adapted from:
SIAM J. Appl. Math., Vol. 51, No 1, pp. 172–186

ON THE FACTORIAL FUNCTIONAL SERIES AND THEIR APPLICATION TO RANDOM MEDIA[1]

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Abstract. Functional series with a point-wise random input (the density field of a random set of points $\boldsymbol{x}_j$) are considered. The series are rearranged so as the so-called factorial fields of the set $\boldsymbol{x}_j$ appear; the obtained series are called factorial. The basic result of the paper states that the factorial series possess virial property. This means that if a random field $u(\boldsymbol{x})$ is expanded as a factorial series, the truncation $u^{(p)}(\boldsymbol{x})$ of the latter after the p-tuple term coincides, in statistical sense, with $u(\boldsymbol{x})$ to the order n^p, where n is the number density of the set $\boldsymbol{x}_j$, $p = 1, 2, \cdots$. The performance of the factorial series in random media problems is illustrated on the example of steady-state diffusion in a random dispersion of spheres whose sink strength differs from that of the matrix. The full statistical solution of this problem, correct to the order c^2, is obtained; in particular, the effective sink-strength of the dispersion is found to the same order c^2, with c being the volume fraction of the spheres.

Key words. functional (Volterra-Wiener) series, random media, correlation functions, effective properties, lossy composites.

AMS (MOS) subject classification. 60H25, 82A42.

1. Introduction. In applications one often has to deal with random media of 'point-wise' nature. The situation in general can be envisaged as a continuum in which, at random locations $\boldsymbol{x}_j$, one finds inhomogeneities of random size and shape. The physical background at the same time could be highly varied: the points $\boldsymbol{x}_j$ could be centers of inclusions in a particulate composite material, centers of solid particles in a fluid suspension, centers of eddies in a turbulent flow, etc. The physical quantities that appear in such media should inherit, in a certain sense, the respective point-wise structures. Moreover, any theory of such media, to be adequate and physically clear, should account from the very beginning for the said structures. With this aim in view, we propose the following simple scheme of arguments.

Let us introduce after Stratonovich [18] the so-called random density field

$$\psi(\boldsymbol{x}) = \sum_j \delta(\boldsymbol{x} - \boldsymbol{x}_j), \tag{1.1}$$

generated by the set $\boldsymbol{x}_j$ of random points. The random field $\psi(\boldsymbol{x})$ is uniquely defined by the set $\boldsymbol{x}_j$ and vice versa. In particular, the multipoint moments (correlation functions) of $\psi(\boldsymbol{x})$ can be easily expressed by means of the multipoint distribution functions (probability densities) f that define the random set $\boldsymbol{x}_j$ (the brackets $\langle\cdot\rangle$ denote ensemble averaging):

$$\langle\psi(\boldsymbol{y})\rangle = f_1(\boldsymbol{y}) = n,$$

$$\langle\psi(\boldsymbol{y}_1)\psi(\boldsymbol{y}_2)\rangle = f_1(\boldsymbol{y}_1)\delta(\boldsymbol{y}_1 - \boldsymbol{y}_2) + f_2(\boldsymbol{y}_1 - \boldsymbol{y}_2), \tag{1.2}$$

[1]Received by the editors December 12, 1988; accepted for publication (in revised form) June 24, 1989.
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$\langle\psi(\boldsymbol{y}_1)\psi(\boldsymbol{y}_2)\psi(\boldsymbol{y}_3)\rangle = f_1(\boldsymbol{y}_1)\delta(\boldsymbol{y}_1-\boldsymbol{y}_2)\delta(\boldsymbol{y}_1-\boldsymbol{y}_3) + 3\{\delta(\boldsymbol{y}_1-\boldsymbol{y}_2)f_2(\boldsymbol{y}_1,\boldsymbol{y}_2)\}_s + f_3(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3),$

etc., see [18], where $\{\cdot\}_s$ denotes symmetrization with respect to all different combination of indices in the brackets. We recall that f_k gives the probability to simultaneously find a point from the set $\boldsymbol{x}_j$ per each of the infinitesimal volumes $\boldsymbol{y}_i < \boldsymbol{y} < \boldsymbol{y}_i + \mathrm{d}^3\boldsymbol{y}_i$, $i = 1,\cdots,k$, as $dP = f_k(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_k)\,\mathrm{d}^3\boldsymbol{y}_1\cdots\mathrm{d}^3\boldsymbol{y}_k$. Hereafter, we assume all random sets and fields statistically homogeneous; then, in particular, $f_1(\boldsymbol{y}) = n$, where n is the number density of the set $\boldsymbol{x}_j$, i.e., the mean number of points $\boldsymbol{x}_j$ per unit volume.

Let $u(\boldsymbol{x})$ be a certain field in the random medium that appears in a given physical problem, say, temperature, displacement, velocity, etc. Provided the respective external impacts and/or boundary conditions are deterministic, u(x) is uniquely defined by the random field $\psi(\boldsymbol{x})$. This allows to consider $\psi(\boldsymbol{x})$ as the 'input' that generates the 'output' $u(\boldsymbol{x})$. Following the general idea of the system theory [17], we develop $u(\boldsymbol{x})$ as functional (Volterra-Wiener) series, generated by the input $\psi(\boldsymbol{x})$

$$u(\boldsymbol{x}) = T_1(\boldsymbol{x}) + \int T_1(\boldsymbol{x}-\boldsymbol{y})\psi(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}$$
$$+\iint T_2(\boldsymbol{x}-\boldsymbol{y}_1,\boldsymbol{x}-\boldsymbol{y}_2)\psi(\boldsymbol{y}_1)\psi(\boldsymbol{y}_2)\,\mathrm{d}^3\boldsymbol{y}_1\mathrm{d}^3\boldsymbol{y}_2 + \cdots \tag{1.3}$$

with certain nonrandom kernels T_i, $i = 0,1,\cdots$. (We assume the medium unbounded so that the integrals hereafter are taken over the whole R^3 if the integration domain is not explicitly indicated.) It is noteworthy that the series (1.3) differs from the usual perturbation series expansion for weakly inhomogeneous media. The reason is that the input in (1.3) is the random density field $\psi(\boldsymbol{x})$ while the input of the perturbation series is the fluctuating part of the respective permittivity field, see, e.g., [10, p.I] and [12, p.I]. Also, the perturbation expansion is applicable for an arbitrary medium, while we have reached (1.3) under the assumption that the medium has a 'point-wise' internal constitution.

The representation (1.3), purely formal at this moment, reflects, so to say, the point-wise origin of the field $u(\boldsymbol{x})$ because, as is seen from (1.1), the one-tuple term is a superposition of fields over the points of the set $\boldsymbol{x}_j$, the two-tuple term is a superposition of fields over all pairs of points from the set $\boldsymbol{x}_j$ and so on. This interpretation of the series (1.3), let us note in passing, makes it tempting to resort immediately to the cluster expansion ideology of Finkel'berg [9] et al. and say that, loosely speaking, the one-tuple term in (1.3) is the contribution to $u(\boldsymbol{x})$ of isolated inhomogeneities, the two-tuple term reflects the pair-wise interactions of the latter, etc. Without entering a more lengthy discussion here, we shall only mention that no matter how appealing this idea may seem intuitively, we prefer a more sound approach in which the kernels T_i in (1.3) are to be specified by means of the equations or variational principles that govern the field $u(\boldsymbol{x})$. And only then, having found the kernels T_i we shall try, if necessary or by reason of pure curiosity, to interpret them as single, double, etc., inhomogeneity interaction fields. We shall illustrate below (§5) how such a program could be accomplished.

Having represented the field $u(\boldsymbol{x})$ as the functional series (1.3), it is natural to truncate the latter after the p-tuple term in order to obtain certain approximations, $u^{(p)}(\boldsymbol{x})$, for $u(\boldsymbol{x})$. Such a truncation immediately brings forth two basic questions, common for all problems in which functional series are employed,

(i) How to rearrange the terms of the series so as the said truncations 'converge' to the field $u(\boldsymbol{x})$

$$u^{(p)}(\boldsymbol{x}) \to u(\boldsymbol{x}), \quad p \to \infty. \tag{1.4}$$

(ii) In what sense the convergence in (1.4) is to be understood.

