The degree spectrum $DS(\mathfrak{A})$ of a countable structure $\mathfrak{A}$ we define to be the set of all enumeration degrees generated by the presentations of $\mathfrak{A}$ on the natural numbers. The co-spectrum of $\mathfrak{A}$ is the set of all lower bounds of $DS(\mathfrak{A})$. In this paper we consider the connections between degree spectra and their co-spectra. We present variants of Selman’s theorem, the minimal pair theorem and quasi-minimal degree theorem for degree spectra. A structure $\mathfrak{A}$ is called total if all presentations of $\mathfrak{A}$ are total sets. For every total structure $\mathfrak{A}$ the set $DS(\mathfrak{A})$ contains only total degrees. We prove that if $DS(\mathfrak{A})$ consists of total degrees above $0'$, then there exists a total structure $\mathfrak{B}$ such that $DS(\mathfrak{B}) = DS(\mathfrak{A})$. We prove a generalized Jump inversion theorem for degree spectra. As an application we receive structures with interesting degree spectra.

1. Preliminaries

Definition 1.1. (Friedberg and Rogers, 1959) We say that $\Psi : 2^\omega \rightarrow 2^\omega$ is an enumeration operator iff for some c.e. set $W_i$ and for each $B \subseteq \omega$

$\Psi(B) = \{x| (\exists D)[(x, D) \in W_i \& D \subseteq B]\}$

Here $\{W_i\}_{i \in \omega}$, $\{D_i\}_{i \in \omega}$ are the standard listings of computably enumerable sets and the finite sets of numbers.

For any sets $A$ and $B$ define $A$ is enumeration reducible to $B$, written $A \leq_e B$, by $A = \Psi(B)$ for some e-operator $\Psi$. Let $A^+ = A \oplus (\omega \setminus A)$. The connection with the Turing reducibility is shown by $A \leq_T B$ iff $A^+ \leq_e B^+$. Let $E_A = \{(i, x)| x \in \Psi_i(A)\}$. The set $J_e(A) = E_A^+$ is called the enumeration jump of $A$ [1, 3]. The enumeration jump $J_e$ is monotone and agrees with the Turing jump $J_T$ in the following sense: $J_T(A)^+ \equiv_e J_e(A^+)$. Let $d_e(A) = \{B \subseteq \omega| A \equiv_e B\}$ and $d_e(A) \leq_e d_e(B) \iff A \leq_e B$.

A set $A$ is called total iff $A \equiv_e A^+$. The Rogers embedding $\iota : D_T \rightarrow D_e$ is defined by $\iota(d_T(A)) = d_e(A^+)$. The enumeration degrees in the range of $\iota$ are called total.

Let $d_e(A)' = d_e(J_e(A))$. The jump is always total and agrees with the Turing jump under the embedding $\iota$.

2. Degree Spectra

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k)$ be a denumerable structure. Enumeration of $\mathfrak{A}$ is every total surjective mapping of $\mathbb{N}$ onto $\mathbb{N}$.

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Given an enumeration $f$ of $\mathcal{A}$ and a subset of $A$ of $\mathbb{N}^a$, let

$$f^{-1}(A) = \{ \langle x_1, \ldots, x_a \rangle : (f(x_1), \ldots, f(x_a)) \in A \}.$$ 

Set $f^{-1}(\mathcal{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq)$.

**Definition 2.1.** (Richter [5]) The Turing Degree Spectrum of $\mathcal{A}$ is the set

$$DS_T(\mathcal{A}) = \{ d_T(f^{-1}(\mathcal{A})) : f \text{ is an one to one enumeration of } \mathcal{A} \}.$$ 

If $a$ is the least element of $DS_T(\mathcal{A})$, then $a$ is called the degree of $\mathcal{A}$.

**Definition 2.2.** [7] The e-Degree Spectrum of $\mathcal{A}$ is the set

$$DS(\mathcal{A}) = \{ d_e(f^{-1}(\mathcal{A})) : f \text{ is an enumeration of } \mathcal{A} \}.$$ 

If $a$ is the least element of $DS(\mathcal{A})$, then $a$ is called the e-degree of $\mathcal{A}$.

The e-degree spectrum is closed upwards: if $a \in DS(\mathcal{A}), b$ is a total e-degree and $a \leq_e b$ then $b \in DS(\mathcal{A})$.

