Quasi-minimal degrees for degree spectra

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Definition. We say that $\Gamma : 2^\mathbb{N} \to 2^\mathbb{N}$ is an enumeration operator iff for some c.e. set $W_i$ for each $B \subseteq \mathbb{N}$

$$\Gamma(B) = \{ x | (\exists D)[\langle x, D \rangle \in W_i \& D \subseteq B] \}.$$ 

Definition. The set $A$ is enumeration reducible to the set $B$ ($A \leq_e B$), if $A = \Gamma(B)$ for some e-operator $\Gamma$. The enumeration degree of $A$ is $d_e(A) = \{ B \subseteq \mathbb{N} | A \equiv_e B \}$. The set of all enumeration degrees is denoted by $\mathcal{D}_e$.

- $0_e = d_e(\emptyset) = \{ W | W \text{ is c.e.} \}$.
- $d_e(A) \lor d_e(B) = d_e(A \oplus B)$.
- $\mathcal{D}_e = \langle \mathcal{D}_e; \leq; \oplus; 0_e \rangle$ is an upper semi-lattice with least element.
The enumeration reducibility

**Definition.** Given a set $A$, denote by $A^+ = A \oplus (\mathbb{N} \setminus A)$. A set $A$ is called *total* iff $A \equiv_e A^+$.

**Theorem.** For any sets $A$ and $B$:
1. $A$ is c.e. in $B$ iff $A \leq_e B^+$.
2. $A \leq_T B$ iff $A^+ \leq_e B^+$.

**Theorem.** ([Selman]) $a \leq_e b$ iff for all total $c$ ($b \leq_e c \Rightarrow a \leq_e c$).
The enumeration jump

**Definition.** For any set \( A \) let \( K_A = \{ \langle i, x \rangle | x \in \Gamma_i(A) \} \). Set \( A' = K_A^+ \).

- Let \( d_e(A)' = d_e(A') \).
- The enumeration jump is always a total degree and agrees with the Turing jump under the standard embedding \( \iota : D_T \to D_e \) by \( \iota(d_T(A)) = d_e(A^+) \).
- \( A \) is \( \Sigma_{n+1} \) if \( A \leq_e (B^+)^{(n)} \).

**Theorem.** [Soskova] For every \( x \in D_e \) there exists a total e-degree \( a \geq x \), such that \( a' = x' \).
Let \( \mathcal{A} = (A; R_1, \ldots, R_k) \) be a countable structure. An enumeration of \( \mathcal{A} \) is every one to one mapping of \( \mathbb{N} \) onto \( A \).

**Definition.** *The degree spectrum of \( \mathcal{A} \) is the set of all Turing degrees which computes the diagram of an isomorphic copy of \( \mathcal{A} \).*

Given an enumeration \( f \) of \( \mathcal{A} \) and a subset of \( B \) of \( A^a \), let

\[
f^{-1}(B) = \{ \langle x_1, \ldots, x_a \rangle \mid (f(x_1), \ldots, f(x_a)) \in B \}.
\]

\[
f^{-1}(\mathcal{A}) = f^{-1}(R_1)^+ \oplus \cdots \oplus f^{-1}(R_k)^+.
\]
Definition. The degree spectrum of $\mathcal{A}$ is the set

$$DS(\mathcal{A}) = \{ a \mid a \in \mathcal{D}_T \& (\exists f)(d_T(f^{-1}(\mathcal{A})) \leq_T a) \}.$$ 

If $a$ is the least element of $DS(\mathcal{A})$ then we call $a$ the degree of $\mathcal{A}$. 
Definition. [Soskov] The co-spectrum of $\mathcal{A}$ is the set

$$CS(\mathcal{A}) = \{ b : b \in D_e \land (\forall a \in DS(\mathcal{A}))(b \leq_e a) \}.$$ 

If $a$ is the greatest element of $CS(\mathcal{A})$ then we call $a$ the co-degree of $\mathcal{A}$.