In some important particular cases the answers to these two questions can be given if the Wiener idea [20] of orthogonalization of the series (1.3) in stochastic sense is invoked. For instance, if the input in (1.3) was the white Gaussian noise it turns out that we are to rearrange the terms in (1.3) so as the multivariate Hermite polynomials appear; then the convergence in (1.4) is L^2 with respect to the Wiener measure [4]. If the input is Poissonian, the Hermite polynomials are to be replaced by the Charlier ones [16]. The Charlier polynomials generate orthogonal functionals also for the case of the so-called perfect disorder of spheres [7]—Poisson-like system in which the points $\boldsymbol{x}_j$ are only forbidden to fall closer than a certain distance of $2a$. For a general point system $\boldsymbol{x}_j$, however, there is no algorithm how to render the functional series (1.3) orthogonal and, moreover, it is not clear whether this is possible at all. That is why the orthogonalization of the series (1.3) should be replaced by a somewhat weaker requirement that can be accomplished for a wide enough class of random point systems $\boldsymbol{x}_j$.

Guided by the well-known virial expansions in theory of liquid state and of theory of effective properties for particulate media, we propose after [7], [13] as such a weaker requirement the virial orthogonality, closely connected with virial convergence of the series (1.3) defined as follows. We say that the series (1.3) is virial-convergent if its truncation $u^{(p)}(\boldsymbol{x})$ gives results for all multipoint moments of $u(\boldsymbol{x})$ that are asymptotically correct to the order n^p at $n \to 0$, i.e., if $u^{(p)}(\boldsymbol{x})$ coincides with $u(\boldsymbol{x})$ in statistical sense [2] to the order n^p, $p = 1, 2, \cdots$; we recall that n is the number density of the set $\boldsymbol{x}_j$.

The central result of this paper is an answer to the basic questions (i) and (ii) for an wide class of point sets $\boldsymbol{x}_j$, introduced in §2. Namely, it appears that we are to expand $u(\boldsymbol{x})$ with respect to certain special fields, generated by the random density field $\psi(\boldsymbol{x})$, and called factorial fields (§2); the respective expansion is called factorial functional series. The truncations $u^{(p)}(\boldsymbol{x})$ of the factorial series then converge to u(x) in the above defined virial sense (§3). In turn, the factorial series could be made n^p-virial orthogonal for any given $p = 1, 2, \cdots$, in the sense that the average value of the product of any pair of functionals of different order has the magnitude $o(n^p)$. Such a virial orthogonalization is not already of crucial importance; however, it simplifies in some cases the procedure of identification of the needed kernels in the factorial series.

2. The factorial fields of a system of random points. Let us imagine a certain process by means of which one produces a family of random sets $\boldsymbol{x}_j$ with different number densities n, $n < \infty$. The points $\boldsymbol{x}_j$ are assumed indistinguishable, so that $\boldsymbol{x}_k \neq \boldsymbol{x}_j$, if $k \neq l$. The statistics of the sets $\boldsymbol{x}_j$, defined by the multipoint distribution functions f_k, will then depend on n as a parameter, i.e. $f_k = f_k(\boldsymbol{Y}_k; n)$, $\boldsymbol{Y}_k = (\boldsymbol{y}_1, \cdots, \boldsymbol{y}_k)$. If the process is 'smooth' enough, we may write

$$f_k(\boldsymbol{Y}_k; n) = \sum_{l=1}^{\infty} f_{kl}(\boldsymbol{Y}_k) n^l, \quad k \geq 1. \tag{2.1}$$

After [12] we introduce the class of point sets that comply with the condition:

(2.2) The distance between the nearest spheres tends to infinity as $n \to 0$.

This means that the process does not put the points $\boldsymbol{x}_j$ in rigid complexes, say, in pairs with a fixed distance between them; thus, there is no the so-called primary clustering of the points [12].

A simple analysis carried out by the author [12, p.II] shows that the assumption (2.2) yields $f_{kl} = 0$ at $l < k$, and therefore

$$f_k(\boldsymbol{Y}_k; n) = n^k f_{kk}(\boldsymbol{Y}_k) + o(n^k). \tag{2.3}$$

Let $\psi(\boldsymbol{x})$ be the random density field (1.1), generated by the random set $\boldsymbol{x}_j$. Consider the random fields

$$\begin{gathered}\Delta_\psi^{(0)} = 1, \quad \Delta_\psi^{(1)}(\boldsymbol{y}) = \psi(\boldsymbol{y}), \cdots, \\ \Delta_\psi^{(k)}(\boldsymbol{y}_1, \cdots, \boldsymbol{y}_k) = \psi(\boldsymbol{y}_1)[\psi(\boldsymbol{y}_1)\delta(\boldsymbol{y}_2 - \boldsymbol{y}_1)] \\ \cdots [\psi(\boldsymbol{y}_k) - \delta(\boldsymbol{y}_k - \boldsymbol{y}_1) - \cdots - \delta(\boldsymbol{y}_k - \boldsymbol{y}_{k-1})],\end{gathered} \tag{2.4}$$

$k = 2, 3, \cdots$, which we call the factorial fields or, briefly, the factorials for the set of random points $\boldsymbol{x}_j$. To the best of our knowledge, the fields (2.4) in this form were first introduced by Christov [5] whose aim was to have a simple general formula for the multivariate Charlier polynomials. It appears, as it will be seen below, that the factorials are of primary importance in the theory of 'point-wise' random media so that a closer look at their properties is warranted.

First, it is easily seen from the definition of the Dirac delta that the factorials are symmetric functions of their arguments. Moreover,

$$\Delta_\psi^{(k)}(\boldsymbol{y}, \cdots, \boldsymbol{y}_k) = \begin{cases} \psi(\boldsymbol{y}_1) \cdots \psi(\boldsymbol{y}_k), & \text{if } \boldsymbol{y}_i \neq \boldsymbol{y}_j, \\ 0, & \text{if } \boldsymbol{y}_i = \boldsymbol{y}_j \text{ for a pair } i \neq j, \end{cases} \tag{2.5}$$

$k = 2, 3, \cdots$ This formula explains, to a certain extent, why the fields $\Delta_\psi^{(k)}$ are called factorial. The exact meaning of the notations in the formula (2.5) needs, however, a careful analysis, because it tacitly includes products of delta-functions and it operates with values of distributions in fixed points. We shall start with a simple heuristic proof of (2.5).

Let $\boldsymbol{y}_i \neq \boldsymbol{y}_j$ for all i, j, $i \neq j$, then all deltas in (2.4) vanish and thus $\Delta_\psi^{(k)}(\boldsymbol{y}_1, \cdots, \boldsymbol{y}_k)$ indeed becomes the product of $\psi(\boldsymbol{y}_1)$ to $\psi(\boldsymbol{y}_k)$. Let now $\boldsymbol{y}_i = \boldsymbol{y}_j$ for a pair i, j, $i \neq j$; due to the above mentioned symmetry of the factorials we can always suppose that $\boldsymbol{y}_1 = \boldsymbol{y}_2$. In virtue of (2.4) we have

$$\begin{gathered}\Delta_\psi^{(k)}(\boldsymbol{y}_1, \cdots, \boldsymbol{y}_k) = \Delta_\psi^{(2)}(\boldsymbol{y}_1, \boldsymbol{y}_2) A(\boldsymbol{y}_1, \cdots, \boldsymbol{y}_k), \\ A = [\psi(\boldsymbol{y}_3) - \delta(\boldsymbol{y}_3 - \boldsymbol{y}_1) - \delta(\boldsymbol{y}_3 - \boldsymbol{y}_2)] \cdots \\ [\psi(\boldsymbol{y}_k) - \delta(\boldsymbol{y}_k - \boldsymbol{y}_1) - \cdots - \delta(\boldsymbol{y}_k - \boldsymbol{y}_{k-1})].\end{gathered}$$

Thus it suffices to show that

$$\Delta_\psi^{(2)}(\boldsymbol{y}_1, \boldsymbol{y}_2) = \sum_i \delta(\boldsymbol{y} - \boldsymbol{x}_i) \left[\sum_j \delta(\boldsymbol{y}_2 - \boldsymbol{x}_j) - \delta(\boldsymbol{y}_2 - \boldsymbol{y}_1) \right] \tag{2.6}$$

vanishes if $\boldsymbol{y}_1 = \boldsymbol{y}_2$. Let first the points $\boldsymbol{y}_1 = \boldsymbol{y}_2$ do not coincide with a point of the set $\boldsymbol{x}_j$, i.e., $\boldsymbol{y}_1 \neq \boldsymbol{x}_i$ for all i; then all $\delta(\boldsymbol{y}_1 - \boldsymbol{x}_i) = 0$ and thus the right-hand side of (2.6) vanishes as well. Now let $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ coincide with a certain point $\boldsymbol{x}_l$ from the set $\boldsymbol{x}_j$; then the only nonvanishing term in the right-hand side of (2.6) could be

$$\delta(\boldsymbol{y}_1 - \boldsymbol{x}_l) \left[\delta(\boldsymbol{y} - \boldsymbol{x}_l) - \delta(\boldsymbol{y}_1 - \boldsymbol{y}_2) \right], \tag{2.7}$$

which, however, also vanishes, since $\boldsymbol{y}_1 = \boldsymbol{y}_2$ and the multiplier in the square brackets is therefore zero.