**Definition 2.3.** The structure $\mathcal{A}$ is called total if for every enumeration $f$ of $\mathcal{A}$ the set $f^{-1}(\mathcal{A})$ is total.

If $\mathcal{A}$ is a total structure then $DS(\mathcal{A}) = \iota(DS_T(\mathcal{A}))$.

Given a structure $\mathcal{A} = (\mathbb{N}, R_1, \ldots, R_k)$, for every $j$ denote by $R^+_j$ the complement of $R_j$ and let $\mathcal{A}^+ = (\mathbb{N}, R_1, \ldots, R_k, R^+_1, \ldots, R^+_k)$. The following are true:

1. $\iota(DS_T(\mathcal{A})) = DS(\mathcal{A}^+)$. 
2. If $\mathcal{A}$ is total then $DS(\mathcal{A}) = DS(\mathcal{A}^+)$. 

Clearly if $\mathcal{A}$ is a total structure then $DS(\mathcal{A})$ consists of total degrees. The vice versa is not always true.

**Example.** Let $K$ be the Kleene’s set and $\mathcal{A} = (\mathbb{N}; G_S, K)$, where $G_S$ is the graph of the successor function. Then $DS(\mathcal{A})$ consists of all total degrees. On the other hand if $f = \lambda x.x$, then $f^{-1}(\mathcal{A})$ is an c.e. set. Hence $K \leq_e f^{-1}(\mathcal{A})$. Clearly $K \leq_e (f^{-1}(\mathcal{A}))^+$. So $f^{-1}(\mathcal{A})$ is not total.

The question here is: if $DS(\mathcal{A})$ consists of total degrees do there exists a total structure $\mathcal{B}$ s.t. $DS(\mathcal{A}) = DS(\mathcal{B})$? We will give a positive answer when all elements of $DS(\mathcal{A})$ are total above $\emptyset'$.

**Definition 2.4.** Let $A$ be a nonempty set of enumeration degrees the co-set of $A$ is the set $co(A)$ of all lower bounds of $A$. Namely

$$co(A) = \{ b : b \in D_e \land (\forall a \in A)(b \leq_a a) \}.$$ 

**Example.** Fix $a \in D_e$ and set $A_a = \{ b \in D_e : a \leq_e b \}$. Then $co(A_a) = \{ b \in D_e : b \leq_e a \}$.

**Definition 2.5.** The co-spectrum of the structure $\mathcal{A}$ is called the set $CS(\mathcal{A}) = co(DS(\mathcal{A}))$.

If $a$ is the greatest element of $CS(\mathcal{A})$ then call $a$ the co-degree of $\mathcal{A}$. If $\mathcal{A}$ has a degree $a$ then $a$ is also the co-degree of $\mathcal{A}$. The vice versa is not always true.

We will give a normal form of the elements of the co-spectrum $CS(\mathcal{A})$. A set $A$ of natural numbers is admissible in $\mathcal{A}$ if for every enumeration $f$ of $\mathcal{A}$, $A \leq_e f^{-1}(\mathcal{A})$. Clearly $a \in CS(\mathcal{A})$ iff $a = d_e(A)$ for some admissible set $A$. 
Every finite mapping of \( \mathbb{N} \) into \( \mathbb{N} \) is called \( \text{finite part} \). For every finite part \( \tau \) and natural numbers \( e, x \), let
\[
\tau \vDash F_e(x) \iff x \in \Psi_e(\tau^{-1}(\mathfrak{A})) \text{ and } \\
\tau \vDash \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \not\vDash F_e(x)).
\]
Given an enumeration \( f \) of \( \mathfrak{A}, e, x \in \mathbb{N} \), set
\[
f \models F_e(x) \iff x \in \Psi_e(f^{-1}(\mathfrak{A})).
\]
An enumeration \( f \) is \emph{generic} if for every \( e, x \in \mathbb{N} \), there exists a \( \tau \subseteq f \) s.t. \( \tau \vdash F_e(x) \lor \tau \vdash \neg F_e(x) \).

