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ON A MICROSTRUCTURAL MODEL OF DAMAGE IN SOLIDS

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ABSTRACT—The aim of the paper is to outline and discuss some ideas and preliminary results concerning microstructural modelling of brittle-like damage in solids. We argue that the basic premises of the simple microelastic theory (independent micro-dilatation of particles), developed by S. Cowin *et al.* and the author, are a natural starting point in developing certain continuum damage mechanics models. In the brittle-like case under study, with the assumption of isotropic deterioration, the damage rate is added to the arguments (strain, damage and damage gradient) of the elastic energy density. The additional field quantity of the theory—the damage parameter in our context—leads to the appearance of hyperstresses and a balance equation for them. It is shown that the latter can be conveniently identified with the well-known Kachanov's law of damage growth. Thus this law, traditionally introduced through purely phenomenological arguments, finds a natural place in the herein proposed damage model as an equation expressing hyperstress equilibrium.

1. INTRODUCTION

In continuum damage mechanics, after the pioneering work of L. Kachanov [1], damage is often treated as a certain scalar quantity—an internal scalar variable for which there is no need to attribute a specific physical meaning. The kinetic laws of damage growth are then prescribed using tacitly mathematical convenience arguments and the respective material constants are specified on the base of experimental data. Though instructive and useful in many cases of practical interest, see [2–4], this approach carries a number of arbitrary assumptions and cannot be considered as an adequate tool in the analysis of more complicated damage and failure problems. To get a more fruitful damage mechanics it is necessary that a bit more of a physical microstructural insight be introduced, see the survey [4] for details and a critical evaluation. One of the possibilities that seems intrinsically correct and appealing in the case of randomly distributed spheroidal cavities is to employ their volume fraction (void ratio) as the measure of damage. (Note that such a measure has been used by a number of authors, starting perhaps with Davison *et al.* [5] for ductile metals.) Accordingly, we shall treat the damage process as a generation and growth of microdefects of the type of microvoids. Since damage is assumed isotropic, these defects introduce above all a certain irreversible volume change of the body points which is independent of their displacement. This fact, in turn, leads us immediately to the basic kinematical premises of an exceptionally simple microstructural continuum theory, introduced and developed by the author [6–9] (under the name dilatation theory of elasticity) and by S. Cowin *et al.* (theory of materials with voids) [10,11], see also [12]. In dilatation elasticity we imagine material points to be micro-volumes which may undergo internal dilatation independent of the macro-dilatation $\nabla \cdot \mathbf{u}$, prescribed by the displacement field $\mathbf{u}(\mathbf{x})$ of the mass centers of the points. Kinematically, this model is a particular case of the more general micromorphic theory of Eringen and Şuhubi [13] (developed at the same time by Mindlin [14] as well) in which the micro-volumes are assumed undergoing internal deformation, described by means of the so-called micro-distortion tensor $\boldsymbol{\alpha}$, independent of the macro-distortion $\nabla\mathbf{u}$, generated by the displacement field $\mathbf{u}(\mathbf{x})$. The independence of dilatation leads in turn to

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appearance of hyperstresses whose balance equation, within the proposed framework, could be directly connected, as we shall argue, to damage evolution law.

The outline of the paper is as follows. The basic equations of the dilatation elasticity are briefly recalled in Section 2. In Section 3, having added the damage rate $\dot{\omega}$ to the basic parameters (strain tensor, damage ω and its gradient $\nabla\omega$), we repeat the thermodynamic reasoning of J. Nunziato and S. Cowin [11] and get as a result the governing equations of our damage model. It is to be pointed out that the damage phenomena treated by the model are, generally speaking, of brittle nature. The reason is that no account is taken for plastic strain or strain rate in damage accumulation—the body fails through a microdefects growth governed, as a matter of fact, by Kachanov's type damage evolution law. In Section 4 we discuss the simplest specification of the model when the internal energy is a quadratic function of its arguments with constant coefficients. Even in this simplest case the respective system of coupled equations for displacement and damage fields is strongly nonlinear. The reason is that the density of the solid depends on damage. To illustrate the performance of this simplest model, we treat at some length an illustrative example concerned with uniaxial tension of a deteriorating rod, inspired by a similar analysis of Passman [12] and Passman and Trucano [15] for their creep-damage model. At a first glance a very interesting feature of our model shows up, namely, the existence of a homogeneous damage field $\omega \equiv \omega_0$, $0 < \omega_0 < 1$. A closer look reveals, unfortunately, that certain unreal material behaviour can be also predicted within the model—for instance, a decrease of the homogeneous time-dependent solution $\omega = \omega(t)$ for which $\omega(0) > \omega_0$. That is why we reject the assumption of constant material parameters as physically unacceptable and assume them, similarly to the density field, depending on damage (Section 5). It turns out possible to choose plausible forms of the material parameters as functions of ω and the strain, so that the equation governing damage field reduces to the well-known Kachanov law of brittle damage accumulation in the unidimensional case. Thus we may conclude that the latter law, within the micromorphic framework, expresses the balance of microstresses.

