Degree Spectra and Conservative Extensions of Abstract Structures

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joint work with

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Outline

- Degree spectra of structures
- Definability on structures
- Conservative \((k, n)\) Extensions
- Jumps of Structures
- Jump inversion theorem for structures
Let $\mathcal{A} = (A; P_1, \ldots, P_k)$ be a denumerable structure. Enumeration of $\mathcal{A}$ is every one to one mapping of $\mathbb{N}$ onto $A$. 

Given an enumeration $f$ of $\mathcal{A}$ and a subset of $X$ of $A^a$, let 

$$f^{-1}(X) = \{\langle x_1, \ldots, x_a \rangle : (f(x_1), \ldots, f(x_a)) \in X\}.$$ 

Set $f^{-1}(\mathcal{A}) = f^{-1}(P_1) \oplus \cdots \oplus f^{-1}(P_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq)$. 

**Definition.** (Richter) *The Degree Spectrum of $\mathcal{A}$ is the set 

$$DS_T(\mathcal{A}) = \{d_T(f^{-1}(\mathcal{A})) : f \text{ is an enumeration of } \mathcal{A}\}.$$*
**Definition.** (Knight) The $n$-th jump spectrum of a structure $\mathcal{A}$ is the set

$$DS_n(\mathcal{A}) = \{ a^{(n)} | a \in DS(\mathcal{A}) \}.$$ 

**Proposition.** (Knight) For every automorphically nontrivial structure $\mathcal{A}$, $DS_n(\mathcal{A})$ is an upwards closed set of degrees.

**Theorem.** (A. Soskova, I. Soskov) Every jump spectrum is a spectrum of a structure, i.e. for every countable structure $\mathcal{A}$ there is a structure $\mathcal{B}$ such that $DS_1(\mathcal{A}) = DS(\mathcal{B})$.

**Theorem.** (A. Soskova, I. Soskov) Let $\mathcal{A}$ and $\mathcal{C}$ be countable structures and $DS(\mathcal{A}) \subseteq DS_1(\mathcal{C})$. There exists a structure $\mathcal{B}$ such that $DS(\mathcal{A}) = DS_1(\mathcal{B})$ and $DS(\mathcal{B}) \subseteq DS(\mathcal{C})$. 

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Degree Spectra and Conservative Extensions
Formally $\Sigma_n^c$-definable sets

Let $L$ be the language of $\mathfrak{A}$. The computable $\Sigma_n^c$ formulas in $L$ are defined inductively:

- A computable $\Sigma_0^c$ ($\Pi_0^c$) formula is a finitary quantifier-free formula in $L$.
- A computable $\Sigma_{n+1}^c$ formula $\Phi(\overline{x})$ is a disjunction of c.e. set of formulas of the form
  \[
  (\exists \overline{Y})\psi(\overline{X}, \overline{Y})
  \]
  $\psi$ is a finite conjunction of $\Sigma_n^c$ and $\Pi_n^c$ formulas
- $\Pi_{n+1}^c$ formulas are the negations of the $\Sigma_{n+1}^c$ formulas.
Consider $O = (\mathbb{N}; =)$ and $S = (\mathbb{N}; G_{\text{Succ}}; =)$, where $G_{\text{Succ}}$ is the graph of the successor function.

$$DS(O) = DS(S)$$

The $\Sigma^c_1(O)$ are all finite and co-finite sets of natural numbers. But all c.e. set are formally $\Sigma^c_1$ definable on $S$. So, the structure $S$ is more powerful than the $O$. 
Consider $\mathcal{O} = (\mathbb{N}; =)$ and $\mathcal{S} = (\mathbb{N}; G_{\text{Succ}}; =)$, where $G_{\text{Succ}}$ is the graph of the successor function.

$$DS(\mathcal{O}) = DS(\mathcal{S})$$

The $\Sigma_1^c(\mathcal{O})$ are all finite and co-finite sets of natural numbers. But all c.e. set are formally $\Sigma_1^c$ definable on $\mathcal{S}$. So, the structure $\mathcal{S}$ is more powerful than the $\mathcal{O}$. 
Definition. The pair \( \alpha = (f_\alpha, R_\alpha) \) is an enumeration of the set \( X \subseteq A \), if \( R_\alpha \) is a set of natural numbers, \( f_\alpha \) is a partial one-to-one mapping of \( \mathbb{N} \) onto \( X \) and \( \text{dom}(f_\alpha) = f_\alpha^{-1}(X) \) is c.e. in \( R_\alpha \). We denote this by \( X \leq \alpha \).

Definition. The pair \( \alpha = (f_\alpha, R_\alpha) \) is an enumeration of \( \mathcal{A} \) if \( \alpha \) is an enumeration of \( A \) and \( f_\alpha^{-1}(\mathcal{A}) \) is computable in \( R_\alpha \). We denote this by \( \mathcal{A} \leq \alpha \).