Soskov proved that every countable ideal of enumeration degrees is a co-spectrum of a structure.
The admissible in $\mathbb{A}$ sets

**Definition.** A set $B$ of natural numbers is admissible in $\mathbb{A}$ if for every enumeration $f$ of $\mathbb{A}$, $B \leq_e f^{-1}(\mathbb{A})$.

Clearly $a \in CS(\mathbb{A})$ iff $a = d_e(B)$ for some admissible in $\mathbb{A}$ set $B$.

Every finite one-to-one mapping of $\mathbb{N}$ into $A$ is called a finite part. For every finite part $\tau$ and natural numbers $e, x$, let

$$\tau \vdash F_e(x) \iff x \in \Gamma_e(\tau^{-1}(\mathbb{A}))$$

and

$$\tau \vdash \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \not\vdash F_e(x)).$$

**Definition.** An enumeration $f$ of $\mathbb{A}$ is generic if for every $e, x \in \mathbb{N}$, there exists a $\tau \subseteq f$ s.t. $\tau \vdash F_e(x) \lor \tau \vdash \neg F_e(x)$. 
Definition. A set $B$ of natural numbers is admissible in $\mathcal{A}$ if for every enumeration $f$ of $\mathcal{A}$, $B \leq_e f^{-1}(\mathcal{A})$.

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Every finite one-to-one mapping of $\mathbb{N}$ into $\mathcal{A}$ is called a finite part. For every finite part $\tau$ and natural numbers $e, x$, let

\[
\tau \models F_e(x) \iff x \in \Gamma_e(\tau^{-1}(\mathcal{A})) \text{ and } \\
\tau \models \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \not\models F_e(x)).
\]

Definition. An enumeration $f$ of $\mathcal{A}$ is generic if for every $e, x \in \mathbb{N}$, there exists a $\tau \subseteq f$ s.t. $\tau \models F_e(x) \lor \tau \models \neg F_e(x)$. 
Definition. A set $B$ of natural numbers is forcing definable in the structure $\mathbb{A}$ iff there exist a finite part $\delta$ and a natural number $e$ s.t.

$$B = \{x | (\exists \tau \supseteq \delta)(\tau \models F_e(x))\}.$$ 

Denote by $D(\mathbb{A})$ the diagram of $\mathbb{A}$.

Proposition. Let $\{B_i\}_{i \in \mathbb{N}}$ be subsets of $\mathbb{N}$ be not forcing definable on $\mathbb{A}$. There exists a 1-generic enumeration $f$ of $\mathbb{A}$ satisfying the following conditions:

1. $f \leq_e D(\mathbb{A})'$.
2. $f^{-1}(\mathbb{A})' \leq_e f \oplus D(\mathbb{A})'$.
3. $B_i \not\leq_e f^{-1}(\mathbb{A})$ for every $i \in \mathbb{N}$.
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3. $B_i \not\leq_e f^{-1}(\mathcal{A})$ for every $i \in \mathbb{N}$.
Definition. A $\Sigma^c_1$ formula with free variables among $W_1$, $\ldots$, $W_r$ is a c.e. disjunction of existential formulae of the form $\exists Y_1 \ldots \exists Y_k \theta(\bar{Y}, \bar{W})$, where $\theta$ is a finite conjunction of atomic and negated atomic formulae.

Definition. A set $B \subseteq \mathbb{N}$ is formally definable on $\mathcal{A}$ if there exists a recursive function $\gamma(x)$, such that $\bigvee_{x \in \mathbb{N}} \Phi_{\gamma(x)}$ is a $\Sigma^c_1$ formula with free variables among $W_1$, $\ldots$, $W_r$ and elements $t_1$, $\ldots$, $t_r$ of $A$ such that the following equivalence holds:

$$x \in B \iff \mathcal{A} \models \Phi_{\gamma(x)}(W_1/t_1, \ldots, W_r/t_r).$$
Theorem. Let $B \subseteq \mathbb{N}$. Then

1. $B$ is admissible in $\mathcal{A}$ ($d_e(B) \in CS(\mathcal{A})$) iff
2. $B$ is forcing definable on $\mathcal{A}$ iff
3. $B$ is formally definable on $\mathcal{A}$.