2. BASIC EQUATIONS

In accordance with the foregoing, we first write down the basic balance equations of the micromorphic theory (under the assumption of small strain) in the particular case when the microdistortion tensor $\boldsymbol{\alpha}$ is spherical

$$\boldsymbol{\alpha} = \omega \mathbf{I}, \tag{2.1}$$

ω is the scalar damage variable and $\mathbf{I}$ is the second rank unit tensor. These equations read

$$\begin{aligned} \rho \dot{\mathbf{v}} &= \nabla \cdot \boldsymbol{\varepsilon} + \rho \mathbf{f}, \\ \rho \overline{\kappa \dot{\omega}}^{\,\cdot} &= \nabla \cdot \mathbf{h} + p - s + \rho \ell, \end{aligned} \tag{2.2}$$

and

$$\rho \dot{e} = \boldsymbol{\sigma} : \boldsymbol{\xi} + (s - p)\dot{\omega} + \mathbf{h} \cdot \nabla \dot{\omega} - \nabla \cdot \mathbf{q} + \rho r, \tag{2.3}$$

which gives the energy balance; the colon denotes contraction with respect to two pairs of indices. The following nomenclature is adopted here:

$\boldsymbol{\xi} = \dot{\boldsymbol{\varepsilon}}$—the strain rate tensor,
$\mathbf{f}$—the body force density,
ℓ—the body microstress density,
$p = \frac{1}{3}\mathrm{tr}\, \boldsymbol{\sigma}$—the hydrostatic pressure, associated with the macrostress tensor $\boldsymbol{\sigma}$,
s—the total hydrostatic pressure which differs from p due to independence of dilatation,
$\mathbf{h}$—the hyperstress field,
$\mathbf{q}$—the heat flux,
r—the body heating.

Note that the density ρ of the solid has now the form

$$\rho = \rho_0(1 - \omega), \tag{2.4}$$

where ρ_0 is the density of the undamaged material. Also, the microinertia coefficient κ in eqn $(2.2)_2$ has the form

$$\kappa = \kappa(\omega), \tag{2.5}$$

as argued in the literature, see [11].

We add also the Clausius-Duhem inequality in its standard form

$$\rho\dot{\eta} \geq -\nabla \cdot \frac{\mathbf{q}}{T} + \frac{\rho r}{T}, \tag{2.6}$$

where η is the entropy and T—the temperature.

3. THE BRITTLE-DAMAGE MODEL OF MICROMORPHIC TYPE

The displacement field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and the damage variable $\omega = \omega(\mathbf{x}, t)$ (interpreted as an independent volume change) describe, according to the foregoing reasoning, the kinematics of a deteriorating solid. Its thermal behaviour is described, in turn, by the entropy density $\eta = \eta(\mathbf{x}, t)$. We shall repeat now the simple thermodynamic reasoning of Nunziato and Cowin [11] with only one point of departure.

By definition, the triple $(\mathbf{u}, \omega, \eta)$ constitutes a *thermokinetic process* in the damaged solid; in what follows the fields $\mathbf{u}$, ω and η will be assumed smooth enough. In the brittle case under study there is no irreversible macro-deformation (creep or plastic), if the level of damage is fixed (i.e., at $\omega =$ const). That is why only the strain tensor $\boldsymbol{\varepsilon}$ should be taken into account. The tensor $\boldsymbol{\varepsilon}$, together with the two other strain characteristics of the dilatation-elastic solid, ω and $\nabla\omega$, define the local strain state of a body point. If we add the entropy at the same point, we get the array

$$\mathcal{S} = \{\boldsymbol{\varepsilon}, \omega, \nabla\omega, \eta\}, \tag{3.1}$$

which we call a *local elastic state* of the damaged solid. The simplest way to invoke damage evolution into the model is to include the time-derivative $\dot{\omega}$ among the local parameters (3.1) and thus to assume that the overall response of the deteriorating solid depends on the parameters $\{\mathcal{S}, \dot{\omega}\}$. In other words, we postulate the following set of constitutive equations

$$\begin{aligned} e = e(\mathcal{S}, \dot{\omega}), \quad & \boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathcal{S}, \dot{\omega}), \\ \mathbf{h} = \mathbf{h}(\mathcal{S}, \dot{\omega}), \quad & s - p = S(\mathcal{S}, \dot{\omega}). \end{aligned} \tag{3.2}$$

Besides, we assume for simplicity that the solid does not conduct heat, i.e., $\mathbf{q} = 0$, and there is no body heating, $r = 0$. The energy balance (2.3) and entropy inequality then simplify

$$\rho\dot{e} = \boldsymbol{\sigma} : \boldsymbol{\xi} + (s - p)\dot{\omega} + \mathbf{h} \cdot \nabla\dot{\omega} + \rho r, \tag{3.3}$$

$$\rho\dot{\eta} \geq 0. \tag{3.4}$$

It is important to point out that the form (3.3) of the energy balance equation is just the above mentioned point of disagreement between the analysis of Nunziato and Cowin [11] and ours. Namely, the authors of [11] take $\rho_R\dot{e}$ in the l.-h. side of eqn (3.3), where ρ_R is the density field in a fixed reference configuration of the body, while here we take the current value ρ of the density. Thus ρ_R is not affected by damage evolution and remain constant in material points, in contrast to the current density ρ which "feels" damage according to eqn (2.4). In our view the assumption $\rho =$ const and thus $\rho = \rho_R$ can be justified only if damage magnitude (i.e., void ratio in our context) is very small, $\omega \ll 1$. In the latter case we would get the linearized version of our model which coincides exactly with that of Nunziato and Cowin [11].