For an enumeration \( \alpha = (f_\alpha, R_\alpha) \) of \( \mathcal{A} \) we denote by \( \alpha^{(n)} = (f_\alpha, R_\alpha^{(n)}) \).
The Degree Spectrum of $\mathcal{A}$ is the set

$$DS(\mathcal{A}) = \{d_T(R_\alpha) \mid \mathcal{A} \leq \alpha\}.$$ 

**Theorem.** (Ash, Knigh, Manasse, Slaman, Chisholm)

For every set $X \subseteq A$,

$$X \in \Sigma^c_{n+1}(\mathcal{A}) \leftrightarrow (\forall \alpha)[\mathcal{A} \leq \alpha \rightarrow X \leq \alpha^{(n)}].$$
Let $\alpha = (f_\alpha, R_\alpha)$ and $\beta = (f_\beta, R_\beta)$ be enumerations of the structures $\mathfrak{A}$ and $\mathfrak{B}$ respectively. We write $\alpha \leq \beta$ if

(i) $R_\alpha \leq_T R_\beta$ and

(ii) the set $E(f_\alpha, f_\beta) = \{(x, y) \mid x \in \text{Dom}(f_\alpha) \& y \in \text{Dom}(f_\beta) \& f_\alpha(x) = f_\beta(y)\}$ is c.e. in $R_\beta$. 
**Definition.** Let $\mathcal{A}$ and $\mathcal{B}$ be countable structures, possibly with different signatures and $A \subseteq B$.

(i) $\mathcal{A} \leq_{k}^{n} \mathcal{B}$ iff for every enumeration $\beta$ of $\mathcal{B}$ there exists an enumeration $\alpha$ of $\mathcal{A}$ such that $\alpha^{(k)} \leq \beta^{(n)}$.

(ii) $\mathcal{A} \geq_{n}^{k} \mathcal{B}$ iff for every enumeration $\alpha$ of $\mathcal{A}$ there exists an enumeration $\beta$ of $\mathcal{B}$ such that $\beta^{(n)} \leq \alpha^{(k)}$.

(iii) $\mathcal{A} \equiv_{n}^{k} \mathcal{B}$ if $\mathcal{A} \leq_{n}^{k} \mathcal{B}$ and $\mathcal{A} \geq_{n}^{k} \mathcal{B}$. We shall say that $\mathcal{B}$ is a $(k, n)$-conservative extension of $\mathcal{A}$.

Note that the relation $\equiv_{n}^{k}$ is not symmetric.
Proposition. Let $\mathcal{A}$ and $\mathcal{B}$ be countable structures with $A \subseteq B$.

(i) If $\mathcal{A} \leq^k_n \mathcal{B}$ then $\text{DS}_n(\mathcal{B}) \subseteq \text{DS}_k(\mathcal{A})$;
(ii) If $\mathcal{A} \geq^k_n \mathcal{B}$ then $\text{DS}_k(\mathcal{A}) \subseteq \text{DS}_n(\mathcal{B})$;
(iii) If $\mathcal{A} \equiv^k_n \mathcal{B}$ then $\text{DS}_k(\mathcal{A}) = \text{DS}_n(\mathcal{B})$;

Corollary.

(i) $k = 1, n = 0$:
   If $\mathcal{A} \equiv^1_0 \mathcal{B}$ then $\text{DS}_1(\mathcal{A}) = \text{DS}(\mathcal{B})$.
(ii) $k = 0, n = 1$:
   If $\mathcal{A} \equiv^0_1 \mathcal{B}$ then $\text{DS}(\mathcal{A}) = \text{DS}_1(\mathcal{B})$. 

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Theorem. Let for $\mathcal{A}$ and $\mathcal{B} : A \subseteq B$. For all $k, n \in \mathbb{N}$,

(i) if $\mathcal{A} \leq^k_n \mathcal{B}$ then $(\forall X \subseteq A)[X \in \Sigma^c_{k+1}(\mathcal{A}) \rightarrow X \in \Sigma^c_{n+1}(\mathcal{B})]$;

(ii) if $\mathcal{A} \geq^k_n \mathcal{B}$ then $(\forall X \subseteq A)[X \in \Sigma^c_{n+1}(\mathcal{B}) \rightarrow X \in \Sigma^c_{k+1}(\mathcal{A})]$;

(iii) if $\mathcal{A} \equiv^k_n \mathcal{B}$ then $(\forall X \subseteq A)[X \in \Sigma^c_{k+1}(\mathcal{A}) \leftrightarrow X \in \Sigma^c_{n+1}(\mathcal{B})]$. 
The opposite direction is not always true:

Example.
Consider $\mathcal{O}_A = (A; =)$ and take $\mathcal{A} = \mathcal{B} = \mathcal{O}_A$.
For every natural number $n$,
$X \subseteq A$ is $\Sigma^c_n(\mathcal{O}_A)$ iff $X$ is a finite or co-finite subset of $A$.
Therefore $\Sigma^c_1(\mathcal{O}_A) = \Sigma^c_n(\mathcal{O}_A)$ and

$$(\forall n)(\forall X \subseteq A)[X \in \Sigma^c_{n+1}(\mathcal{O}_A) \rightarrow X \in \Sigma^c_1(\mathcal{O}_A)].$$

But $(\forall n)[\mathcal{O}_A \leq^n_0 \mathcal{O}_A]$ is evidently not true.
Let $\mathcal{A} = (A; P_1, \ldots, P_k)$ and $\bar{0} \notin A$.

Set $A_0 = A \cup \{\bar{0}\}$.

Let $\langle ., . \rangle$ be a pairing function s.t. none of the elements of $A$ is a pair and $A^*$ be the least set containing $A_0$ and closed under $\langle ., . \rangle$.

Let $0^* = \bar{0}$ and $(n + 1)^* = \langle \bar{0}, n^* \rangle$, $\mathbb{N}^* = \{n^* \mid n \in \mathbb{N}\}$.

The decoding functions: $L(\langle s, t \rangle) = s \& R(\langle s, t \rangle) = t$

$L(\bar{0}) = R(\bar{0}) = 0^*$ ($\forall t \in A)[L(t) = R(t) = 1^*]$. 
**Definition.** Moschovakis’ extension of $\mathcal{A}$ is the structure

$$\mathcal{A}^* = (A^*, P_1, \ldots, P_k, A_0, G_{\langle,\rangle}, G_L, G_R).$$

**Proposition.** $\mathcal{A} \equiv_n^n \mathcal{A}^*$ for every $n \in \mathbb{N}$.

**Proposition.** For every two structures $\mathcal{A}, \mathcal{B}$ with $A \subseteq B$ and natural numbers $n, k$

$\mathcal{A} \equiv_n^n \mathcal{B}$ iff $\mathcal{A}^* \equiv_n^n \mathcal{B}^*$. 
Theorem. (S. Vatev)

Let $\mathcal{A}$ and $\mathcal{B}$ be countable structures with $A^* \subseteq B$ and $k, n \in \mathbb{N}$. If $(\forall X \subseteq A^*) [X \in \Sigma^c_{k+1}(A^*) \rightarrow X \in \Sigma^c_{n+1}(B)]$ then $\mathcal{A} \leq_n^k \mathcal{B}$.

Corollary. For any two countable structures $\mathcal{A}$, $\mathcal{B}$ with $A \subseteq B$ and $n, k \in \mathbb{N}$,

$$\mathcal{A} \leq_n^k \mathcal{B} \iff (\forall X \subseteq A^*) [X \in \Sigma^c_{k+1}(A^*) \rightarrow X \in \Sigma^c_{n+1}(B^*)].$$
A new predicate $K_{2^1}$ (analogue of Kleene’s set).

For $e, x \in \mathbb{N}$ and finite part $\tau$, let

$$
\tau \models F_e(x) \iff x \in W_e^{\tau^{-1}(2^1)}
$$

$$
\tau \models \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \not\models F_e(x))
$$

$$
K_{2^1} = \{ \langle \delta, e, x \rangle : (\exists \tau \supseteq \delta)(\tau \models F_e(x)) \}.
$$

$\mathcal{A}' = (\mathcal{A}^*, K_{2^1})$.

Theorem. $\text{DS}_1(\mathcal{A}) = \text{DS}(\mathcal{A}')$.

Proposition. $\mathcal{A} \equiv_0^{1} \mathcal{A}'$. 
For every $e, x, n \in \mathbb{N}$ and for every finite part $\tau : \mathbb{N} \to A$, we define the forcing relations $\Vdash_n$:

\[
\begin{align*}
\tau \Vdash_0 F_e(x) & \iff x \in W_e^\tau^{-1}(\emptyset) \\
\tau \Vdash_{n+1} F_e(x) & \iff (\exists v)[\langle x, v \rangle \in W_e \land (\forall u \in D_v)[(u = \langle e_u, x_u, 1 \rangle \land \tau \Vdash_n F_{e_u}(x_u)) \lor (u = \langle e_u, x_u, 0 \rangle \land \tau \Vdash_n \neg F_{e_u}(x_u))]] \\
\tau \Vdash_n \neg F_e(x) & \iff (\forall \rho \supseteq \tau)(\rho \not\Vdash_n F_e(x)).
\end{align*}
\]
The set $K_{n}^{2\mathfrak{A}}$

**Definition.**

$$K_{n}^{2\mathfrak{A}} = \{ \langle \delta, e, x \rangle \mid (\exists \tau \supseteq \delta)[\tau \models_n F_{e}(x)] \}.$$

**Proposition. (S. Vatev)**

(i) $K_{n}^{2\mathfrak{A}} \in \Sigma_{n+1}^{c}(\mathfrak{A}^{*})$ and $A^{*} \setminus K_{n}^{2\mathfrak{A}} \in \Sigma_{n+2}^{c}(\mathfrak{A}^{*})$.

