Estimates for the sedimentation speed

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The classical problem for a cloud of identical spheres settling steadily in an unbounded incompressible fluid is revisited and a rigorous upper bound on the sedimentation speed is derived. Within the frame of the Rayleigh analogy between equations of linear (incompressible) elasticity and Stokesian hydrodynamics a more general Robin's type problem in elastostatics is first discussed, in which an array of rigid spheres, cemented into an elastic matrix, are displaced due to a given force. Instead of a variational principle a Bounjakowsky-Schwarz type inequality is applied in order to avoid the quest for trial fields that satisfy the boundary conditions. The energy of the system is thus explicitly estimated which generates a lower bound on the effective constant α^* that interconnects the force with the mean displacement of the spheres. In the incompressible case, invoking the Rayleigh analogy, a rigorous bound on the familiar hindered settling speed of the dispersion immediately follows. The bound turns out to coincide with the 'far-field' approximation for the sedimentation speed, proposed by Brady & Durlofsky.

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1. Random elastostatic Robin's problems

Consider a macroscopic piece $\mathcal{V}$ of an elastic dispersion of spheres, i.e. an (isotropic) elastic matrix (phase 2) that contains a random array of small rigid spherical particles (phase 1). The array is assumed statistically homogeneous and isotropic. The regions, occupied by the phases, are denoted $\mathcal{K}_1$ and $\mathcal{K}_2$ and their characteristic functions are $I_1(\boldsymbol{x})$ and $I_2(\boldsymbol{x})$, respectively. Hence $\eta_k = \langle I_k(\boldsymbol{x})\rangle$ is the volume fraction of the phase 'k', $k = 1, 2$. There are no displacements on the boundary of $\mathcal{V}$:

$$\boldsymbol{u}\Big|_{\partial\mathcal{V}} = 0. \tag{1.1}$$

In statistical setting, to be employed in what follows, the condition (1.1) will be replaced by

$$\langle \boldsymbol{u}(\boldsymbol{x})\rangle = 0 \tag{1.2}$$

with the brackets $\langle\cdot\rangle$ denoting ensemble averaging. In such a setting the piece $\mathcal{V}$ with a specific distribution of particles will be treated as one of the realizations of the random dispersion.

Assume that a constant force $\boldsymbol{F}_0$ is applied to each sphere. As a result displacement $\boldsymbol{u}(\boldsymbol{x})$ and stress tensor $\boldsymbol{T}_\sigma(\boldsymbol{x})$ fields appear in the matrix. The displacement satisfies the Lamé equations

$$L[\boldsymbol{u}] = \mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} = 0, \quad \boldsymbol{x}\in K_2, \tag{1.3}$$

$L[\cdot]$ is Lamé's operator.

In turn, the condition that the force acting on the spheres is $\boldsymbol{F}_0$ reads

$$\boldsymbol{F}_0 = \int_{\mathcal{S}_\alpha} \boldsymbol{\sigma}_n \, \mathrm{d}s, \quad \boldsymbol{\sigma}_n = \boldsymbol{T}_\sigma \cdot \boldsymbol{n}, \tag{1.4}$$

$\alpha = 1, 2, \ldots$ Here $\mathcal{S}_\alpha$ is the surface of the α-th sphere $\mathcal{V}_\alpha$ and $\boldsymbol{n}$ is its outward unit vector.

Due to the applied force $\boldsymbol{F}_0$ the spheres, assumed cemented into the matrix, undergo rigid displacements $\boldsymbol{U}_\alpha$

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{U}_\alpha, \quad \boldsymbol{x} \in \boldsymbol{V}_\alpha, \tag{1.5}$$

$\alpha = 1, 2, \ldots$. Using (1.5) we extend the displacement field $\boldsymbol{u}(\boldsymbol{x})$, defined in the matrix by means of Lamé equations (1.3), over the whole $\mathbb{R}^3$, by taking $\boldsymbol{u}(\boldsymbol{x}) \equiv \boldsymbol{U}_\alpha$ within the α-th sphere $\mathcal{V}_\alpha$.