The foregoing reasoning has the obvious flaw that it employs the 'naive' definition of the Dirac delta and, in particular, the notion of a value of a distribution in a fixed point. To remove this flaw and to expose at the same time the exact meaning of the formula (2.5), we recall the sequential approach in theory of distributions [1]. According to this approach, a distribution $G(\boldsymbol{y}_1, \boldsymbol{y}_2)$, in our case, on $R^3 \times R^3$, is an equivalency class, defined by a fundamental sequence $G_m(\boldsymbol{y}_1, \boldsymbol{y}_2)$; we denote $G = [G_m]$. (A fundamental sequence is a certain derivative, $G_m = \Phi_m^{(\alpha)}$, of the sequence $\Phi_m(\boldsymbol{y}_1, \boldsymbol{y}_2)$ that converges almost uniformly, i.e., uniformly on each closed and bounded subset of $R^3 \times R^3$; α is a fixed multi-index, see [1, Ch.2] for details.)

Let $D = \{(\boldsymbol{y}_1, \boldsymbol{y}_2) | \boldsymbol{y}_1 = \boldsymbol{y}_2\}$ be the diagonal of the Cartesian product $R^3 \times R^3$. We say that the distribution $G(\boldsymbol{y}_1, \boldsymbol{y}_2)$ has a trace, $G \mid_D = G(\boldsymbol{y}, \boldsymbol{y})$, on D if the restrictions $G_m \mid_D = [G_m](\boldsymbol{y}, \boldsymbol{y})$ form a fundamental sequence on D: $G(\boldsymbol{y}, \boldsymbol{y}) = [G_m(\boldsymbol{y}, \boldsymbol{y})]$, i.e., if $G(\boldsymbol{y}, \boldsymbol{y})$ define a distribution on D. (This definition of the trace of a distribution is an obvious generalization of the definition for the value $f(x_0)$ of a distribution at the point $x = x_0$ in the one-dimensional case, as given in [1, Ch.3, §12.2].) It is now easy to show that the trace of the second order factorial exists on D and vanishes

$$\Delta_\psi^{(2)}(\boldsymbol{y}_1, \boldsymbol{y}_2) \Big|_D = \Delta_\psi^{(2)}(\boldsymbol{y}, \boldsymbol{y}) = 0.$$

Indeed, let us define the three-dimensional delta-function $\delta(\boldsymbol{y})$ through the fundamental sequence $\delta_m(\boldsymbol{y})$ for which $\delta_m(\boldsymbol{y})$ have conical shape, vanishing at $|\boldsymbol{y}| > 1/m$, with the respective height at $\boldsymbol{y} = 0$ such that $\int \delta_m(\boldsymbol{y})\, \mathrm{d}^3\boldsymbol{y} = 1$, $m = 1, 2, \cdots$ According to (2.6), the fundamental sequence that defines $\Delta_\psi^{(2)}(\boldsymbol{y}_1, \boldsymbol{y}_2)$ has the form

$$\Delta_\psi^{(2)}(\boldsymbol{y}_1, \boldsymbol{y}_2) = \sum_i \delta_m(\boldsymbol{y}_1 - \boldsymbol{x}_i) \left[\sum_j \delta_m(\boldsymbol{y}_2 - \boldsymbol{x}_j) - \delta_m(\boldsymbol{y}_2 - \boldsymbol{y}_1) \right]. \tag{2.6a}$$

As a matter of fact, we shall repeat now the above 'naive' reasoning, but applied to (2.6a) instead of (2.6). Let $\boldsymbol{y}_1 = \boldsymbol{y}_2 = \boldsymbol{y}$ and $\boldsymbol{y}_1 \neq \boldsymbol{x}_i$ for all i. Then, starting with a certain m_0, $\delta_m(\boldsymbol{y} - \boldsymbol{x}_i) = 0$, for all i, because the sets $\boldsymbol{x}_j$ cannot possess points of condensation in R^3 due to their statistical homogeneity and the assumption $n < \infty$. Consequently, the traces of the functions (2.6a) on D vanish at $m > m_0$ which just means that the trace $\Delta_\psi^{(2)}(\boldsymbol{y}, \boldsymbol{y}) = 0$.

Let now $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ coincide with the point $\boldsymbol{x}_i$ from the set $\boldsymbol{x}_j$. Then, starting with a certain m_0, the only nonvanishing term in the right-hand side of (2.6a) could be

$$\delta_m(\boldsymbol{y}_1 - \boldsymbol{x}_i) \left[\delta_m(\boldsymbol{y}_2 - \boldsymbol{x}_i) - \delta_m(\boldsymbol{y}_2 - \boldsymbol{y}_1) \right], \tag{2.6b}$$

which obviously also vanishes, since $\boldsymbol{y}_1 = \boldsymbol{y}_2 = \boldsymbol{x}_i$. Consequently, the traces of the functions (2.6a) vanish on D at $m > m_0$ and thus $\Delta_\psi^{(2)}(\boldsymbol{y}, \boldsymbol{y}) = 0$ in this case as well.

Consider next the average values $\left\langle \Delta_\psi^{(k)}(\boldsymbol{y}_1, \cdots, \boldsymbol{y}_k) \right\rangle$ of the factorials, making use of (2.5) and (1.2). We note that if $\boldsymbol{y}_i \neq \boldsymbol{y}_j$ for all $i \neq j$, then the deltas in the r.h. sides of the formulae (1.2) vanish, so that

$$\left\langle \Delta_\psi^{(k)}(\boldsymbol{y}, \cdots, \boldsymbol{y}_k) \right\rangle = \begin{cases} f_k(\boldsymbol{y}_1, \cdots, \boldsymbol{y}_k), & \text{if } \boldsymbol{y}_i \neq \boldsymbol{y}_j, \\ 0, & \text{if } \boldsymbol{y}_i = \boldsymbol{y}_j \text{ for a pair } i \neq j. \end{cases} \tag{2.8}$$

Note that the last formula can be written in the simpler form

$$\left\langle \Delta_\psi^{(k)}(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_k)\right\rangle = f_k(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_k), \tag{2.8a}$$

because the probability densities $f_1(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_k)=0$, if $\boldsymbol{y}_i=\boldsymbol{y}_j$ for a pair $i\neq j$. The reason is that the points $\boldsymbol{x}_j$ cannot coincide, because they are undistinguishable by assumption, so that the probability for two of them to occupy one and the same spatial position is zero.