A set \( A \) of natural numbers is \emph{forcing definable in the structure \( \mathfrak{A} \) iff there exist finite part \( \delta \) and natural number \( e \) s.t.}
\[
A = \{ x | (\exists \tau \supseteq \delta)(\tau \vdash F_e(x)) \}.
\]

**Theorem 2.1.** Let \( A \subseteq \mathbb{N} \) and \( d_e(B) \in DS(\mathfrak{A}) \). Then the following are equivalent:

1. \( A \) is admissible.
2. \( A \leq_e f^{-1}(\mathfrak{A}) \) for all generic enumerations \( f \) of \( \mathfrak{A} \).
3. \( A \leq_e f^{-1}(\mathfrak{A}) \) for all generic enumerations \( f \) of \( \mathfrak{A} \) s.t. \( (f^{-1}(\mathfrak{A}))' \equiv_e B' \).
4. \( A \) is forcing definable.

**Example.** (Richter 1981, [5]) Let \( \mathfrak{A} = (\mathbb{N}; <) \) be a linear ordering. Then \( DS(\mathfrak{A}) \) contains a minimal pair of degrees and hence \( CS(\mathfrak{A}) = \{ 0_e \} \). Clearly \( 0_e \) is the co-degree of \( \mathfrak{A} \). Therefore if \( \mathfrak{A} \) has a degree \( a \), then \( a = 0_e \).

**Definition 2.6.** Let \( n \geq 0 \). The \( n \)-th jump spectrum of a structure \( \mathfrak{A} \) is defined by \( DS_n(\mathfrak{A}) = DS(\mathfrak{A}) \backslash DS_{n-1}(\mathfrak{A}) \). The co-degree of \( \mathfrak{A} \) is \( 0'_e \).

**Example.** (Knight 1986, [2]) Consider again a linear ordering \( \mathfrak{A} \). Then \( CS_1(\mathfrak{A}) \) consists of all \( \Sigma_2^0 \) sets. The co-degree of \( \mathfrak{A} \) is \( 0'_e \).

**Example.** (Slaman 1998, Whener 1998) There exists an \( \mathfrak{A} \) s.t.
\[
DS(\mathfrak{A}) = \{ a : a \text{ is total and } 0_e < a \}.
\]
Clearly the structure \( \mathfrak{A} \) has co-degree \( 0_e \) but has not a degree.

**Example.** (based on Coles, Dawney, Slaman - 1998) Let \( G \) be a torsion free Abelian group of rank 1, i.e. \( G \) is a subgroup of \( Q \). Let \( a \neq 0 \in G \) and let \( p \) be a prime number.

Let \( h_p(a) \) be the greatest \( k \) s.t. \( (\exists x \in G)(p^k.x = a) \). Let
\[
\chi(a) = (h_{p_0}(a), h_{p_1}(a), \ldots) \text{ and } \\
S_n = \{ (i,j) : j \leq \text{the } i\text{-th member of } \chi(a) \}.
\]
For \( a,b \neq 0 \in G, S_a \equiv_e S_b \).

Set \( s_G = d_e(S_a) \). Then \( DS(G) = \{ b : b \text{ is total and } s_G \leq_e b \} \).

- The co-degree of \( G \) is \( s_G \).
- \( G \) has a degree iff \( s_G \) is total
- If \( 1 \leq n \), then \( s^{(n)}_G \) is the \( n \)-th jump degree of \( G \).

For every \( d \in D_e \) there exists a \( G, \text{s.t. } s_G = d \). Hence every principle ideal of enumeration degrees is \( CS(G) \) for some \( G \).

We can represent every countable non-principle countable ideal as co-spectra.
Example. Let $B_0, \ldots, B_n, \ldots$ be a sequence of sets of natural numbers. Set $A = (\mathbb{N}; f; \sigma)$,

$$f(\langle i, n \rangle) = \langle i + 1, n \rangle;$$

$$\sigma = \{\langle i, n \rangle : n = 2k + 1 \lor n = 2k \land i \in B_k\}.$$ 

Then $CS(A) = I(d_e(B_0), \ldots, d_e(B_n), \ldots)$

Definition 2.7. Consider a subset $A$ of $D_e$. Say that $A$ is upwards closed if for every $a \in A$ all total degrees greater than $a$ are contained in $A$.

Let $A$ be an upwards closed set of degrees. Note that if $B \subseteq A$ then $co(A) \subseteq co(B)$. Selman [6] proved that if $A_t = \{a : a \in A \land a$ is total} then $co(A) = co(A_t)$.

Proposition 2.1 (Selman’s Theorem for Degree Spectra). Let $b$ be an arbitrary enumeration degree and $n > 0$. Set $A_{b,n} = \{a : a \in A \land b \leq e^{a(n)}\}$. Then $co(A) = co(A_{b,n})$.