We next subtract (3.4) (multiplied by T) from (3.3) and, using the first of the constitutive assumptions (3.2) (that for the internal energy density e), rearrange the result in the form

$$\begin{aligned}&\Big[\boldsymbol{\sigma}-\rho\frac{\partial e}{\partial \boldsymbol{\varepsilon}}\Big]:\boldsymbol{\xi}+\Big[s-p-\rho\frac{\partial e}{\partial \omega}\Big]\dot{\omega}\\&\quad+\rho\frac{\partial e}{\partial \omega}\ddot{\omega}+\Big[\mathbf{h}-\rho\frac{\partial e}{\partial \nabla\omega}\Big]\cdot\nabla\dot{\omega}+\rho\Big[T-\frac{\partial e}{\partial \eta}\Big]\dot{\eta}\geq 0.\end{aligned}\tag{3.5}$$

This inequality leads as usual to certain restrictions concerning the constitutive equations (3.2). Namely it appears that

$$\frac{\partial e}{\partial \dot{\omega}}=0,\quad \text{i.e.,}\quad e=e(\boldsymbol{\varepsilon},\omega,\nabla\omega,\eta),\tag*{(3.6)$_1$}$$

$$\boldsymbol{\sigma}=\rho\frac{\partial e}{\partial \boldsymbol{\varepsilon}},\quad \mathbf{h}=\rho\frac{\partial e}{\partial \nabla\omega},\quad T=\rho\frac{\partial e}{\eta}.\tag*{(3.6)$_2$}$$

Only the microstress s can thus depend on the damage rate $\dot{\omega}$; moreover

$$s-p=\rho\frac{\partial e}{\partial \omega}+F(\mathcal{S},\dot{\omega})\tag{3.7}$$

with the only restriction on the function F that

$$F(\mathcal{S},\dot{\omega})\dot{\omega}\geq 0,\tag{3.8}$$

as it follows from (3.5) and (3.6); this is the dissipation inequality of the model under study. It remains now to choose plausible forms of the functions e and F and to combine them with the balance equations (2.2) in order to get a coupled system for the unknown fields $\mathbf{u}(\mathbf{x},t)$ and $\omega(\mathbf{x},t)$ in the damaged solid. Prior to this, however, we note that only stationary processes in the solid will be dealt with in the sequel, i.e., $\mathbf{v}=$ const and $\dot{\omega}=$ const. Moreover, the damage rate $\dot{\omega}$ and damage "acceleration" $\ddot{\omega}$ should be small except for a very short interval before final failure of the body and that is why we may neglect the term $\dot{\omega}^2$ as compared to $\dot{\omega}$. Consequently, the microinertia term in eqn $(2.2)_2$:

$$\dot{\overline{\kappa\dot{\omega}}}=\kappa\ddot{\omega}+\kappa'(\omega)\dot{\omega}^2$$

should be also neglected, so that the balance equations (2.2), at the absence of body sources, reduce to their static form

$$\nabla\cdot\boldsymbol{\sigma}=0,\quad \nabla\cdot\mathbf{h}+p-s=0,\tag{3.9}$$

Note that a similar in spirit micromorphic model was introduced by Passman and Trucano [15] who, in order to account for creep-damage interrelation, introduced the strain-rate tensor $\boldsymbol{\xi}$, instead of the strain $\boldsymbol{\varepsilon}$, in the array (3.1).

A final remark in this Section is warranted. Assume the damage field homogeneous, $\omega=\omega(t)$. Then $\mathbf{h}=0$, see $(3.6)_2$, and thus $s-p=0$ as it follows from the balance equation $(3.9)_2$. Eqn (3.7) now yields

$$\rho\frac{\partial e}{\partial \omega}+F(\mathcal{S},\dot{\omega})=0.\tag{3.10}$$

Let

$$F=f\dot{\omega},\quad f=f(\mathcal{S},\dot{\omega})\geq 0,\tag{3.11}$$

where f is a positive and arbitrary, for the moment, function to be discussed and specified later on.

Assuming ρ damage independent (which is a crude approximation, as we shall argue in the next Section), we recast (3.10) as

$$\dot{\omega} = 2Y/f, \quad Y = -\frac{1}{2}\frac{\partial(\rho e)}{\partial \omega},$$

so that Y is the so-called damage energy release rate. As is well-known the quantity Y plays an important role in the recent development of continuum damage mechanics, especially in formulating the plastic damage models, based on the notion of damage surface, see [16,17] and the references therein. Thus the damage rate in our micromorphic model (at constant ρ) appears proportional to the damage energy release rate—the thermodynamic force, conjugated to the damage variable.