In a bit more general context (sets of marked random points) the formula (2.8a) was announced by Christov [5], who used for its proof the technique of generating functionals for point random sets [6]. The formula (2.8) turns out to be of crucial importance in our study of 'point-wise' random media. To demonstrate this we need first of all certain formulae for the products of the factorial fields. We start with the identity

$$\Delta_\psi^{(k)}(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_k)\Delta_\psi^{(1)}(\boldsymbol{y}_{k+1})$$

$$= [\delta(\boldsymbol{y}_{k+1}-\boldsymbol{y}_1)+\cdots\delta(\boldsymbol{y}_{k+1}-\boldsymbol{y}_k)]\Delta_\psi^{(k)}(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_k) \tag{2.9}$$

$$+\Delta_\psi^{(k+1)}(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_k,\boldsymbol{y}_{k+1}),$$

which is readily deduced from the definition (2.4). This identity illustrates well the more general relation which can be similarly derived, namely,

$$\Delta_\psi^{(k)}(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_k)\Delta_\psi^{(k)}(\boldsymbol{y}_{k+1},\cdots,\boldsymbol{y}_{k+l})$$

$$= D_1\Delta_\psi^{(m)}(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_m)+D_2\Delta_\psi^{(m+1)}(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_m,\boldsymbol{y}_{m+l})+\cdots \tag{2.9a}$$

$$+\Delta_\psi^{(k+l)}(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_k,\boldsymbol{y}_{k+1},\cdots,\boldsymbol{y}_{k+l}),$$

where $m=\max(k,l)$, D_1, $D_2,\cdots$, etc., are certain products of the Dirac deltas $\delta(\boldsymbol{y}_i-\boldsymbol{y}_j)$.

Upon averaging (2.9) and making use of (2.8), we can conclude that for the random sets $\boldsymbol{x}_j$ that obey the basic assumption (2.2) and, consequently, $f_k\sim n^k$ (cf. (2.3)), we have

$$\left\langle \Delta_\psi^{(k)}(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_k)\,\Delta_\psi^{(l)}(\boldsymbol{y}_{k+1},\cdots,\boldsymbol{y}_{k+l})\right\rangle = O(n^m), \tag{2.10a}$$

$m=max(k,l)$. The reasoning is fully similar also in the case of products of several factorials and the eventual result reads

$$\left\langle \Delta_\psi^{(k_1)}\Delta_\psi^{(k_2)}\cdots\Delta_\psi^{(k_p)}\right\rangle = O(n^m), \tag{2.10b}$$

where $m=\max(k_1,\cdots,k_p)$; for the sake of brevity, we omit the arguments of $\Delta_\psi^{(k_j)}$.

3. The factorial functional series and its virial property. Let us rearrange the terms of the series (1.3) replacing the products $\psi(\boldsymbol{y}_1)\cdots\psi(\boldsymbol{y}_1)$ by the factorials $\Delta_\psi^{(k)}(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_k)$, defined in (2.4). Thus we represent $u(\boldsymbol{x})$ as

$$u(\boldsymbol{x}) = T_0(\boldsymbol{x})+\int T_1(\boldsymbol{x}-\boldsymbol{y}_)\Delta_\psi^{(1)}(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}$$

$$+\iint T_2(\boldsymbol{x}-\boldsymbol{y}_1,\boldsymbol{x}-\boldsymbol{y}_2)\Delta_\psi^{(2)}(\boldsymbol{y}_1,\boldsymbol{y}_2)\,\mathrm{d}^3\boldsymbol{y}_1\mathrm{d}^3\boldsymbol{y}_2+\cdots; \tag{3.1}$$

for the sake of brevity, we use the same notations T_i for the kernels. The series (3.1) will be called in what follows factorial. Due to the definition (2.4), the kernels T_i in (3.1) are linear combinations of the kernels T_i in (1.3) for $j>i$, $i=1,2,\cdots$. This means, let us point out, that an infinite reordering should take place when rearranging the series (1.3) into its factorial form (3.1).

Let $R_p(\boldsymbol{x})$ denotes the series (3.1) truncated after the p-tuple term, so that

$$u(\boldsymbol{x})=u^{(p)}(\boldsymbol{x})+R_p(\boldsymbol{x}),$$

$$R_p(\boldsymbol{x})=\sum_{k=p+1}^{\infty}\int\cdots\int T_k(\boldsymbol{x}-\boldsymbol{y}_1,\cdots,\boldsymbol{x}-\boldsymbol{y}_k)\,\mathrm{d}^3\boldsymbol{y}_1\cdots\mathrm{d}^3\boldsymbol{y}_k. \tag{3.2}$$

Due to the properties (2.10) of the factorials, it is easily seen that

$$\langle u(\boldsymbol{x}_1)\cdots u(\boldsymbol{x}_l)\rangle=\left\langle u^{(p)}(\boldsymbol{x}_1)\cdots u^{(p)}(\boldsymbol{x}_l)\right\rangle+o(n^p). \tag{3.3}$$

$p,l=0,1,\cdots$, i.e., all multipoint moments for $u(\boldsymbol{x})$ and for the truncation $u^{(p)}(\boldsymbol{x})$ differ by quantities of order higher than n^p. In other words, the random fields $u(\boldsymbol{x})$ and $u^{(p)}(\boldsymbol{x})$ are identical, in statistical sense, to the asymptotical order n^p.

Thus we can give the following simple answers to the basic questions (i) and (ii) (see §1) for the class of point random sets that satisfy the condition (2.2), namely:

Rearrange the terms in (1.3) so as to obtain the factorial functional expansion (3.1) of the field $u(\boldsymbol{x})$. Then the truncations $u^{(p)}(\boldsymbol{x})$ of the series (3.1) converge to $u(\boldsymbol{x})$ in the above introduced virial sense (cf. (3.3)).

This result makes clear the basic difference between the usual perturbation expansions for random media and the factorial series (3.1). Namely, the role of the small parameter for the perturbation expansion is played by $\delta\kappa=\max_x|\kappa(\boldsymbol{x})|/\langle\kappa\rangle$ instead of the number density n of the inclusions or, which is the same, the volume fraction c of the particulate constituent for a two-phase 'point-wise' medium. The truncation of the said expansion after the p-tuple term yields full statistical description of the expanded field, correct to the asymptotical order $(\delta\kappa)^p$, see [10, p.I] and [12, p.I], but not to the order c^p, if the volume fraction c of the particulate constituents is small. The latter is well seen from the perturbation evaluation of the effective permittivity, κ^*, for the Miller cell material [15], performed in [10, p.I]. In this case $\delta\kappa=[\kappa]/\langle\kappa\rangle$, $[\kappa]=\kappa_2-\kappa_1$, where κ_2 and κ_1 are the permittivities of the constituents with volume fractions c and $1-c$ respectively. It appears that if the cells are spherical

$$\kappa^*=\langle\kappa\rangle\left\{1-\frac{\langle\kappa'^2\rangle}{3\langle\kappa\rangle^2}+\frac{\langle\kappa'^3\rangle}{3\langle\kappa\rangle^3}\right\}+o((\delta\kappa)^3), \tag{3.4a}$$

$$\left\langle\kappa'^2\right\rangle=c(1-c)[\kappa]^2,\quad\left\langle\kappa'^3\right\rangle=c(1-c)(1-2c)[k]^3, \tag{3.4b}$$

where the second and third terms in the right-hand side of (3.4a) are the contributions of the two- and three-tuple terms of the perturbation expansion respectively. From (3.4) it is now

obvious that the latter two terms contribute quantities of the order $O(c)$ to κ^* so that the perturbation expansion is not virial in the above sense even for Miller cell materials.

It is to be noted that the notion of viriality for a functional expansion was introduced in [7]; the central result of this work states that for a special random system—the so-called perfect disorder of spheres—the multivariate Charlier polynomials, generated by the random density field $\psi(\boldsymbol{x})$, yield a virial and, moreover, virial-orthogonal functional series. This result was generalized in [13], where it was shown that for the class of point sets that comply with (2.2), the Charlier polynomials generate virial series whose virial-orthogonality can be achieved by means of a kind of orthogonalization. The above proved viriality of the factorial series puts everything on its own place: the fundamental notion appear to be the factorial fields; their orthogonalization in stochastic sense will make the factorial series virial-orthogonal and will bring forth certain generalizations of the Charlier polynomials, as we shall see in §4.

One more remark concerning the factorial series (3.1): As is seen from (1.1) and (2.5), the p-tuple term in (3.1) is the sum

$$\sum T_p(\boldsymbol{x}-\boldsymbol{x}_{k_1},\cdots,\boldsymbol{x}-\boldsymbol{x}_{k_p}), \tag{3.5}$$

taken over all subsets comprising p different points from the set $\boldsymbol{x}_j$, i.e., $\boldsymbol{x}_{k_l}\neq\boldsymbol{x}_{k_n}$ at $l\neq n$. This is the basic difference from the initial series (1.3) in which the p-tuple term corresponds to the sum (3.5) taken over all subsets including those with coinciding points. We thus may say using, somewhat loosely, the terminology of Hori and Yonezawa [10, p.III], that the factorial series (3.1) represents a natural consequence of the initial series (1.3) if the 'exclusion effect' is taken into account in the sense that the points in the sums (3.5) are forbidden to coincide. The importance of a similar 'exclusion effect was demonstrated by Hori and Yonezawa when evaluating the effective permittivity of the so-called completely random medium. It should be also pointed out that the 'exclusion effect' is one of the premises of the cluster expansion evaluation of the effective properties of particulate media, in which the result of the mutual interaction, say, of pairs of particles is represented as a sum over all particles that do not coincide with the reference one, see, e.g., [9].