If $1 \leq n$ and $c \in DS_n(A)$ then $CS(A) = co(\{b \in DS(A) : b^{(n)} = c\})$.

Example. (Upwards closed set for which the Theorem is not true) Let $B \subseteq A$ and $A \nsubseteq B'$. Denote by $D = \{a : d_e(A) \leq_e a\} \cup \{a : d_e(B) \leq_e a\}$.

Set $A = \{a : a \in D \land a' = d_e(B)\}$.

• $d_e(B)$ is the least element of $A$ and hence $d_e(B) \in co(A)$.
• $d_e(B) \nleq d_e(A)$ and hence $d_e(B) \notin co(D)$.

Theorem 2.2 (The minimal pair theorem). Let $c \in DS_2(\mathfrak{A})$. There exist $f, g \in DS(\mathfrak{A})$ s.t. $f, g$ are total, $f'' = g'' = c$ and $CS(\mathfrak{A}) = co(\{f, g\})$.

Notice that for every enumeration degree $a$ there exists a structure $\mathfrak{A}_a$ s.t. $DS(\mathfrak{A}) = \{x \in D_T : a \leq_e x\}$. As a corollary we receive

Corollary 2.1. (Rozinas) For every $b \in D_e$ there exist total $f, g$ below $b''$ which are a minimal pair over $b$.

The next example shows that not every upwards closed set of enumeration degrees has a minimal pair:

[Diagram of the minimal pair theorem]
Definition 2.8. Let $\mathcal{A}$ be a set of enumeration degrees. The degree $q$ is quasi-minimal with respect to $\mathcal{A}$ if:

- $q \not\in \text{co}(\mathcal{A})$.
- If $a$ is total and $a \geq q$, then $a \in \mathcal{A}$.
- If $a$ is total and $a \leq q$, then $a \in \text{co}(\mathcal{A})$.

If $q$ is quasi-minimal with respect to $\mathcal{A}$, then $q$ is an upper bound of $\text{co}(\mathcal{A})$.

Theorem 2.3. For every structure $\mathcal{A}$ there exists a quasi-minimal with respect to $\text{DS}(\mathcal{A})$ degree.

Corollary 2.2. (Slaman and Sorbi) Let $I$ be a countable ideal of enumeration degrees. There exist an enumeration degree $q$ s.t. 

1. If $a \in I$ then $a < e q$.
2. If $a$ is total and $a \leq e q$ then $a \in I$.

Let $B \subseteq A$ be sets of degrees. Then $B$ is a base of $\mathcal{A}$ if $(\forall a \in A)(\exists b \in B)(b \leq a)$.

Theorem 2.4. Let $A$ be an upwards closed set of degrees possessing a quasi-minimal degree. Suppose that there exists a countable base $B$ of $A$ such that all elements of $B$ are total. Then $A$ has a least element.

As a corollary we have that a total structure $\mathfrak{A}$ has a degree if and only if $\text{DS}(\mathfrak{A})$ has a countable base.

If we consider the set of two incomparable degrees and the cones of all total degrees over them then this set is an example of a upwards closed set which is not a degree spectrum, since it has a countable base but it has no degree.

3. Jump spectra

Definition 3.1. The $n$-th jump spectrum of a structure $\mathfrak{A}$ is the set

$$\text{DS}_n(\mathfrak{A}) = \{a^{(n)} \mid a \in \text{DS}(\mathfrak{A})\}.$$ 

If $a$ is the least element of $\text{DS}_n(\mathfrak{A})$ then $a$ is called $n$-th jump degree of $\mathfrak{A}$. For every $\mathfrak{A}$, $\text{DS}_1(\mathfrak{A}) \subseteq \text{DS}(\mathfrak{A})$. It is not known if for every $\mathfrak{A}$, $\text{DS}_1(\mathfrak{A}) \subset \text{DS}(\mathfrak{A})$. Probably the answer is "no".

We will show that every jump spectrum is spectrum of a total structure. Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_n)$. Consider a new element $\bar{0} \notin \mathbb{N}$. Denote by $\mathbb{N}_0 = \mathbb{N} \cup \{\bar{0}\}$. Let $\langle .. \rangle$ be a pairing function s.t. none of the elements of $\mathbb{N}_0$ is a pair and $\mathbb{N}^*$ be the least set containing $\mathbb{N}_0$ and closed under $\langle .. \rangle$.