4. THE SIMPLEST SPECIFICATION OF THE MODEL

4.1. *The governing equations*

We shall now concentrate on the simplest plausible choice for the functions e and F in order to illuminate certain characteristic features, introduced by the micromorphic premises in damage picture.

Obviously, the simplest form of e in the isotropic case under study is quadratic

$$\rho e = \frac{1}{2}\lambda \mathrm{tr}\, \boldsymbol{\varepsilon}^2 + \mu \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} - \xi_1 \omega \mathrm{tr}\, \boldsymbol{\varepsilon} + \frac{1}{2}\xi_2 \omega^2 + \frac{1}{2}\xi_3 |\nabla \omega|^2. \tag{4.1}$$

Besides

$$\mu > 0,\ k = \lambda + \frac{2}{3}\mu > 0,\ \xi_2 > 0,\ \xi_3 > 0,\ k\xi_2 - \xi_1^2 > 0 \tag{4.2}$$

in order to assure positive definiteness of e.

Note that ρ itself depends on ω, see (2.4), so that it is more convenient to deal with ρe, as we did in (4.1).

As a very crude approximation, we first assume all material functions in (4.1) constant. Our first task will be to show that such an assumption leads to certain unrealistic predictions for damage process and should be therefore rejected in favor to the assumption that the material parameters, similarly to density field ρ, are sensitive to damage magnitude.

Under the assumption of constant parameters in the potential function (4.1) we get, using (3.6) and (3.7), that

$$\begin{gathered} \boldsymbol{\sigma} = \lambda \mathbf{I} \mathrm{tr}\, \boldsymbol{\varepsilon} + 2\mu \boldsymbol{\varepsilon} - \xi_1 \omega \mathbf{I}, \quad \mathbf{h} = \xi_3 \nabla \omega, \\ s - p = f\dot{\omega} + \xi_2 \omega - \xi_1 \mathrm{tr}\, \boldsymbol{\varepsilon} \\ + \frac{1}{1-\omega}\left\{ \frac{1}{2}\lambda \mathrm{tr}^2 \boldsymbol{\varepsilon} + \mu \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} - \xi_1 \omega \mathrm{tr}\, \boldsymbol{\varepsilon} + \frac{1}{2}\xi_2 \omega^2 + \frac{1}{2}\xi_3 |\nabla \omega|^2 \right\}; \end{gathered} \tag{4.3}$$

the expression in the curly brackets in the formula for $s - p$ is just ρe, see (4.1). This cumbersome, at a first glance, formula follows from (4.1):

$$\rho \frac{\partial e}{\partial \omega} = \frac{\partial(\rho e)}{\partial \omega} - e\frac{\partial \rho}{\partial \omega} = \frac{\partial(\rho e)}{\partial \omega} + \frac{\rho_0}{\rho}(\rho e), \tag{4.4}$$

since ρ *depends* on ω, according to (2.4). The latter dependence is thus responsible for the obvious and strong nonlinearity of the basic equations of the micromorphic damage model under study, even in the simplest case of quadratic energy function ρe with constant coefficients. A linearization could be achieved under the assumption of small damage magnitudes, $\omega \ll 1$. This assumption, however, is not acceptable in the framework of the continuum damage mechanics which associates the failure with critical damage values close (or simply equal) to 1. Moreover, damage has a tendency to localize around stress concentrators (like micro-holes, micro-cracks, etc.) so that it can be expected to attain considerable values in their vicinities even in the brittle case.

When coupled with the balance equations (3.9), the constitutive ones (3.6) would yield a nonlinear system of partial differential equations for the unknown fields of displacement and damage, namely

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} - \xi_1\nabla\omega = 0,$$
$$f\dot\omega = \xi_3\Delta\omega - \xi_2\omega + \xi_1\nabla\cdot\mathbf{u} \tag{4.5}$$
$$-\frac{1}{1-\omega}\left\{\frac{1}{2}\lambda(\nabla\cdot\mathbf{u})^2 + \mu\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} - \xi_1\omega\nabla\cdot\mathbf{u} + \frac{1}{2}\xi_2\omega^2 + \frac{1}{2}\xi_3|\nabla\omega|^2\right\}.$$

The classical boundary conditions in stress and/or displacement are then imposed:

$$\boldsymbol{\sigma}\Big|_S = \boldsymbol{\sigma}_n^0 \quad \text{or} \quad \mathbf{u}\Big|_S = \mathbf{u}^0, \tag{4.6$_1$}$$

where $\mathbf{n}$ is the outward unit normal vector to the boundary S of the solid. A boundary condition in hyperstresses is also needed; a natural choice is the requirement of no normal hyperstresses on S [11]:

$$h_n\Big|_S = 0, \quad h_n = \mathbf{h}\cdot\mathbf{n}. \tag{4.6$_2$}$$

As far as the initial conditions are concerned, we first require

$$\omega(\mathbf{x},0) = 0 \tag{4.6$_3$}$$

—no damage in the initial state ($t = 0$) of the body. The initial condition for the displacement is then