4. Identification of the kernels. The identification of the kernels T_i in the factorial series (3.1) can be performed by means of the procedure, successfully employed in [5], [7], [13] et al. in some particular situations. The procedure consists in the following. Suppose we want to determine the field $u(\boldsymbol{x})$ to the order n^p only, $p=1,2,\cdots$. Then the truncation $u^{(p)}(\boldsymbol{x})$ of the factorial series is solely needed and thus the kernels $T_0,T_1,\cdots,T_p$ are to be specified. Consider the equation that governs $u(\boldsymbol{x})$, insert there $u^{(p)}(\boldsymbol{x})$ instead of $u(\boldsymbol{x})$, multiply by 1, $\Delta_\psi^{(1)}(0),\cdots,\Delta_\psi^{(p)}(0,\boldsymbol{z}_1,\cdots,\boldsymbol{z}_{p-1})$ and average the results. Making use of the formulae for the averages of the respective products of the factorials (they are straightforward consequences of (1.2) and (2.4) (cf., e.g., (4.2) below), and truncating them to the same order n^p , we get a system of $p+1$ equations for the needed kernels T_0 to T_p. The concrete form of this system is specific for the problem under study; however, three important features, common for all such systems and independent of the physical background of the problems and their governing equations, could be traced out.

First, the kernels T_i are defined over unbounded regions and, therefore, instead of boundary conditions we should require that they be finite everywhere and have summable squares on R^3.

Second, the systems for the kernels T_0 to T_p can be split into sets of independent equations that can be solved one after another, if the needed kernels are expanded in powers of n as

follows:

$$T_i = T_{i0} + nT_{i1} + \cdots + n^k T_{ik}, \tag{4.1}$$

$k = p - i$, $i = 0, 1, \cdots, p$, i.e., if the solution is looked for again in a truncated virial form.

Third, the leading coefficients T_{i0} in (3.1), i.e., the zero-density limit of the kernels T_i, appear to be connected, at least in the case of linear governing equations, with the respective single, double, triple, etc., point interaction fields in the medium, in a certain sense imposed by the physical context of these equations. This fact implies that the k-tuple term of the factorial series which is accountable for the n^k-contribution to the random field $u(\boldsymbol{x})$ results, loosely speaking, from interactions of groups of l points up to $l = k$. In this sense the factorial series (3.1) could be viewed as a cluster (or group) expansion for $u(\boldsymbol{x})$ because the latter is broken up in a sum of consecutive terms that result from interactions within successively larger groups of 'inhomogeneities', represented by the points of the random set $\boldsymbol{x}_j$. Such cluster expansions have been widely used, e.g., in the theory of random dispersions, see [8], [9], [11] et al. However, in all these works the cluster expansions concern the effective properties only and not the full statistical description of the random fields under study, and they are introduced by a certain heuristic reasoning unconvincing, to the author's view, in general. In our approach such a group or cluster interpretation emerges in a natural way as a kind of a by-product of the virial property of the factorial series (3.1).

We shall illustrate the above said in §5 on the example of a steady-state diffusion problem in a random dispersion of spheres—a classical random medium of 'point-wise' type. The analysis will be detailed for $p = 2$, i.e., to the order n^2. The reasons for such a choice of p are twofold. First, it suffices to demonstrate the needed technique. Second, the length of calculations is kept within reasonable limits and tangible results are reached at the same time.

The n^2-analysis below, and in similar other problems [13], [14], is facilitated if an n^2-orthogonal system of basic fields is employed. The latter comprises the following linear combinations of the factorials

$$D_\psi^{(0)} = 1, \quad D_\psi^{(1)}(\boldsymbol{y}) = \Delta_\psi^{(1)}(\boldsymbol{y}) - n = \psi'(\boldsymbol{y}),$$

$$D_\psi^{(2)}(\boldsymbol{y}_1, \boldsymbol{y}_2) = \Delta_\psi^{(2)}(\boldsymbol{y}_1, \boldsymbol{y}_2) - ng_0(\boldsymbol{y}_{12}) \left[D_\psi^{(1)}(\boldsymbol{y}_1) + D_\psi^{(1)}(\boldsymbol{y}_2) \right] - n^2 g_0(\boldsymbol{y}_{12}),$$

$$D_\psi^{(p)}(\boldsymbol{y}_1, \cdots, \boldsymbol{y}_p) = \Delta_\psi^{(p)}(\boldsymbol{y}_1, \cdots, \boldsymbol{y}_p), \quad p = 3, 4, \cdots. \tag{4.2}$$

Here $\psi'(\boldsymbol{y})$ is the fluctuating part of $\psi(\boldsymbol{y})$ and

$$g_0(\boldsymbol{y}) = f_{22}(\boldsymbol{y}) = g(\boldsymbol{y}) + O(n), \tag{4.3}$$

so that $g_0(\boldsymbol{y})$ is the leading term, i.e., the zero-density limit, in the virial expansion of the usual radial distribution function $g(\boldsymbol{y}) = f_2(\boldsymbol{y})/n^2$ for the random set $\boldsymbol{x}_j$. Note that if the system $\boldsymbol{x}_j$ is Poissonian, then $g = g_0 = 1$ and $D_\psi^{(i)} = C_\psi^{(i)}$, $i = 0, 1, 2$, where $C_\psi^{(i)}$ are the Charlier polynomials [16]. As a consequence of (1.2) and (2.4), it can be easily verified that

$$\left\langle D_\psi^{(1)}(\boldsymbol{y}) \right\rangle = 0, \quad \left\langle D_\psi^{(2)}(\boldsymbol{y}_1, \boldsymbol{y}_2) \right\rangle = o(n^2),$$

$$\left\langle D_\psi^{(2)}(\boldsymbol{y}_1) D_\psi^{(2)}(\boldsymbol{y}_2, \boldsymbol{y}_3) \right\rangle = o(n^2). \tag{4.4}$$

These relations, together with (2.10a), obviously suffice to claim that the system (4.2) indeed is n^2-orthogonal.

Besides (4.4), we shall need in §5 the following formulae

$$\left\langle \Delta_\psi^{(1)}(\boldsymbol{y}) \right\rangle = n, \quad \left\langle \Delta_\psi^{(2)}(\boldsymbol{y}_1, \boldsymbol{y}_2) \right\rangle = n^2 g_0(\boldsymbol{y}_{12}),$$

$$\left\langle D_\psi^{(1)}(\boldsymbol{y}_1)\Delta_\psi^{(2)}(\boldsymbol{y}_2, \boldsymbol{y}_3) \right\rangle = n^2 g_0(\boldsymbol{y}_{23})[\delta(\boldsymbol{y}_{21}) + \delta(\boldsymbol{y}_{31})],$$

$$\left\langle D_\psi^{(1)}(\boldsymbol{y}_1)\Delta_\psi^{(1)}(\boldsymbol{y}_2)\Delta_\psi^{(1)}(\boldsymbol{y}_3) \right\rangle = n\delta(\boldsymbol{y}_{21})\delta(\boldsymbol{y}_{31}) - n^2\{\delta(\boldsymbol{y}_{21})R_0(\boldsymbol{y}_{21})\}_s + n^2[\delta(\boldsymbol{y}_{12}) + \delta(\boldsymbol{y}_{13})],$$

$$\left\langle D_\psi^{(2)}(\boldsymbol{y}_1, \boldsymbol{y}_2)\Delta_\psi^{(1)}(\boldsymbol{y}_3)\Delta_\psi^{(1)}(\boldsymbol{y}_4) \right\rangle = \left\langle D_\psi^{(2)}(\boldsymbol{y}_1, \boldsymbol{y}_2)\Delta_\psi^{(2)}(\boldsymbol{y}_3, \boldsymbol{y}_4) \right\rangle$$

$$= n^2 g_0(\boldsymbol{y}_{21}) \left[\delta(\boldsymbol{y}_{31})\delta(\boldsymbol{y}_{42}) + \delta(\boldsymbol{y}_{32})\delta(\boldsymbol{y}_{41})\right], \tag{4.5}$$

$$\left\langle D_\psi^{(2)}(\boldsymbol{y}_1, \boldsymbol{y}_2)\Delta_\psi^{(2)}(\boldsymbol{y}_3, \boldsymbol{y}_4)\Delta_\psi^{(1)}(\boldsymbol{y}_5 \right\rangle$$

$$= n^2 g_0(\boldsymbol{y}_{12})[\delta(\boldsymbol{y}_{51}) + \delta(\boldsymbol{y}_{52})][\delta(\boldsymbol{y}_{31})\delta(\boldsymbol{y}_{42}) + \delta(\boldsymbol{y}_{41})\delta(\boldsymbol{y}_{32})],$$