The mean displacement of the spheres is

$$\overline{\boldsymbol{U}}_0 = \frac{1}{N} \sum_\alpha \boldsymbol{U}_\alpha \tag{1.6}$$

for a given realization, containing N spheres or, in statistical setting,

$$\overline{\boldsymbol{U}}_0 = \frac{1}{\eta_1} \left\langle I_1(\boldsymbol{x}) \boldsymbol{u}(\boldsymbol{x}) \right\rangle. \tag{1.7}$$

Moreover,

$$\boldsymbol{F}_0 = \alpha^* \overline{\boldsymbol{U}}_0 \tag{1.8}$$

with a certain effective (macroscopic) constant α^* that depends on the shear modulus μ and Poisson ratio ν of the matrix, as well as on the statistical properties of the array of rigid spheres.

The problem (1.2), (1.3), (1.4) will be called *the random elastostatic Robin's problem.* The reason is that it represents an obvious multi-particle generalization of the electrostatic Robin's problem. The constancy of the potential in the latter context on the conductors' surfaces—the spheres $\mathcal{S}_\alpha$—corresponds to rigid displacement of the balls $\mathcal{V}_\alpha$; the lack of electric force within $\mathcal{V}_\alpha$ is 'translated' as absence of stresses there, see, e.g., Lur'e (1970), Sec. IV.4.7. In the simplest particular case a single rigid sphere is cemented within unbounded elastic medium and undergoes a fixed displacement $\boldsymbol{U}_0$ due to the applied force $\boldsymbol{F}_0$. Then

$$\boldsymbol{F}_0 = A_0 \boldsymbol{U}_0, \quad A_0 = \frac{24\pi\mu(1-\nu)}{5-6\nu} \, a, \tag{1.9}$$

see again Lur'e (1970), Eq. (V.3.3.9); hereafter a stands for the radius of the spheres.

It is noted that a problem that differs a bit from (1.2), (1.3), (1.4), was formulated by the author (Markov, 1991) who named it also random elastostatic Robin's problem. There the spheres undergo fixed displacements $\boldsymbol{U}_0$, one and the same for all of them, and one looks for the mean force $\overline{\boldsymbol{F}}_0$ that generates them.† This problem will not be discussed here.

† In fluid context, see below, this corresponds to the steady flow through a fixed bed of spherical obstacles.

As is well known there exists a formal analogy between the Lamé equations for an elastic solid and the quasi-static Stokesian equations for a viscous incompressible fluid (provided the solid is incompressible as well, i.e. its Poisson ratio $\nu = \frac{1}{2}$). This analogy was pointed out by Lord Rayleigh (1894), but rarely exploited in the literature. A notable exception is the work of by Hill & Power (1956) who employed it to obtain variational estimates on the drag of some bodies of complicated shape in a viscous flow. Within the frame of Rayleigh's analogy the Robin's problem (1.2), (1.3), (1.4) exactly corresponds to the classical sedimentation problem: a cloud of rigid particles, subject to a constant (gravitational) force, falls steadily throughout an unbounded fluid at rest at infinity. The counterpart of (1.8) in this case reads

$$\boldsymbol{F}_0 = \alpha^* \overline{\boldsymbol{V}}_0, \tag{1.10}$$

where $\overline{\boldsymbol{V}}_0$ is the mean steady-state velocity of the particles and $\boldsymbol{F}_0$ is the force applied to them; α^* is interpreted as a sedimentation constant. Recall that in the dilute case, when each sphere can be viewed as single, the Stokes formula holds, so that

$$\alpha^* = 6\pi\mu a. \tag{1.11}$$

Eq. (1.11) follows immediately from (1.9) at $\nu = \frac{1}{2}$, as it should be. The only difference is that μ in (1.11) is interpreted now as the fluid viscosity.

It is noted that all considerations below can be performed using fluid context from the very beginning. The Lamé equation is to be replaced then with the Stokes one, taking into account the incompressibility condition $\nabla \cdot \boldsymbol{v} = 0$ and the corresponding Lagrange multiplier—the pressure p. The Kelvin-Somigliana tensor will then generate Stokeslets in §§3 and 4, etc. In author's view however it seems more consistent logically to start with the more general Robin's problem and only in the final stage let $\nu = \frac{1}{2}$ and invoke fluid context.

2. The basic inequality and some of its consequences

Let $\boldsymbol{T}_e$ and $\boldsymbol{T}'_e$ be arbitrary (symmetric) tensor fields of second rank, statistically homogeneous, not necessarily generated as small strain tensors by means of certain displacement fields. Let in turn $\boldsymbol{H}$ be a fourth-rank tensor of elastic moduli, i.e. $\boldsymbol{H}$ is strictly

positive-definite and possesses the familiar internal and external symmetries.