$$\mathbf{u}(\mathbf{x},0)\Big|_S = \mathbf{U}(\mathbf{x}), \tag{4.6$_4$}$$

where $\mathbf{U}(\mathbf{x})$ is the displacement field in the virgin elastic body ($\omega = 0$) with elastic moduli λ and μ, corresponding to the boundary condition (4.6)$_1$. A rigorous proof of the existence and uniqueness theorem for the initial-boundary-value problem (4.5)–(4.6) would be of interest. It is to be noted that such a theorem has been recently proved in [18] for the linear version of the system (4.5):

$$\begin{aligned}&\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} - \xi_1\nabla\omega = 0,\\ &\xi_3\Delta\omega - \xi_2\omega + \xi_1\nabla\cdot\mathbf{u} = f\dot\omega\end{aligned} \tag{4.7}$$

in the static case ($\dot\omega = 0$). The foregoing reasoning indicates, however, that the system (4.7) oversimplifies the damage phenomena and cannot be expected to provide an adequate picture of the latter.

Let us point out in passing that a number of classical problems has been solved for the system (4.7), see, e.g., [19-21] *et al.*, under the assumption of material constants independent of void ratio. Some of these solutions could be used to identify the additional material parameters ξ_i employing the idea of mimicking the behaviour of a porous solid with that of a homogeneous and microstructured one, see [22, 11, 7]. For example, it was shown in this way that for a dilatation-elastic solid, governed by the system (4.7) in the static case ($\dot\omega = 0$) one has $\xi_1 = k$.

4.2. *Tension of a rod*

Consider now a simple solution of the system (4.5), concerned with the uniaxial tension of a rod with a circular cross-section. In the cylindrical coordinate system (r, φ, x) with the x-axis along the rod axis, the solution due to the obvious symmetry has the form

$$u_x = u_x(x), \quad u_r = u_r(r), \quad u_\varphi = 0, \quad \omega = \omega(x),$$

so that

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \quad \varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\varphi = \frac{u_r}{r}.$$

From eqn $(4.5)_1$, together with the boundary condition (4.5), one easily obtains $u_r = Ar$, $A = A(x)$ and therefore $\varepsilon_r = \varepsilon_\varphi$. The Hooke law $(4.3)_1$ then yields

$$\lambda\theta + 2\mu\varepsilon_x - \xi_1\omega = \sigma_x, \quad \lambda\theta + 2\mu\varepsilon_r - \xi_1\omega = 0, \tag{4.8}$$

$\theta = \operatorname{tr}\boldsymbol{\varepsilon} = \varepsilon_x + 2\varepsilon_r$, since $\sigma_r \equiv 0$; σ_x is the fixed tensile stress.

Upon solving eqns (4.8) as a system with respect to the strain components ε_x and ε_r, we get

$$\sigma_x = \widetilde{E}\varepsilon_x, \quad \varepsilon_r = -\widetilde{\nu}\varepsilon_x,$$

where $\widetilde{E}$ and $\widetilde{\nu}$ are the apparent values of the Young modulus and Poisson ratio for the damaged solid given, respectively, by the formulae

$$\widetilde{E} = E(1-\Omega), \quad \widetilde{\nu} = \frac{\nu - \Omega}{1+\Omega}; \quad \Omega = \frac{\xi_1\omega}{3k\varepsilon_x}. \tag{4.9}$$

Since the Young modulus $\widetilde{E}$ decreases with deterioration, we should have $\Omega > 0$. That is why the Poisson ratio $\widetilde{\nu}$ should also decrease and the parameter ξ_1 is positive. It is to be noted as well that (4.9) supplies us with a simple method of experimental determination of ξ_1: it obviously suffices, for a given tensile strain ε_x, to measure the apparent Young modulus $\widetilde{E}$ (say, by an ultrasonic test) and to find the void ratio ω (say, through the change of the specimen density, cf. (2.4)).

Eqns (4.9) imply the following relation

$$\Delta\nu = \frac{\Delta E}{E}(1+\nu) \tag{4.10}$$

between the increments $\Delta E = E - \widetilde{E}$ and $\Delta\nu = \nu - \widetilde{\nu}$ of the moduli due to damaging. This relation is, however, a consequence of the specific model adopted, more specifically, of the Hooke law (4.8) with constant material parameters, in the dilatation elasticity theory and may thus serve as a test for applicability of the model. It is easily seen, e.g., that (4.10) does not fit at all the experimental data, reported by Cordebois and Sidoroff [22] which is no wonder—the data concern ductile damage created by a considerable plastic deformation.