$\boldsymbol{y}_{ij} = \boldsymbol{y}_i - \boldsymbol{y}_j$, $R_0(\boldsymbol{y}) = 1 - g_0(\boldsymbol{y})$; they are correct to the order n^2 and represent straightforward consequences of (1.2), (2.4) and (4.3).

5. Example: sink-strength of a random lossy dispersion of spheres. Consider the steady-state diffusion equation

$$\Delta\phi(\boldsymbol{x}) - k^2(\boldsymbol{x})\phi(\boldsymbol{x}) + K = 0 \tag{5.1}$$

that governs, at the expense of some simplifying assumptions, the concentration $\phi(\boldsymbol{x})$ of a diffusing species, generated with the rate K, in an unbounded lossy medium. The medium is microinhomogeneous so that the sink strength parameter $k^2(\boldsymbol{x})$ varies rapidly and randomly with position. The physical context of (5.1) comes, e.g., from the field of irradiation damage, see [3], [19] for details and references. The basic problem, from the microscopical point of view, consists in evaluating through the given statistics of the random field $k^2(\boldsymbol{x})$ the effective (or overall) sink strength k^2 of the medium, defined after [19] as

$$k^{*2} = \frac{K}{\langle\phi(\boldsymbol{x})\rangle}. \tag{5.2}$$

Note that in the mentioned work [19] a number of novel and interesting results (self-consistent approximations, variational bounds, etc.) concerning k^{*2} are given. Our aim here is much more modest—simply to illustrate the performance of the factorial series in the study of the random equation (5.1).

Consider a dispersion of equisized and nonoverlapping spheres randomly distributed throughout an infinite matrix. Let $\boldsymbol{x}_j$ be the set of sphere centers; k_f^2 and k_m^2 denote the sink strengths of the spheres and of the matrix, respectively. The field $k^2(\boldsymbol{x})$ for the dispersion has then the simple form

$$k = k_m^2 + [k^2]\int h(\boldsymbol{x} - \boldsymbol{y})\Delta_\psi^{(1)}(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}, \tag{5.3}$$

where $h(\boldsymbol{x}) = 1$, if $|\boldsymbol{x}| < a$ and vanishes otherwise, a is the radius of the spheres, $[k^2] = k_f^2 - k_m^2$, and $\Delta_\psi^{(1)}(\boldsymbol{y}) = \psi(\boldsymbol{y})$ is the first-order factorial, i.e., simply the random density field for the point system $\boldsymbol{x}_j$ (cf. (2.4) and (1.1)). The requirement of nonoverlapping is incorporated in the properties of the multipoint distribution functions f_p, $p > 2$, for the set $\boldsymbol{x}_j$: they all should vanish if the distance between any pair of their arguments is less than the sphere diameter $2a$. In particular, for the zero-density limit $g_0(\boldsymbol{y})$ of the radial distribution function, defined in (4.3), we have

$$g_0(\boldsymbol{y}) = 0, \quad \text{if } |\boldsymbol{y}| < 2a. \tag{5.4}$$

We look for the solution to the random equation (5.1) in the form of the truncated factorial series

$$\begin{aligned} \phi^{(2)}(\boldsymbol{x}) &= T_0(\boldsymbol{x}) + \int T_1(\boldsymbol{x}-\boldsymbol{y})\Delta_\psi^{(1)}(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y} \\ &+ \iint T_2(\boldsymbol{x}-\boldsymbol{y}_1, \boldsymbol{x}-\boldsymbol{y}_2)\Delta_\psi^{(2)}(\boldsymbol{y}_1,\boldsymbol{y}_2)\,\mathrm{d}^3\boldsymbol{y}_1\mathrm{d}^3\boldsymbol{y}_2 \end{aligned} \tag{5.5}$$

with nonrandom kernels T_0, T_2, T_2 that depend also on the number density n of the spheres, i.e., of the set x of their centers. As shown in §3, the series (5.5), for the class of sets $\boldsymbol{x}_j$ that satisfy (2.2), gives full statistical description of the field $\phi(\boldsymbol{x})$ that is asymptotically correct to the order n^2, provided the kernels T_0, T_2, T_2 are properly identified. Moreover, the representation (5.3) for $k^2(\boldsymbol{x})$ together with (2.10) yield

$$\begin{aligned} &\Big\langle k^2(\boldsymbol{x}_1)\cdots k^2(\boldsymbol{x}_r)\,\phi(\boldsymbol{x}_{r+1})\cdots\phi(\boldsymbol{x}_{r+p})\Big\rangle \\ &= \Big\langle k^2(\boldsymbol{x}_1)\cdots k^2(\boldsymbol{x}_r)\,\phi^{(2)}(\boldsymbol{x}_{r+1})\cdots\phi^{(2)}(\boldsymbol{x}_{r+p})\Big\rangle + o(n^2), \end{aligned}$$

$r, l = 0, 1, \cdots$, and this just means that the truncation $\phi^{(2)}(\boldsymbol{x})$ is the solution of the random problem (5.1) in statistical sense, asymptotically correct to the order n^2, see [2, p.4].

Let us first average eqn (5.5). Making use of (1.2), (2.4) and (4.5), we get

$$\begin{aligned} \Big\langle \phi^{(2)}(\boldsymbol{x})\Big\rangle &= T_0(\boldsymbol{x}) + n\int T_1(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y} \\ &+ n^2\iint T_2(\boldsymbol{y}_1,\boldsymbol{y}_2)g_0(\boldsymbol{y}_1-\boldsymbol{y}_2)\,\mathrm{d}^3\boldsymbol{y}_1\mathrm{d}^3\boldsymbol{y}_2, \end{aligned} \tag{5.6}$$

which suggests that T_0 depends on the number density n of the spheres only, since $\Big\langle \phi^{(2)}(\boldsymbol{x})\Big\rangle =$ const due to the assumed statistical homogeneity of the dispersion.

To identify the kernels T_0, T_2 and T_2 we follow the general scheme described in §4. We insert (5.3) and (5.5) into (5.1), multiply by 1, $D_\psi^{(1)}(0)$, $D_\psi^{(2)}(0,\boldsymbol{z})$ and average the results. Keeping in mind (1.2), (2.4), (4.2), (4.4) and (4.5), we get the following equations

$$\begin{aligned} &T_0\left(k_m^2 + [k^2]Vn\right) + k_m^2 n\int T_1(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y} \\ &+[k^2]\left\{n\int h(\boldsymbol{y})T_1(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y} + n^2\iint h(\boldsymbol{y}_1)T_1(\boldsymbol{y}_2)g_0(\boldsymbol{y}_1-\boldsymbol{y}_2)\,\mathrm{d}^3\boldsymbol{y}_1\mathrm{d}^3\boldsymbol{y}_2\right\} \\ &+k_m^2 n^2\iint T_2(\boldsymbol{y}_1,\boldsymbol{y}_2)g_0(\boldsymbol{y}_1-\boldsymbol{y}_2)\,\mathrm{d}^3\boldsymbol{y}_1\mathrm{d}^3\boldsymbol{y}_2 \end{aligned}$$