Definition 3.2. Moschovakis’ extension [4] of $\mathfrak{A}$ is the structure

$$\mathfrak{A}^* = (\mathbb{N}^*, R_1, \ldots, R_n, \mathbb{N}_0, G_{\langle .. \rangle}).$$

It is easy to see that $\text{DS}(\mathfrak{A}) = \text{DS}(\mathfrak{A}^*)$.

Let $K_\mathfrak{A} = \{\langle \delta, e, x \rangle : (\exists \tau \supseteq \delta)(\tau \models F_e(x))\}$. And $\mathfrak{A}' = (\mathfrak{A}^*, K_\mathfrak{A}, \mathbb{N}^* \setminus K_\mathfrak{A})$.

Theorem 3.1. The structure $\mathfrak{A}'$ is total and $\text{DS}_1(\mathfrak{A}) = \text{DS}(\mathfrak{A}')$.

We will present an analogue of the Jump Inversion Theorem for degree spectra. Consider two structures $\mathfrak{A}$ and $\mathfrak{B}$. Suppose that

$$\text{DS}(\mathfrak{B})_t = \{a \mid a \in \text{DS}(\mathfrak{B}) \text{ and } a \text{ is total} \} \subseteq \text{DS}_1(\mathfrak{A}).$$
Theorem 3.2. There exists a structure $\mathcal{C}$ s.t. $DS(\mathcal{C}) \subseteq DS(\mathfrak{A})$ and $DS_1(\mathcal{C}) = DS(\mathfrak{A})$.

Corollary 3.1. Let $DS(\mathfrak{B}) \subseteq DS(\mathfrak{A})$. Then there exists a structure $\mathcal{C}$ s.t. $DS(\mathcal{C}) \subseteq DS(\mathfrak{B})$ and $DS(\mathfrak{B}) = DS(\mathcal{C})$.

Corollary 3.2. Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $0'$. Then there exists a total structure $\mathcal{C}'$ such that $DS(\mathfrak{B}) = DS(\mathcal{C}')$.

Theorem 3.3. Let $n \geq 1$. Suppose that $DS(\mathfrak{B}) \subseteq DS_n(\mathfrak{A})$. There exists a structure $\mathcal{C}$ s.t. $DS_n(\mathcal{C}) = DS(\mathfrak{B})$.

Corollary 3.3. Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $0^{(n)}$. Then there exists a total structure $\mathcal{C}$ s.t. $DS_n(\mathcal{C}) = DS(\mathfrak{B})$.

Example. Let $n \geq 0$. There exists a total structure $\mathcal{C}$ s.t. $\mathcal{C}$ has a $n + 1$-th jump degree $0^{(n+1)}$ but has no $k$-th jump degree for $k \leq n$.

Example. Let $n \geq 0$. There exists a total structure $\mathcal{C}$ such that $DS_n(\mathcal{C}) = \{a|0^{(n)} < e \leq a\}$.

It is sufficient to construct a structure $\mathfrak{B}$ satisfying:

1. $DS(\mathfrak{B})$ has no least element.
2. $0^{(n+1)}$ is the least element of $DS_1(\mathfrak{B})$.
3. All elements of $DS(\mathfrak{B})$ are total and above $0^{(n)}$.

Consider a set $B$ satisfying: $B$ is quasi-minimal above $0^{(n)}$ and $B' \equiv_e 0^{(n+1)}$.

Let $G$ be a subgroup of the additive group of the rationales s.t. $S_G \equiv_e B$. Recall that $DS(G) = \{a|d_e(S_G) \leq_e a \text{ and } a \text{ is total}\}$. Then $d_e(S_G)'$ is the least element of $DS_1(G)$.

Example. Let $n \geq 0$. There exists a total structure $\mathcal{C}$ such that $DS_n(\mathcal{C}) = \{a|0^{(n)} < e \leq a\}$.

It is sufficient to construct a structure $\mathfrak{B}$ such that the elements of $DS(\mathfrak{B})$ are exactly the total $e$-degrees greater than $0^{(n)}$.

This is done by Whener’s construction using a special family of sets: There exists a family $\mathcal{F}$ of sets of natural number s.t. for every $X$ strictly above $0^{(n)}$ there exists a recursive in $X$ set $U$ satisfying the equivalence:

$$F \in \mathcal{F} \iff (\exists a)(F = \{x | (a,x) \in U\})$$

But there is no c.e. in $0^{(n)}$ such $U$.

References


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