4.3. *A simplest damage distribution in the rod*

We shall look now for certain simple damage fields in the strained rod. The simplest one is obviously the homogeneous time-independent field $\omega \equiv \omega_0$. For such a solution we have from $(4.5)_2$

$$2\xi_1\theta - 2\xi_2\omega_0 + \xi_2\omega_0^2 - \lambda\theta^2 - 2\mu(\varepsilon_x^2 + 2\varepsilon_r^2) = 0 \tag{4.11}$$

and, besides,

$$\varepsilon_x = \frac{\sigma_x}{E} + \frac{\xi_1\omega}{3k}, \quad \varepsilon_r = -\frac{\nu}{E}\sigma_x + \frac{\xi_1\omega}{3k},$$
$$\theta = \varepsilon_x + 2\varepsilon_r = \frac{\sigma_x + 3\xi_1\omega}{3k}, \tag{4.12}$$

as it follows from (4.8). Upon inserting the expressions for ε_x, ε_r and θ in (4.11), we get the quadratic equation

$$g(\omega) = A\omega^2 - 2B\omega + C = 0,$$
$$A = 3(\xi_2 k - \xi_1^2), \quad B = A + \xi_1\sigma_x, \quad C = 2\xi_1\sigma_x - \frac{\sigma_x^2}{1-2\nu}, \tag{4.13}$$

whose solutions should be the needed constant damage fields. But $A > 0$ as a consequence of (4.2) and $g(1) = -A - \dfrac{\sigma_x^2}{1-2\nu} < 0$, so that eqn $(4.13)_1$ has a root in the interval $(0, 1)$ (a natural condition for the latter to possess a "damage" meaning) if and only if $g(0) = C > 0$, i.e.,

$$0 < \sigma_x < \sigma_x^0, \quad \sigma_x^0 = 2\xi_1(1-2\nu). \tag{4.14}$$

The second root of eqn $(4.13)_1$ is also real in this case but it is greater than 1 and therefore has no damage interpretation. We thus conclude that there is a certain stress "limit" σ_x^0 with the property that at tensile stresses $\sigma_x < \sigma_x^0$ a constant damage field ω_0 can exist in the rod. A similar conclusion was reached by Passman and Trucano [15] in their creep-damage model of dilatation-elastic, in our terminology, type. Moreover, these authors even claimed the existence of three different real roots of the respective equation for ω_0 and thus the possibility of coexistence of three different damage fields in the body under simple tension. It should be pointed out, however, that no account was taken in [15] for the natural condition that the roots should lie in the interval $(0, 1)$. (It could be immediately seen that one of the constant solutions in [15] is negative and has no damage meaning as a result.)

The stress "limit" σ_x^0 admits an appealing interpretation on a very rough heuristic level, connected with the Griffith theory of cracks. Let ℓ be the mean length of the microdefects imagined for the moment as crack-like objects. For a given tensile stress let ℓ^* be the critical length of a Griffith crack in the solid. If $\ell < \ell^*$, the defects will remain stable under the tension—they will just open a bit without propagating and thus a certain stationary constant damage field will take place in the specimen as predicted by the foregoing considerations. The stress "limit" σ_x^0 then corresponds to the minimal critical length ℓ^* of the order of several interatomic distances. Therefore σ_x^0 can be identified with the stress that ruptures the rod instantly; the latter stress may be found having extrapolated the time-to-rupture curve in the brittle region to $t = 0$ when it intersects the ordinate axis.

4.4. *The failure of the simplest model*

Let us face now a more realistic situation when, at fixed tensile stress σ_x, a homogeneous time-dependent damage field $\omega = \omega(t)$ appears in the rod. Repeating the simple arguments of Section 4.3, we get

$$\dot{\omega} = \frac{g(\omega)}{6kf}, \tag{4.15}$$

where $g(\omega)$ is the quadratic function, defined in (4.13).

Let $\sigma_x < \sigma_x^0$, where σ_x^0 is the stress "limit" (4.14), and $\omega = \omega_0$ is the corresponding to σ_x homogeneous damage field that solves eqn $(4.13)_1$, $0 < \omega_0 < 1$. As is easily seen from eqn (4.20), if the initial damage magnitude $\omega(0) < \omega_0$, then $\omega(t)$ monotonically increases (as it should have been expected), approaching the value ω_0. If, however, $\omega(0) > \omega_0$, then $\omega(t)$ monotonically decreases approaching from above the same value ω_0. This unrealistic behaviour suffices to indicate clearly that the simplest micromorphic model, corresponding to the quadratic form (4.1) of the internal energy function with constant coefficients, is unrealistic as far as damage phenomena are concerned.

5. DAMAGE DEPENDENT MATERIAL CONSTANTS

To get a more realistic model we should assume all material parameters depending on damage magnitude. This is a natural assumption since the elastic moduli, in particular, are strongly affected by damage, as is well known. Moreover, the current values of the latter in a deteriorating solid could serve as a convenient measure of damage [3].

The balance equation $(3.9)_2$ for microstresses, with (2.4), (3.7) and (4.4) taken into account, yields

$$f\dot{\omega} = \nabla \cdot (\xi_3 \nabla\omega) - \left[\frac{\partial(\rho e)}{\partial\omega} + \frac{\rho e}{1-\omega}\right]. \tag{5.1}$$

Let the internal energy have the same simplest form (4.1), but with damage dependent coefficients $\lambda = \lambda(\omega)$, ..., $\xi_3 = \xi_3(\omega)$. Eqn (5.1) then becomes

$$\begin{aligned} f\dot{\omega} = \nabla \cdot (\xi_3 \nabla\omega) &- \frac{1}{2}\left[\lambda' + \frac{\lambda}{1-\omega}\right] \operatorname{tr}^2 \boldsymbol{\varepsilon} \\ &- \left[\mu' + \frac{\mu}{1-\omega}\right] \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} - \frac{1}{2}\left(2\xi_2\omega + \left[\xi_2' + \frac{\xi_2}{1-\omega}\right]\omega^2\right) \\ &+ \left(\xi_1 + \left[\xi_1' + \frac{\xi_1}{1-\omega}\right]\omega\right) \operatorname{tr} \boldsymbol{\varepsilon} - \frac{1}{2}\left[\xi_3' + \frac{\xi_3}{1-\omega}\right] |\nabla\omega|^2; \end{aligned} \tag{5.2}$$

the primes denote differentiation with respect to ω.