Corollary. If $\mathcal{B}$ is an isomorphic structure of $\mathcal{A}$ then a set $X \subseteq \mathbb{N}$ is forcing definable on $\mathcal{A}$ if and only if $X$ is forcing definable on $\mathcal{B}$.
Definition. The $n$th jump spectrum of $\mathcal{A}$ is the set
\[ DS_n(\mathcal{A}) = \{ a^{(n)} \mid a \in DS(\mathcal{A}) \}. \]

Definition. The $n$th jump co-spectrum $CS_n(\mathcal{A})$ of $\mathcal{A}$ is the set
\[ CS_n(\mathcal{A}) = \{ b \mid b \in D_e \& (\forall a \in DS_n(\mathcal{A}))(b \leq a) \}. \]
**Definition.** Let $B \subseteq A$ be sets of degrees. Then $B$ is a base of $A$ if

$$(\forall a \in A)(\exists b \in B)(b \leq a).$$

**Theorem.** A structure $\mathfrak{A}$ has a degree if and only if $DS(\mathfrak{A})$ has a countable base.

Suppose that the sequence of e-degrees $\{b_i\}_i$ is a base for $DS(\mathfrak{A})$. Assume that no $b_i$ is an e-degree of $\mathfrak{A}$. Then for every $i$, $b_i \notin CS(\mathfrak{A})$.

Let $B_i \in b_i$ for every $i \in \mathbb{N}$. Then all the sets $B_i$ have no forcing normal form.

We can construct a generic enumeration $f$ of $\mathfrak{A}$, omitting all $B_i$, i.e. $B_i \nleq_e f^{-1}(\mathfrak{A})$.

This contradicts with fact that $\{b_i\}_i$ is a base for $DS(\mathfrak{A})$. 
An upwards closed set of degrees which is not a degree spectra of a structure
The minimal pair theorem

**Theorem.** [Soskov] There exist \( f, g \in DS(\mathcal{A}) \) such that

\[
(\forall b \in D_e)(b \leq f \ \& \ b \leq g \Rightarrow b \in CS(\mathcal{A})).
\]
The quasi-minimal degree

**Definition.** [Medvedev (1955)] An e-degree \( a \) is said to be quasi-minimal if

- \( a \neq 0_e \);
- \((\forall \text{ total } b)[b \leq a \rightarrow b = 0_e]\). 

**Definition.** [Slaman, Sorbi] Given any \( I \subseteq D_e \), we say that an e-degree \( a \) is \( I \)-quasi-minimal if

- \((\forall c \in I)[c < a]\);
- \((\forall \text{ total } c)[c \leq a \iff (\exists b \in I)[c \leq b]]\).
**Definition.** Let $\mathcal{A}$ be a set of enumeration degrees. The degree $q$ is quasi-minimal with respect to $\mathcal{A}$ if:

- $q \notin \text{co}(\mathcal{A})$.
- If $a$ is total and $a \geq q$, then $a \in \mathcal{A}$.
- If $a$ is total and $a \leq q$, then $a \in \text{co}(\mathcal{A})$.

From Selman’s theorem it follows that if $q$ is quasi-minimal with respect to $\mathcal{A}$, then $q$ is an upper bound of $\text{co}(\mathcal{A})$.

**Theorem.** [Soskov] For every structure $\mathfrak{A}$ there exists a quasi-minimal with respect to $DS(\mathfrak{A})$ degree.
Let $\bot \not\in A$.

**Definition.** A *partial finite part* is a finite mapping of $\mathbb{N}$ into $A \cup \{\bot\}$.

Let $\tau$ be a partial finite part and let $f$ be a partial enumeration, by $\tau \subseteq f$ we denote that for all $x$ in $\text{dom}(\tau)$ either $\tau(x) = \bot$ and $f(x)$ is not defined or $\tau(x) \in A$ and $f(x) = \tau(x)$.

