set $U$ satisfying the equivalence:

$$F \in \mathcal{F} \iff (\exists a)(F = \{x|(a, x) \in U\}).$$

But there is no c.e. in $0^{(n)}$ such $U$. 
Let $S$ be the set of all sequences of sets of natural numbers. For $\mathcal{B} = \{B_n\}_{n<\omega} \in S$ call the jump class of $\mathcal{B}$ the set

$$J_\mathcal{B} = \{d_T(X) \mid (\forall n)(B_n \text{ is c.e. in } X^{(n)} \text{ uniformly in } n)\}.$$

$A$ is $\omega$-enumeration reducible to $\mathcal{B}$ ($A \leq_\omega \mathcal{B}$) if $J_\mathcal{B} \subseteq J_A$

$A \equiv_\omega \mathcal{B}$ if $J_A = J_\mathcal{B}$.
Let \( B = \{ B_n \}_{n<\omega} \in S \).

**Definition.** A jump sequence \( P(B) = \{ P_n(B) \}_{n<\omega} \):

1. \( P_0(B) = B_0 \)
2. \( P_{n+1}(B) = (P_n(B))' \oplus B_{n+1} \)

**Theorem.** [Soskov, Kovachev] \( A \leq_\omega B \), if \( A_n \leq_e P_n(B) \) uniformly in \( n \).
\(d_\omega(B) = \{A \mid A \equiv_\omega B\}\)
\(D_\omega = \{d_\omega(B) \mid B \in S\}\).

If \(A \subseteq \mathbb{N}\) denote by \(A \uparrow \omega = \{A, \emptyset, \emptyset, \ldots\}\).

For every \(A, B \subseteq \mathbb{N}\):

\[A \leq_e B \iff A \uparrow \omega \leq_\omega B \uparrow \omega.\]

The mapping \(\kappa(d_e(A)) = d_\omega(A \uparrow \omega)\) gives an isomorphic embedding of \(D_e\) to \(D_\omega\).
Definition. For every $A \in S$ the $\omega$-enumeration jump of $A$ is

$A' = \{ P_{n+1}(A) \}_{n<\omega}$

Let $d_\omega (A)' = d_\omega (A')$.

$A^{(k)} = \{ P_{n+k}(A) \}_{n<\omega}$ for each $k$.

$d_\omega (A)^{(k)} = d_\omega (A^{(k)})$. 
Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be given structures.

**Definition.** The relative spectrum $\text{RS}(\mathcal{A}, \mathcal{A}_1 \ldots, \mathcal{A}_n)$ of the structure $\mathcal{A}$ with respect to $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is the set

$$\{d_e(f^{-1}(\mathcal{A})) \mid f \text{ is an enumeration of } \mathcal{A} \& \sum_{k \leq n} f^{-1}(\mathcal{A}_k) \leq_e f^{-1}((\mathcal{A})^{(k)})\}.$$ 

It turns out that almost all properties of the degree spectra remain true for the relative spectra.
Let $\mathcal{B} = \{B_n\}_{n<\omega}$ be a fixed sequence of sets.

**Definition.** The enumeration $f$ of the structure $\mathfrak{A}$ is acceptable with respect to $\mathcal{B}$, if for every $n$,

$$f^{-1}(B_n) \leq_e f^{-1}(\mathfrak{A})^{(n)}$$

uniformly in $n$.

Denote by $\mathcal{E}(\mathfrak{A}, \mathcal{B})$ - the class of all acceptable enumerations.

**Definition.** The $\omega$-degree spectrum of $\mathfrak{A}$ with respect to $\mathcal{B} = \{B_n\}_{n<\omega}$ is the set

$$\text{DS}(\mathfrak{A}, \mathcal{B}) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \in \mathcal{E}(\mathfrak{A}, \mathcal{B})\}$$
It is easy to find a structure $\mathcal{A}$ and a sequence $\mathcal{B}$ such that $\text{DS}(\mathcal{A}, \mathcal{B}) \neq \text{DS}(\mathcal{A})$.

The notion of the $\omega$-degree spectrum is a generalization of the relative spectrum: $\text{RS}(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n) = \text{DS}(\mathcal{A}, \mathcal{B})$, where $\mathcal{B} = \{B_k\}_{k < \omega}$,

- $B_0 = \emptyset$,
- $B_k$ is the positive diagram of the structure $\mathcal{A}_k$, $k \leq n$
- $B_k = \emptyset$ for all $k > n$. 
Proposition. \( \text{DS}(\mathcal{A}, \mathcal{B}) \) is upwards closed with respect to total e-degrees.

Definition. The \( k \)th \( \omega \)-jump spectrum of \( \mathcal{A} \) with respect to \( \mathcal{B} \) is the set

\[
\text{DS}_k(\mathcal{A}, \mathcal{B}) = \{ a^{(k)} \mid a \in \text{DS}(\mathcal{A}, \mathcal{B}) \}.
\]

Proposition. \( \text{DS}_k(\mathcal{A}, \mathcal{B}) \) is upwards closed with respect to total e-degrees.
For every $A \subseteq D_\omega$ let $co(A) = \{b | b \in D_\omega \& (\forall a \in A)(b \leq_\omega a)\}$.