Denote

$$\boldsymbol{A} = \sqrt{\boldsymbol{H}} : \boldsymbol{T}_e, \quad \boldsymbol{A}' = \sqrt{\boldsymbol{H}} : \boldsymbol{T}'_e$$

and invoke the obvious inequality

$$\big\langle (\boldsymbol{A} - \boldsymbol{A}') : (\boldsymbol{A} - \boldsymbol{A}') \big\rangle = \big\langle \boldsymbol{A} : \boldsymbol{A} - 2\boldsymbol{A} : \boldsymbol{A}' + \boldsymbol{A}' : \boldsymbol{A}' \big\rangle \geq 0. \tag{2.1}$$

The semicolon indicates contraction with respect to two pair of indices. In virtue of the assumed symmetry properties of the tensors $\boldsymbol{A}$, $\boldsymbol{A}'$ and $\boldsymbol{H}$ one has

$$\boldsymbol{A} : \boldsymbol{A} = \left(\sqrt{\boldsymbol{H}} : \boldsymbol{T}_e\right) : \left(\sqrt{\boldsymbol{H}} : \boldsymbol{T}_e\right) = \boldsymbol{T}_e : \boldsymbol{H} : \boldsymbol{T}_e$$

and similarly for $\boldsymbol{A} : \boldsymbol{A}'$ and $\boldsymbol{A}' : \boldsymbol{A}'$. Thus (2.1) can be recast as

$$\langle \boldsymbol{T}_\sigma : \boldsymbol{T}_e \rangle \geq 2 \langle \boldsymbol{T}'_\sigma : \boldsymbol{T}_e \rangle - \langle \boldsymbol{T}'_\sigma : \boldsymbol{T}'_e \rangle, \tag{2.2}$$

where

$$\boldsymbol{T}_\sigma = \boldsymbol{H} : \boldsymbol{T}_e, \quad \boldsymbol{T}'_\sigma = \boldsymbol{H} : \boldsymbol{T}_e \tag{2.3}$$

are the 'stress' fields associated with $\boldsymbol{T}_e$ and $\boldsymbol{T}'_e$, respectively, by means of the 'elastic' tensor $\boldsymbol{H}$.

Let $\boldsymbol{T}'_e = \lambda\widetilde{\boldsymbol{T}}_e$, where $\widetilde{\boldsymbol{T}}_e$ is a fixed (but otherwise arbitrary) symmetric tensor field and λ is an adjustable scalar. For such a choice of $\boldsymbol{T}'_e$ the right-hand side of (2.2) becomes a quadratic function of λ whose optimization yields

$$\langle \boldsymbol{T}_\sigma : \boldsymbol{T}_e \rangle \geq \frac{\left\langle \widetilde{\boldsymbol{T}}_\sigma : \boldsymbol{T}_e \right\rangle^2}{\left\langle \widetilde{\boldsymbol{T}}_\sigma : \widetilde{\boldsymbol{T}}_e \right\rangle}. \tag{2.4}$$

Equality sign is achieved if the tensors $\widetilde{\boldsymbol{T}}_e$ and $\boldsymbol{T}_e$ are proportional.

The inequality in (2.4) is the starting point of our study. It obviously represents a certain tensorial counterpart of the familiar Bounjakowsky-Schwarz inequality.

Observe that (2.4) resembles a variational statement as far as the functional

$$\mathcal{F}\left[\boldsymbol{T}_e, \widetilde{\boldsymbol{T}}_e\right] = \langle \boldsymbol{T}_\sigma : \boldsymbol{T}_e \rangle - \frac{\left\langle \widetilde{\boldsymbol{T}}_\sigma : \boldsymbol{T}_e \right\rangle^2}{\left\langle \widetilde{\boldsymbol{T}}_\sigma : \widetilde{\boldsymbol{T}}_e \right\rangle} \to \min \tag{2.5}$$