$$+2[k^2]n^2\iint h(\boldsymbol{y}_1)T_2(\boldsymbol{y}_1,\boldsymbol{y}_2)g_0(\boldsymbol{y}_1-\boldsymbol{y}_2)\,\mathrm{d}^3\boldsymbol{y}_1\mathrm{d}^3\boldsymbol{y}_2 = K = 0, \tag{5.7a}$$

$$\begin{aligned}
&\Delta T_1(\boldsymbol{x}) - k_0^2(\boldsymbol{x})T_1(\boldsymbol{x}) - [k^2]\left[h(\boldsymbol{x}) - n\int h(\boldsymbol{x}-\boldsymbol{y})R_0(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}\right]T_0\\
&\qquad -n\int\left\{\Delta T_1(\boldsymbol{x}-\boldsymbol{y}) - k_0^2(\boldsymbol{x}-\boldsymbol{y})\right\}R_0(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}\\
&-n[k^2]\left\{h(\boldsymbol{x})\int T_1(\boldsymbol{x}-\boldsymbol{y})g_0(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y} + T_1(\boldsymbol{x})\int h(\boldsymbol{x}-\boldsymbol{y})g_0(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}\right\}\\
&+2n\left\{\int \Delta T_2(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{x})g_0(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y} - k_0^2(\boldsymbol{x})\int T_2(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{x})g_0(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}\right.\\
&\qquad \left.-[k^2]\int T_2(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{x})h(\boldsymbol{x}-\boldsymbol{y})g_0(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}\right\} = 0,
\end{aligned} \tag{5.7b}$$

$$\begin{aligned}
&2\Delta T_2(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{x}) - 2[k_0^2(\boldsymbol{x}) + [k^2]h(\boldsymbol{x}-\boldsymbol{z})]T_2(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{x})\\
&\qquad -[k^2]\left\{h(\boldsymbol{x})T_1(\boldsymbol{x}-\boldsymbol{z}) + h(\boldsymbol{x}-\boldsymbol{z})T_1(\boldsymbol{x})\right\} = 0;
\end{aligned} \tag{5.7c}$$

here $k_0^2(\boldsymbol{x}) = k_m^2 + [k^2]h(\boldsymbol{x})$, so that $k_0^2(\boldsymbol{x}) = k_f^2$, if $|\boldsymbol{x}| < a$ and $k_0^2(\boldsymbol{x}) = k_m^2$, otherwise, $V_a = \frac{4}{3}\pi a^3$. Everywhere in (5.7) the differentiation is with respect to $\boldsymbol{x}$; $\boldsymbol{y}$ and $\boldsymbol{z}$ play the role of parameters.

The equations (5.7) form a coupled system for the needed kernels $T_0(n)$, $T_1(\boldsymbol{y};n)$, $T_2(\boldsymbol{y}_1,\boldsymbol{y}_2;n)$; we underline here that the latter depend on the number density as well. To the order n^2, we look for the solution of the system (5.7) in the virial form

$$\begin{aligned}
T_0(n) &= T_{00} + nT_{01} + n^2T_{02},\\
T_1(\boldsymbol{y};n) &= T_{10}(\boldsymbol{y}) + nT_{11}(\boldsymbol{y}),\\
T_2(\boldsymbol{y}_1,\boldsymbol{y}_2;n) &= T_{20}(\boldsymbol{y}_1,\boldsymbol{y}_2);
\end{aligned} \tag{5.8}$$

the reason is that in the expressions for the averaged statistical characteristics of $\phi^{(2)}(\boldsymbol{x})$ the kernel T_1 always appears multiplied by n, and T_2—by n^2 (cf. (2.4), (4.1), (4.2), (4.5), (5.5) and (5.6)). The virial coefficients in the right-hand sides of (5.8) do not depend already on the number density n.

On introducing (5.8) into the system (5.7) and equaling the coefficients of the same degree of n we get

$$k_m^2T_{00} - K = 0, \quad i.e., \quad T_{00} = K/k_m^2, \tag{5.9a}$$

$$k_m^2T_{01} + V_a[k^2]T_{10} + k_m^2\int T_{10}(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y} + [k^2]\int h(\boldsymbol{y})T(y)T_{10}(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y} = 0, \tag{5.9b}$$

$$k_m^2T_{02} + V_a[k^2]T_{01} + k_m^2\int T_{11}(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}$$

$$+[k^2]\left\{\int h(\boldsymbol{y})T_{11}(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}+\iint h(\boldsymbol{y}_1)g_0(\boldsymbol{y}_1-\boldsymbol{y}_2)T_{10}(\boldsymbol{y}_2)\,\mathrm{d}^3\boldsymbol{y}_1\mathrm{d}^3\boldsymbol{y}_2\right\}$$

$$+k_m^2\iint T_{20}(\boldsymbol{y}_1,\boldsymbol{y}_2)g_0(\boldsymbol{y}_1-\boldsymbol{y}_2)\,\mathrm{d}^3\boldsymbol{y}_1\mathrm{d}^3\boldsymbol{y}_2$$

$$+2[k^2]\iint h(\boldsymbol{y}_1)g_0(\boldsymbol{y}_1-\boldsymbol{y}_2)T_{20}(\boldsymbol{y}_1,\boldsymbol{y}_2)\,\mathrm{d}^3\boldsymbol{y}_1\mathrm{d}^3\boldsymbol{y}_2=0, \tag{5.9c}$$

$$\Delta T_{11}(\boldsymbol{x})-k_0^2(\boldsymbol{x})T_{10}(\boldsymbol{x})-[k^2]T_{00}h(\boldsymbol{x})=0; \tag{5.9d}$$

$$\Delta T_{11}(\boldsymbol{x})-k_0^2(\boldsymbol{x})T_{11}(\boldsymbol{x})-[k^2]h(\boldsymbol{x})T_{01}+T_{00}[k^2]\int h(\boldsymbol{x}-\boldsymbol{y})R_0(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}$$

$$-\int\left[\Delta T_{10}(\boldsymbol{x}-\boldsymbol{y})-k_0^2(\boldsymbol{x}-\boldsymbol{y})T_{10}(\boldsymbol{x}-\boldsymbol{y})\right]R_0(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}$$

$$-[k^2]\left\{T_{10}(\boldsymbol{x})\int h(\boldsymbol{x}-\boldsymbol{y})g_0(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}+h(\boldsymbol{x})\int T_{10}(\boldsymbol{x}-\boldsymbol{y})g_0(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}\right\}$$

$$+2\int\left[\Delta T_{20}(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{x})-[k_0^2(\boldsymbol{x})+[k^2]h(\boldsymbol{x}-\boldsymbol{y})]T_{20}(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{x})\right]g_0(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}=0, \tag{5.9e}$$

$$2\Delta T_{20}(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{x})-2[k_0^2(\boldsymbol{x})+[k^2]h(\boldsymbol{x}-\boldsymbol{z})]T_{20}(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{x})$$

$$-[k^2]\left\{h(\boldsymbol{x})T_{10}(\boldsymbol{x}-\boldsymbol{z})+h(\boldsymbol{x}-\boldsymbol{z})T_{10}(\boldsymbol{x})\right\}=0. \tag{5.9f}$$

The solution of the system (5.9) is already easy. We note that (5.9e) drastically simplifies in virtue of (5.9d) and (5.9f), becoming

$$\Delta T_{11}(\boldsymbol{x})-k_0^2(\boldsymbol{x})T_{11}(\boldsymbol{x})-[k^2]T_{01}(\boldsymbol{x})h(\boldsymbol{x})=0. \tag{5.10}$$

Let $H^{(1)}(\boldsymbol{x})$ be the bounded spherically symmetric solution of the equation

$$\Delta H^{(1)}(\boldsymbol{x})-\left[k_m^2+[k^2]h(\boldsymbol{x})\right]H^{(1)}(\boldsymbol{x})-[k^2]h(\boldsymbol{x})=0, \tag{5.11}$$

continuous together with its normal derivative on the sphere $|\boldsymbol{x}|=a$. It is easily seen that

$$H^{(1)}(\boldsymbol{x})=\frac{[k^2]}{k_f^2}\begin{cases}A_1\dfrac{a_f\sinh\rho_f}{\rho_f\sinh a_f}, & \text{if } |\boldsymbol{x}|\le a,\\[2ex] A_2\dfrac{a_m}{\rho_m}\exp\left(a_m-\rho_m\right), & \text{if } |\boldsymbol{x}|>a,\end{cases} \tag{5.12a}$$

where

$$A_1=\frac{1+a_m}{a_m+a_f\coth a_f}\,,\quad A_2=\frac{1-a_f\coth a_f}{a_m+a_f\coth a_f}\,, \tag{5.12b}$$

with the nondimensional parameters introduced as follows

$$a_f=ak_f,\quad a_m=ak_m,\quad \rho_f=rk_f,\quad \rho_m=rk_m,\quad r=|\boldsymbol{x}|.$$

The field $H^{(1)}(\boldsymbol{x})$ is the single-sphere solution in the problem under study; it gives the concentration of the diffusing species in an unbounded matrix, generated by sources of power proportional to the jump $[k^2]$, and located within the spherical inhomogeneity of radius a at the origin (cf. (5.11)).