**Definition.** The $\omega$-co-spectrum of $\mathcal{A}$ with respect to $\mathcal{B}$ is the set

$$CS(\mathcal{A}, \mathcal{B}) = co(DS(\mathcal{A}, \mathcal{B})).$$

**Definition.** The $k$th $\omega$-co-spectrum of $\mathcal{A}$ with respect to $\mathcal{B}$ is the set

$$CS_k(\mathcal{A}, \mathcal{B}) = co(DS_k(\mathcal{A}, \mathcal{B})).$$
Properties of the co-sets of omega degrees of upwards closed sets

Let $\mathcal{A} \subseteq D_e$ be an upwards closed set with respect to total e-degrees.

**Proposition.** $\text{co}(\mathcal{A}) = \text{co}(\{a : a \in A \& a \text{ is total}\})$.

**Corollary.** $\text{CS}(\mathcal{A}, \mathcal{B}) = \text{co}(\{a \mid a \in \text{DS}(\mathcal{A}, \mathcal{B}) \& a \text{ is a total e-degree}\})$. 
Let $\mathcal{A} \subseteq \mathcal{D}_e$ be an upwards closed set with respect to total e-degrees and $k > 0$.

There exists $b \in \mathcal{D}_e$ such that

$$\text{co}(\mathcal{A}) \neq \text{co} \left( \{ a : a \in \mathcal{A} \& b \leq a^{(k)} \} \right).$$

Let $n > 0$. There is a structure $\mathfrak{A}$, a sequence $\mathcal{B}$ and $c \in \text{DS}_n(\mathfrak{A}, \mathcal{B})$ such that

$$\text{CS}(\mathfrak{A}, \mathcal{B}) \neq \text{co}(\{ a \in \text{DS}(\mathfrak{A}, \mathcal{B}) \mid a^{(n)} = c \}).$$
Theorem. For every structure $\mathcal{A}$ and every sequence $\mathcal{B} \in S$ there exist total enumeration degrees $f$ and $g$ in $DS(\mathcal{A}, \mathcal{B})$ such that for every $\omega$-enumeration degree $a$ and $k \in \mathbb{N}$:

$$a \leq_\omega f^{(k)} \& a \leq_\omega g^{(k)} \Rightarrow a \in CS_k(\mathcal{A}, \mathcal{B}) .$$
Corollary. $CS_k(\mathcal{A}, \mathcal{B})$ is the least ideal containing all $k$th $\omega$-jumps of the elements of $CS(\mathcal{A}, \mathcal{B})$.

- $I = CS(\mathcal{A}, \mathcal{B})$ is a countable ideal;
- $CS(\mathcal{A}, \mathcal{B}) = I(f) \cap I(g)$;
- $I^{(k)}$ - the least ideal, containing all $k$th $\omega$-jumps of the elements of $I$;
- (Hristo Ganchev)
  $I = I(f) \cap I(g) \implies I^{(k)} = I(f^{(k)}) \cap I(g^{(k)})$ for every $k$;
- $I(f^{(k)}) \cap I(g^{(k)}) = CS_k(\mathcal{A}, \mathcal{B})$ for each $k$;
- Thus $I^{(k)} = CS_k(\mathcal{A}, \mathcal{B})$. 
Countable ideals of $\omega$-enumeration degrees

There is a countable ideal $I$ of $\omega$-enumeration degrees for which there is no structure $\mathcal{A}$ and sequence $\mathcal{B}$ such that $I = CS(\mathcal{A}, \mathcal{B})$.

- $\mathcal{A} = \{0, 0', 0'', \ldots, 0^{(n)}, \ldots\}$;
- $I = I(\mathcal{A}) = \{a \mid a \in D_\omega \& (\exists n)(a \leq_\omega 0^{(n)})\}$ - a countable ideal generated by $\mathcal{A}$.

Assume that there is a structure $\mathcal{A}$ and a sequence $\mathcal{B}$ such that $I = CS(\mathcal{A}, \mathcal{B})$

Then there is a minimal pair $f$ and $g$ for $DS(\mathcal{A}, \mathcal{B})$, so $I^{(n)} = I(f^{(n)}) \cap I(g^{(n)})$ for each $n$.

- $f \geq 0^{(n)}$ and $g \geq 0^{(n)}$ for each $n$.

Then by Enderton and Putnam [1970], Sacks [1971]: $f'' \geq 0^{(\omega)}$ and $g'' \geq 0^{(\omega)}$.

Hence $I'' \neq I(f'') \cap I(g'')$. A contradiction.
Theorem. For every structure $\mathcal{A}$ and every sequence $\mathcal{B}$, there exists $F \subseteq \mathbb{N}$, such that $q = d_\omega(F \uparrow \omega)$ and:

1. $q \notin \text{CS}(\mathcal{A}, \mathcal{B})$;
2. If $a$ is a total e-degree and $a \geq_\omega q$ then $a \in \text{DS}(\mathcal{A}, \mathcal{B})$;
3. If $a$ is a total e-degree and $a \leq_\omega q$ then $a \in \text{CS}(\mathcal{A}, \mathcal{B})$. 
Questions:

- Is it true that for every structure $\mathcal{A}$ and every sequence $\mathcal{B}$ there exists a structure $\mathcal{B}'$ such that $DS(\mathcal{B}') = DS(\mathcal{A}, \mathcal{B})$?
- If for a countable ideal $I \subseteq D_\omega$ there is an exact pair then are there a structure $\mathcal{A}$ and a sequence $\mathcal{B}$ so that $CS(\mathcal{A}, \mathcal{B}) = I$?
Thank you!