**Definition.** A subset $B$ of $\mathbb{N}$ is *partially forcing definable* on $\mathcal{A}$ if there exist an $e \in \mathbb{N}$ and a partial finite part $\delta$ such that for all natural numbers $x$,

$$x \in B \iff (\exists \tau \supseteq \delta)(\tau \vdash F_e(x)).$$

**Lemma.** Let $B \subseteq \mathbb{N}$ be partially forcing definable on $\mathcal{A}$. Then $d_e(B) \in CS(\mathcal{A})$. 

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The quasi-minimal degree

**Proposition.**

1. For every partial generic $f$, $f^{-1}(A) \nleq_e D(A)$. Hence $d_e(f^{-1}(A)) \notin CS(A)$.

2. There exists a partial generic enumeration $f \leq_e D(A)'$ such that $f^{-1}(A) \leq_e D(A)'$.

3. If $B \leq_e f^{-1}(A)$ for all partial generic enumerations $f$, then $B$ is partially forcing definable on $A$.

**Theorem.** Let $f$ be a partial generic enumeration of $A$. Then $d_e(f^{-1}(A))$ is quasi-minimal with respect to $DS(A)$.

**Corollary.** [Slaman and Sorbi] Let $I$ be a countable ideal of enumeration degrees. There exists an enumeration degree $q$ s.t.

1. If $a \in I$ then $a <_e q$.

2. If $a$ is total and $a \leq_e q$ then $a \in I$. 
Proposition. For every countable structure \( \mathcal{A} \) there exist continuum many quasi-minimal degrees with respect to \( DS(\mathcal{A}) \).

Suppose that all quasi-minimal degrees with respect to \( DS(\mathcal{A}) \) are \( q_0, q_1, \ldots, q_n, \ldots \) and let \( X_i \in q_i, \) for all \( i \in \mathbb{N} \). Then all \( q_i \) are not in \( CS(\mathcal{A}) \) and hence every \( X_i \) is not forcing definable on \( \mathcal{A} \). Then we could build a partial generic enumeration \( f \) of \( \mathcal{A} \) such that \( X_i \not\leq_e f^{-1}(\mathcal{A}) \). Thus \( d_e(f^{-1}(\mathcal{A})) \) is quasi-minimal with respect to \( DS(\mathcal{A}) \) and not in \( \{q_i\} \).
**Theorem.** [Ganchev] Let $B \subseteq \mathbb{N}$ and $Q$ be a total set such that $B' \leq Q$. There exists a partial set $F$ called quasi-minimal over $B$ with the following properties:

1. $B < F$;
2. $F' \equiv Q$.
3. For every total $X \leq F$ we have that $X \leq B$.

**Lemma.** There exists a partial 1-generic enumeration $f$ of $\mathcal{A}$, such that $f^{-1}(\mathcal{A})' \leq D(\mathcal{A})'$ and $\langle f \rangle \leq D(\mathcal{A})'$.

**Theorem.** The first jump spectrum of every structure $\mathcal{A}$ consists exactly of the enumeration jumps of the quasi-minimal degrees.

**Corollary.** [McEvoy] For every total $e$-degree $a \geq_e 0'_e$ there is a quasi-minimal degree $q$ with $q' = a$. 

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Proof.

- Let $g^{-1}(\mathcal{A})' \in DS_1(\mathcal{A})$. Denote by $B = g^{-1}(\mathcal{A})$.
- $\mathcal{B} = (\mathbb{N}, g^{-1}(R_1), \ldots, g^{-1}(R_n))$.
- There is a partial 1-generic enumeration $f$ of $\mathcal{B}$ such that $f^{-1}(\mathcal{B})' \leq B'$.
- There is a partial set $F$, such that $f^{-1}(\mathcal{B}) < F$, $F' \equiv B'$, $(\forall$ total $X)(X \leq F \Rightarrow X \leq f^{-1}(\mathcal{B}))$.
- Set $q = d_e(F)$.
- $q$ is a quasi-minimal with respect to $DS(\mathcal{A})$. 
Proposition. [Jockusch] For every total e-degree \( a \) there are quasi-minimal degrees \( p \) and \( q \) such that \( a = p \vee q \).