(ii) $K_{n}^{2\mathfrak{A}} \notin \Sigma_{n}^{c}(\mathfrak{A}^{*})$. 
Definition. For every natural number $n$, we define the $n$-th jump of the structure $\mathcal{A}$ in the following way:

$$\mathcal{A}^{(0)} = \mathcal{A} \text{ and } \mathcal{A}^{(n+1)} = (\mathcal{A}^*, K_n^{2n}).$$
Proposition. For every $\mathcal{A}$ and natural number $n$,

(i) $\mathcal{A} \equiv_0^n \mathcal{A}^{(n)}$;

(ii) $\mathcal{A}^{(n)} \leq_0^0 \mathcal{A}^{(n+1)}$ and $\mathcal{A}^{(n)} \not\equiv_0^0 \mathcal{A}^{(n+1)}$.

Since $\mathcal{A} \equiv_n^k \mathcal{B}$ implies $DS_k(\mathcal{A}) = DS_n(\mathcal{B})$, we get the following.

Corollary. For every $\mathcal{A}$, $DS(\mathcal{A}^{(n)}) = DS_n(\mathcal{A})$. 
The Jump Inversion Theorem

**Theorem.** Let $\mathcal{A}$ and $\mathcal{C}$ be countable structures and $DS(\mathcal{A}) \subseteq DS_1(\mathcal{C})$. There exists a structure $\mathcal{B} = \mathcal{A}^{\exists \forall} \oplus \mathcal{C}$ such that $DS(\mathcal{A}) = DS_1(\mathcal{B})$ and $DS(\mathcal{B}) \subseteq DS(\mathcal{C})$.

**Remark.** Similar results by:
- A. Montalban (2009) by different approach with complete set of $\Pi_n^c$ formulas.
- A. Stukachev (2009) for $\Sigma$ reducibility with Marker’s extentions.

Stukachev proves an analogue of this theorem for the semilattices of $\Sigma$-degrees of structures with arbitrary cardinalities.

**Theorem.** (Stukachev) Let $\mathcal{A}$ be a structure such that $0' \leq_{\Sigma} \mathcal{A}$. There exists a structure $\mathcal{B}$ such that $\mathcal{A} \equiv_{\Sigma} \mathcal{B}'$.

We can prove a similar to Stukachev’s result.
Proposition. If $O_A \leq_0^1 A$, then $A \equiv_1^0 A^\exists^A$.

Theorem. Let $O_A \leq_k^0 A$ for some $k \in \mathbb{N}$. There exists a structure $B = A^\exists^A$ such that $A \equiv_0^0 B^{(k)}$.

Remark. Note that $O_A \leq_0^k A$ iff the elements of $DS(A)$ are above $0^{(k)}$. 
Proposition. Let $\mathcal{O}_A \leq_k^0 \mathcal{A}$ for some $k \in \mathbb{N}$. There exists a structure $\mathcal{B}$ such that for every $n \in \mathbb{N}$, $\mathcal{A} \equiv^B_k \mathcal{B}^{(n)}$.

Corollary. Let $\mathcal{O}_A \leq_k^0 \mathcal{A}$ for some $k \in \mathbb{N}$. There exists a countable structure $\mathcal{B}$ such that

$$(\forall n \in \mathbb{N})(\forall X \subseteq A)[X \in \Sigma^c_{n+1}(\mathcal{A}) \iff X \in \Sigma^c_{k+1}(\mathcal{B}^{(n)})].$$

Corollary. If $\mathcal{O}_A \leq_k^0 \mathcal{A}$ for some $k \in \mathbb{N}$ then for each $n \in \mathbb{N}$, there is a structure $\mathcal{B}$ such that

$$(\forall X \subseteq A)[X \in \Sigma^c_{n+1}(\mathcal{A}) \iff X \in \Sigma^c_{k+1}(\mathcal{B})].$$
Some problems

- The definition of $\mathcal{A} \equiv^k_n \mathcal{B}$ is not symmetric since we suppose that $\mathcal{A} \subseteq \mathcal{B}$. How to define the similar relation more symmetric and for arbitrary $\mathcal{A}$ and $\mathcal{B}$?
- How to relativize the Jump Inversion Theorem for structures?
- The Jump inversion Theorem for structures for arbitrary constructive ordinal $\alpha$. 

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Degree Spectra and Conservative Extensions


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