is minimized here and $\widetilde{\boldsymbol{T}}_e$ can be thought as a 'trial' field that approximates the 'true' one $\boldsymbol{T}_e$. In such statements, however, the appropriate functionals depend as a rule only on the 'trial' field and they are extremized when the latter 'hits' the true one. Unlike this the functional $\mathcal{F}\left[\boldsymbol{T}_e, \widetilde{\boldsymbol{T}}_e\right]$ here depends *both* on the 'trial' and 'true' fields. This fact implies that practical results may be obtained from (2.4) only if the 'mixed' term $\left\langle \widetilde{\boldsymbol{T}}_\sigma : \boldsymbol{T}_e \right\rangle$ in (2.5) can be somehow calculated without full knowledge of the 'true' field $\boldsymbol{T}_e$. If this is done, then (2.5) may reduce to a variational statement which compares the energy $\langle \boldsymbol{T}_\sigma : \boldsymbol{T}_e \rangle$ of the actual field with that of a 'trial' one, $\left\langle \widetilde{\boldsymbol{T}}_\sigma : \widetilde{\boldsymbol{T}}_e \right\rangle$.

Assume for example that the trial fields $\widetilde{\boldsymbol{u}}(\boldsymbol{x})$ satisfy the same boundary conditions as the true one:

$$\widetilde{\boldsymbol{u}} \Big|_{\partial\mathcal{K}} = \boldsymbol{u} \Big|_{\partial\mathcal{K}}. \tag{2.6}$$

Then, for a given realization of the dispersion,

$$\left\langle \widetilde{\boldsymbol{T}}_\sigma : \boldsymbol{T}_e \right\rangle = \left\langle \boldsymbol{T}_\sigma : \widetilde{\boldsymbol{T}}_e \right\rangle$$
$$= \frac{1}{\mathrm{Vol}\,(\mathcal{V})} \int_{\partial\mathcal{K}} \boldsymbol{\sigma}_n \cdot \widetilde{\boldsymbol{u}}\, \mathrm{d}s = \frac{1}{\mathrm{Vol}\,(\mathcal{V})} \int_{\partial\mathcal{K}} \boldsymbol{\sigma}_n \cdot \boldsymbol{u}\, \mathrm{d}s = \langle \boldsymbol{T}_\sigma : \boldsymbol{T}_e \rangle$$

after an obvious integration by parts with (1.1) taken into account. Hence (2.4) reduces to the classical variational principle

$$\left\langle \widetilde{\boldsymbol{T}}_\sigma : \widetilde{\boldsymbol{T}}_e \right\rangle \geq \langle \boldsymbol{T}_\sigma : \boldsymbol{T}_e \rangle \tag{2.7}$$

that states the minimum of the elastic energy (or of the dissipation in fluid context) for the true deformation (respectively flow).

It is however difficult to construct tractable trial fields that satisfy boundary conditions like (2.6), especially in Robin's or sedimentation like problems when the displacements (or speeds) of the spheres are the unknowns. And here comes the advantage of employing inequalities of the type of (2.4) with an appropriate choice of trial field, since in this way one may avoid necessity of fulfilling the boundary conditions. That such a possibility is not fictious for the Robin's problem under study will be demonstrated below (§§ 3 and 4). But first of all the 'mixed' term $\left\langle \widetilde{\boldsymbol{T}}_\sigma : \boldsymbol{T}_e \right\rangle$ in (2.5) should be explicitly calculated for an appropriate choice of trial fields, without full knowledge of the 'true' field.

To this end we start by choosing, naturally enough, the tensor $\boldsymbol{H}$ in (2.3) as the elastic tensor of the matrix, i.e. the components of $\boldsymbol{H}$ in a Cartesian system read $H_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$.

It is also natural to take $\boldsymbol{u}(\boldsymbol{x})$ as the actual displacement field in the matrix that solves the Robin's problem, i.e. $\boldsymbol{u}(\boldsymbol{x})$ satisfies the equations (1.2), (1.3), (1.4).

Choose

$$\boldsymbol{T}_e = \tfrac{1}{2}(\nabla\boldsymbol{u} + \boldsymbol{u}\nabla) I_2(\boldsymbol{x}); \tag{2.8}$$

recall that $I_2(\boldsymbol{x})$ is the characteristic function of the region $\mathcal{K}_2$ occupied by the matrix. With such choices of $\boldsymbol{T}_e$ and $\boldsymbol{H}$ the left-hand side of (2.4) can be transformed as follows:

$$\left.\begin{aligned} \langle I_2(\boldsymbol{x})\boldsymbol{T}_\sigma : \boldsymbol{T}_e\rangle &= \langle I_2(\boldsymbol{x})\boldsymbol{T}_\sigma : \nabla\boldsymbol{u}\rangle \\ = \langle \nabla\cdot[I_2(\boldsymbol{x})\boldsymbol{T}_\sigma\cdot\boldsymbol{u}]\,\rangle &- \langle \boldsymbol{u}\cdot\nabla\cdot[I_2(\boldsymbol{x})\boldsymbol{T}_\sigma]\,\rangle\,, \quad \boldsymbol{x}\in\mathcal{K}_2. \end{aligned}\right\} \tag{2.9}$$