We notice that eqn (5.2) drastically simplifies if we choose

$$\lambda(\omega) = \lambda_0(1-\omega), \quad \mu(\omega) = \mu_0(1-\omega), \quad \xi_i(\omega) = \xi_i^0(1-\omega), \quad i = 1, 2, 3, \tag{5.3}$$

where the subscript (or superscript) '0' indicates the values of the respective parameters for the undamaged solid ($\omega = 0$). In this case all expressions in the square brackets in (5.2) vanish and hence

$$f\dot{\omega} = \nabla \cdot (\xi_3 \nabla\omega) + \xi_1 \operatorname{tr} \boldsymbol{\varepsilon} - \xi_2\omega, \tag{5.4}$$

which coincides with eqn $(4.7)_2$ if $\nabla\omega = 0$.

Eqns (5.3) imply that we have assumed the internal energy of the damaged solid in the form

$$\rho e = (1-\omega)\left[\frac{1}{2}\lambda_0 \operatorname{tr} \boldsymbol{\varepsilon}^2 + \mu_0 \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} - \xi_1^0 \omega \operatorname{tr} \boldsymbol{\varepsilon} + \frac{1}{2}\xi_2^0\omega^2 + \frac{1}{2}\xi_3^0|\nabla\omega|^2\right], \tag{5.5}$$

or

$$\rho_0 e = \frac{1}{2}\lambda_0 \operatorname{tr} \boldsymbol{\varepsilon}^2 + \mu_0 \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} - \xi_1^0 \omega \operatorname{tr} \boldsymbol{\varepsilon} + \frac{1}{2}\xi_2^0\omega^2 + \frac{1}{2}\xi_3^0|\nabla\omega|^2$$

which is just the form of the internal energy chosen by Nunziato and Cowin [11] (assuming tacitly that the undamaged state of the solid is taken as the reference one: $\rho_R = \rho_0$). For a homogeneous damage field the basic equations of our model coincide then with those of [11], see (4.7). However, for a nonhomogeneous damage distribution our equations are strongly nonlinear even under the simplest assumptions (5.3). At the same time eqns (4.7) of Nunziato and Cowin [11], see also [19], remain the same no matter whether the damage field is homogeneous or not. This represents an essential difference due to the fact, let us point out once more, that we employ the damage dependent density field ρ in the energy balance equation unlike the authors of [11] who take ρ as damage independent.

For a homogeneous damage field, $\nabla\omega = 0$, eqn (5.4) simplifies even more

$$f\dot{\omega} = \xi_1 \operatorname{tr} \boldsymbol{\varepsilon} - \xi_2\omega. \tag{5.6}$$

However, if the strain is deviatoric, $\mathrm{tr}\,\boldsymbol{\varepsilon} = 0$, as it is the case in a pure torsion, eqn (5.6) predicts damage decrease in loading (since both f and ξ_2 are positive) which is unacceptable. Hence the assumption (5.3), though highly appealing due to extreme simplicity of its corollaries like (5.4) and (5.6), should be rejected.

One of the reasons for failure of the assumption (5.3) lies perhaps in the fact that it corresponds, in particular, to the so-called Voigt values of the effective elastic moduli of the damaged solid treated here as a composite material comprising a matrix with voids [24]. As is well known the Voigt values $\mu_V = \mu_0(1-\omega)$, $\lambda_V = \lambda_0(1-\omega)$ provide elementary upper bounds on the elastic moduli of a porous solid, grossly overestimating the latter. A number of more refined theories predicting the effective properties of porous solids (and, more general, of composites) have been proposed in literature, see, e.g. [22, 24], but a detailed discussion would not be warranted in the present study. We shall only point out the simple empirical relations

$$\mu = \mu_0(1-\omega)^2, \quad \lambda = \lambda_0(1-\omega)^2 \tag{5.7}$$

for the moduli of the porous (damaged) solid which at any case describe much more realistically their dependence upon the damage variable ω. Since nothing in this moment can be said about the microstructural parameters ξ_i we shall stick to the Voigt type relations (5.3) for them: $\xi_i = \xi_i^0(1-\omega)$, $i = 1, 2, 3$.