Theorem. For every element \( a \) of the jump spectrum of a structure \( \mathcal{A} \) there exists quasi-minimal with respect to \( DS(\mathcal{A}) \) degrees \( p \) and \( q \) such that \( a = p \vee q \).
Suppose that \( \mathcal{A} = (\mathbb{N}; R_1, \ldots, R_n) \).
Denote by \( \Delta \) the set of all finite parts.
For each \( \tau \in \Delta \) and \( x \in \mathbb{N} \) by \( \tau \ast x \) we denote an extension of \( \tau \)
such that \( \tau \ast x(\text{lh}(\tau)) = x \).
Let \( f : \Delta \rightarrow \Delta \) and \( \{y_n\}_n \) be a sequence of natural numbers.
If \( \tau_0 = \emptyset \), \( \tau_{n+1} = f(\tau_n \ast y_i) \), then we denote by \( f(\{y_n\}_n) = \bigcup_n \tau_n \).
Let \( P \) be a set of enumerations of \( \mathcal{A} \).

**Lemma.** [Ganchev] If \( f \) is computable in the total set \( Q \) and such
that for every sequence \( \{y_n\}_n \) computable in \( Q \), \( f(\{y_n\}_n) \in P \), then
there exist enumerations \( g, h \in P \) of \( \mathcal{A} \) such that \( Q \equiv_e \langle g \rangle \oplus \langle h \rangle \).
Let \( q \) be an enumeration of \( Q \) such that \( \langle q \rangle \leq_e Q \). We construct two sequences of finite parts \( \{\tau_n\}_n \) and \( \{\sigma_n\}_n \) by the following rule:

1. \( \tau_0 = \sigma_0 = \emptyset \);
2. \( y_n = \langle lh(\sigma_n), q(2n) \rangle \);
3. \( \tau_{n+1} = f(\tau_n \ast y_n) \);
4. \( z_n = \langle lh(\tau_n), q(2n + 1) \rangle \);
5. \( \sigma_{n+1} = f(\sigma_n \ast z_n) \).

Define \( g = f(\{y_n\}_n) \) and \( h = f(\{z_n\}_n) \).
A method of splitting a total set

**Theorem.** For every element \( a \) of the jump spectrum of a structure \( \mathcal{A} \) there exists quasi-minimal with respect to \( DS(\mathcal{A}) \) degrees \( p \) and \( q \) such that \( a = p \lor q \).

**Proof.**

- Let \( a = d_T(g^{-1}(\mathcal{A}')') \in DS_1(\mathcal{A}) \). Denote by \( B = g^{-1}(\mathcal{A}) \).
- \( \mathcal{B} = (\mathbb{N}, g^{-1}(R_1), \ldots, g^{-1}(R_n)) \).
- Construct a partial 1-generic enumeration \( f \) of \( \mathcal{B} \) such that \( f^{-1}(\mathcal{B}')' \leq B' \).
- Let \( P \) be the class of all partial generic enumerations \( g \) of \( \mathcal{A} \), s.t. \( \langle g \rangle \) is quasi-minimal over \( f^{-1}(\mathcal{B}) \), i.e \( f^{-1}(\mathcal{B}) \prec \langle g \rangle \), \( \langle g \rangle' \equiv B' \), \( (\forall \text{ total } X)(X \leq \langle g \rangle \Rightarrow X \leq f^{-1}(\mathcal{B})) \).
- Applying the lemma there are \( p = d_e(\langle g \rangle) \) and \( q = d_e(\langle h \rangle) \) are quasi-minimal over \( f^{-1}(\mathcal{B}) \) and hence quasi-minimal for \( DS(\mathcal{A}) \) and \( a = p \lor q \).


Thank you!