From (5.9b), (5.9d), (5.10) and (5.11) we readily obtain

$$T_{10}(\boldsymbol{x}) = T_{00}H^{(1)}(\boldsymbol{x}), \quad T_{01} = 0, \quad T_{11}(\boldsymbol{x}) = 0, \tag{5.13}$$

because $T_{11}(\boldsymbol{x}) = T_{01}H^{(1)}(\boldsymbol{x})$ (cf. (5.10) and (5.11)). Thus the kernel $T_1(\boldsymbol{x})$ of the one-tuple term in (5.5) appears to be proportional, to the order $o(n^2)$, to the single-sphere solution H (x) in the sink-strength problem under study.

Consider finally the equation (5.9f) for the kernel T_{20}. Making use of (5.9d), it may be easily verified that

$$T_{20}(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{x}) = T_{00}H_{20}(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{x}),$$

$$2H_{20}(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{x}) = H^{(2)}(\boldsymbol{x};\boldsymbol{z}) - H^{(1)}(\boldsymbol{x}) - H^{(1)}(\boldsymbol{x}-\boldsymbol{z}), \tag{5.14}$$

where the field $H^{(2)}(\boldsymbol{x};\boldsymbol{z})$ is the bounded everywhere solution of the equation

$$\Delta H^{(2)}(\boldsymbol{x};\boldsymbol{z}) - \left[k_m^2 + [k^2]\,(h(\boldsymbol{x}) + h(\boldsymbol{x}-\boldsymbol{z}))\right] H^{(2)}(\boldsymbol{x};\boldsymbol{z})$$

$$-[k^2]\,(h(\boldsymbol{x}) + h(\boldsymbol{x}-\boldsymbol{z})) = 0,$$

continuous together with its normal derivatives on the spheres $|\boldsymbol{x}| = a$ and $|\boldsymbol{x}-\boldsymbol{z}| = a$.

Obviously, the field $H^{(2)}(\boldsymbol{x};\boldsymbol{z})$ is the two-sphere solution of our problem in the sense that it represents the concentration of the diffusing species in an unbounded matrix, generated by sources of power proportional to the jump $[k^2]$, and located within two spherical inhomogeneities of radii a—one at the origin, the other at the point $\boldsymbol{z}$, $|\boldsymbol{z}| \geq 2a$ (cf. (5.15)). According to (5.15), the kernel of the two-tuple term in (5.5), to the order $o(n^2)$, is proportional to the field that should be added to the single-sphere solutions $H^{(1)}(\boldsymbol{x})$ and $H^{(1)}(\boldsymbol{x}-\boldsymbol{z})$, generated by the two spheres (at the origin and at the point $\boldsymbol{z}$ respectively), in order to obtain the two-sphere solution $H^{(2)}(\boldsymbol{x};\boldsymbol{z})$. Note that the foregoing interpretation of the virial coefficients $T_{10}(\boldsymbol{x})$ and $T_{20}(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{z})$ is fully similar to that in the heat conduction problem for the dispersion; however, $T_{11}(\boldsymbol{x})$ does not vanish in the latter case and its evaluation constitutes the most difficult problem there, see [7], [13], and [14] for details.

The relations (5.8), (5.9a), (5.9c), (5.13) and (5.14) determine the kernels in (5.5) to the order n^2. Thus the solution $\phi(\boldsymbol{x})$ of the random equation (5.1), in statistical sense, is obtained to the order n as the truncated factorial series $\phi^{(2)}(\boldsymbol{x})$ in (5.5). This allows to calculate all statistical characteristics of $\phi(\boldsymbol{x})$ to the same order n^2, replacing it by $\phi^{(2)}(\boldsymbol{x})$. For example,

$$\langle\phi(\boldsymbol{x})\rangle = \left\langle\phi^{(2)}(\boldsymbol{x})\right\rangle + o(c^2)$$

$$= \frac{K}{k_m^2}\,(1 + b_1 c + b_2 c^2) + o(c^2),$$

$$b_1 = -\frac{[k^2]}{k_f^2}\left(1 - 3\frac{[k^2]}{k_f^2}\,\frac{1+a_m}{a_f^2}\,\frac{1 - a_f\coth a_f}{a_m + a_f\coth a_f}\right), \tag{5.16}$$

$$b_2 = -\frac{[k^2]}{k_m^2 V_a^2}\int h(\boldsymbol{y}_1)\left\{\int g_0(\boldsymbol{y}_1-\boldsymbol{y}_2)H^{(1)}(\boldsymbol{y}_2)\,\mathrm{d}^3\boldsymbol{y}_2 + 2I_{20}(\boldsymbol{y}_1)\right\}\mathrm{d}^3\boldsymbol{y}_1,$$

where

$$I_{20}(\boldsymbol{y}) = \int g_0(\boldsymbol{y}-\boldsymbol{z})H_{20}(\boldsymbol{y},\boldsymbol{z})\,\mathrm{d}^3\boldsymbol{z}, \tag{5.17}$$

and $c = nV_a$ is the volume fraction of the spheres. In turn, (5.2) and (5.16) allow us to evaluate the effective sink strength of the dispersion to the order $o(c^2)$, i.e., $o(n^2)$,

$$k^{*2} = k^2\left(1 + a_1 c + a_2 c^2\right) + o(c^2),$$

$$a_1 = -b_1, \quad a_2 = -\frac{b_2}{1+b_1}. \tag{5.17}$$

Note that to the order c, i.e., for dilute dispersions, the relations (5.18) coincides with the self-consistent formula of Brailsdorf and Bullough [3], see [19, p.I].

With the same ease the multipoint correlation functions for the field $\phi(\boldsymbol{x})$ can be found to the order c^2 in a closed form by means of the function $g_0(\boldsymbol{y})$ and of the integral $I_{20}(\boldsymbol{y})$, defined in (5.17). Of course, the explicit evaluation of these functions and, in particular, of the average $\langle\phi(\boldsymbol{x})\rangle$ in (5.16), need above all explicit expressions for the two-sphere solution $H^{(2)}(\boldsymbol{x};\boldsymbol{z})$ and for $I_{20}(\boldsymbol{y})$. This is not an easy task; however, the difficulties to be encountered already concern deterministic problems and are of purely analytical nature.

The foregoing example demonstrates well that the factorial series are indeed a powerful tool for analyzing random problems for 'point-wise' media, because they allow to obtain stochastic solutions with controlled asymptotic accuracy in powers of the number density of the inclusions. From this example it is also clear that the term 'point-wise' does not restrict the analysis to media with inclusions of infinitesimal size. (Cellular media of this type were considered in [10, p.III].) We could rather say that the 'point-wise' media, as understood here, are media whose internal constitution is generated by a random set of points $\boldsymbol{x}_j$; each point, in turn, is endowed with 'marks' that may represent the size, certain shape factors, orientations, etc., for the inclusion associated with the point.

Another nontrivial example of successful application of the factorial series is supplied by the classical problem of heat conduction through a random dispersion of spheres. As a matter of fact this is done, in a brief form, in [13], though the multi-variate Charlier polynomials are the starting point there instead of the simpler and more fundamental factorial fields for the random point sets $\boldsymbol{x}_j$. The detailed statistical c^2-solution of the heat conduction problem is considered elsewhere [14] because it needs a more involving and lengthy analysis of the respective system for the kernels T_1 and T_2.

Acknowledgement. It is a pleasant author's duty to thank Dr C. I. Christov for many stimulating discussions concerning the topic of this work.

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