The foregoing assumptions allow us to write the balance equation (5.2) in the form

$$f\dot{\omega} = \nabla \cdot (\xi_3 \nabla\omega) + \frac{1}{2}\lambda_0(1-\omega)\mathrm{tr}^2\boldsymbol{\varepsilon} + \mu_0(1-\omega)\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} + \xi_1 \mathrm{tr}\,\boldsymbol{\varepsilon} - \xi_2\omega. \tag{5.8}$$

The parameter $f = f(\mathcal{S}, \dot{\omega})$, see (3.11). We may neglect the dependence of f upon the damage rate $\dot{\omega}$ since it is multiplied by $\dot{\omega}$ which itself is assumed small. Hereafter we shall deal with homogeneous damage fields, $\nabla\omega = 0$, so that

$$f = f(\boldsymbol{\varepsilon}, \omega). \tag{5.9}$$

Moreover, due to isotropy assumption, f is an scalar-valued isotropic function of the strain tensor $\boldsymbol{\varepsilon}$.

Let us recall that the damage growth in brittle solids in $3D$ loading is appropriately described by means of the phenomenological Kachanov's law

$$\dot{\omega} = B\left(\frac{\sigma_{eq}}{1-\omega}\right)^n, \tag{5.10}$$

where B and n are material constants easily specified through the respective time-to-rupture curves in the brittle region, see [2,3]. In (5.10) σ_{eq} is the so-called equivalent stress which is a certain scalar-valued isotropic function of the stress tensor $\boldsymbol{\sigma}$, expressed usually as a linear combination of the first and second invariants of $\boldsymbol{\sigma}$ and the maximum tensile stress σ_{max}. The specific form of the σ_{eq} is traditionally obtained using the respective isochronous surfaces for a solid, see for details and further references [2, 3, 25] *et al.*

A natural requirement is then to want eqns (5.7) and (5.10) to be equivalent, since they have the one and the same structure of evolution laws determining the damage growth rate. This allows us to specify the material function f:

$$f = f(\boldsymbol{\varepsilon}, \omega) = \frac{(1-\omega)^{n+1}}{B\sigma_{eq}^n}\left[\frac{1}{2}\lambda_0(1-\omega)\mathrm{tr}^2\boldsymbol{\varepsilon} + \mu_0(1-\omega)\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} + \xi_1^0 \mathrm{tr}\,\boldsymbol{\varepsilon} - \xi_2^0\omega\right], \tag{5.11}$$

having used the assumption (5.3) for the microstructure parameters dependence on damage; the equivalent stress σ_{eq} is to be expressed by means of the strain tensor $\boldsymbol{\varepsilon}$ using the Hooke law $(4.3)_1$

of our model. Eqn (5.11) assures that the microstress balance equation of the proposed model reduces to Kachanov's law of damage growth for homogeneous damage fields.

It is should be noted, however, that the direct comparison between the balance equation (5.8) and Kachanov's law (5.10) is, strictly speaking, incorrect. The reason is that two different interpretations of damage are inherent in these two equations. In eqn (5.8) ω is the volume fraction of the microvoids while in eqn (5.10) ω is the relative decrease of the cross-sections of a rod due to deterioration. Thus in uniaxial tension $\sigma_x/(1-\omega)$ is the real tensile stress acting on the rod. A simple remedy could be proposed if one recalls the studies of the strength of porous solids, summarized, e.g., in [26, Chap. 3]. One of the simplest model there, due to Pines *et al.* (1958) claims that an array of identical micro-pores with volume concentration ω leads to a relative decrease of $\frac{3}{2}\omega$ for the cross-section of a rod (provided the pore distribution is spatially uniform.) Thus the real (effective) tensile stress in the microporous rod will be $\sigma_x/\left(1-\frac{3}{2}\omega\right)$ and Kachanov's law (5.10) should be replaced by the relation

$$\dot{\omega} = B'\left(\frac{\sigma_{eq}}{1-\frac{3}{2}\omega}\right)^{n'}, \tag{5.12}$$

with new material constants B' and n'. The latter may be again specified by means of the respective time-to-rupture curves for the specimen—the same procedure as that for the classical Kachanov's law (5.10), with the only difference that the failure criterion $\omega = 1$ should be replaced by the criterion $\omega = 2/3$, since the denominator of (5.12) vanishes at this value of ω. (Note also that $\omega = 2/3$ is very close to the packing value for an array of identical spheres.) The comparison between (5.12) and (5.10) will bring forth now a certain relation for the function $f(\varepsilon, \omega)$, a bit different from (5.11), but the basic idea—identification of Kachanov's damage growth equation as a balance of hyperstresses in the simple dilatation elastic model—remains the same.

6. CONCLUDING REMARKS

When treating damage as a certain internal variable with specific micromechanical meaning, namely, as an additional volume change due to appearance and growth of the microdefects, it is natural to invoke a simple microstructural theory of elasticity. The appropriate choice seems to be the model of an elastic solid with independent dilatation of particles, developed by Cowin et al. and the author. The additional field quantity of the theory—the damage parameter in our context—leads to the appearance of hyperstresses and a balance equation for them. We argue that the latter can be conveniently identified with the well-known Kachanov's law of damage growth. Thus this law, traditionally introduced through purely phenomenological arguments, finds a cosy and natural place in the herein proposed damage model as an equation expressing hyperstress equilibrium. Such an interpretation of Kachanov's law seems new and appealing. But the usefulness of the proposed model and the limits of its applicability (if any) for more realistic continuum damage mechanics problems obviously need extensive further study.

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