The second term in the right-hand side of (2.9) vanishes due to the absence of body sources in the matrix. The first term there, for a given realization of the random dispersion, reads:

$$\left.\begin{aligned} \langle \nabla\cdot[I_2(\boldsymbol{x})\boldsymbol{T}_\sigma\cdot\boldsymbol{u}]\,\rangle &= \frac{1}{\mathrm{Vol}\,(\mathcal{V})}\int_{\partial\mathcal{K}} \boldsymbol{\sigma}_n\cdot\boldsymbol{u}\,\mathrm{d}s \\ = \frac{1}{\mathrm{Vol}\,(\mathcal{V})}\sum_\alpha \left(\boldsymbol{U}_\alpha\cdot\int_{\mathcal{S}_\alpha}\boldsymbol{\sigma}_n\mathrm{d}s\right) &= n\overline{\boldsymbol{U}}_0\cdot\boldsymbol{F}_0, \end{aligned}\right\} \tag{2.10}$$

in virtue of (1.1), (1.5) and (1.4); $n = N/\mathrm{Vol}\,(\mathcal{V})$ is the number density of the spheres. Hence

$$\langle \boldsymbol{T}_\sigma : \boldsymbol{T}_e\rangle = n\alpha^*\overline{U}_0^2, \tag{2.11}$$

as it follows from (1.8).

Choose next

$$\widetilde{\boldsymbol{T}}_e = \tfrac{1}{2}(\nabla\widetilde{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}\nabla), \tag{2.12}$$

where the 'trial' displacement field $\widetilde{\boldsymbol{u}}(\boldsymbol{x})$ is defined over the whole $\mathbb{R}^3$, has zero mean value

$$\langle\widetilde{\boldsymbol{u}}(\boldsymbol{x})\rangle = 0, \tag{2.13}$$

and satisfies the inhomogeneous Lamé equation

$$\nabla\cdot\widetilde{\boldsymbol{T}}_\sigma = L\,[\widetilde{\boldsymbol{u}}] = -\overline{\boldsymbol{U}}_0, \quad \boldsymbol{x}\in\mathcal{K}_2, \tag{2.14}$$

in the matrix. Within the spheres it is assumed that

$$\left\langle \nabla \cdot \widetilde{\boldsymbol{T}}_\sigma \right\rangle_{\mathcal{V}_\alpha} = \frac{1}{V_a} \int_{\mathcal{V}_\alpha} \nabla \cdot \widetilde{\boldsymbol{T}}_\sigma \, \mathrm{d}\boldsymbol{x} = \boldsymbol{W}, \quad \alpha = 1, 2, \dots, \tag{2.15}$$

where $\mathcal{V}_\alpha$ is the region occupied by the α-th sphere, $V_a = \frac{4}{3}\pi a^3$, and $\boldsymbol{W}$ is a constant vector. Note that $\boldsymbol{W}$ cannot be arbitrary: due to (2.13),

$$\left\langle \nabla \cdot \widetilde{\boldsymbol{T}}_\sigma \right\rangle = \left\langle [I_1(\boldsymbol{x}) + I_2(\boldsymbol{x})] \, \nabla \cdot \widetilde{\boldsymbol{T}}_\sigma \right\rangle = -\eta_2 \overline{\boldsymbol{U}}_0 + \left\langle I_1(\boldsymbol{x}) \nabla \cdot \widetilde{\boldsymbol{T}}_\sigma \right\rangle = 0.$$

But

$$\left\langle I_1(\boldsymbol{x}) \nabla \cdot \widetilde{\boldsymbol{T}}_\sigma \right\rangle = \frac{N V_a}{\mathrm{Vol}\,(\mathcal{V})} \left\langle \nabla \cdot \widetilde{\boldsymbol{T}}_\sigma \right\rangle_{\mathcal{V}_\alpha} = \eta_1 \boldsymbol{W}, \quad \forall\, \alpha.$$

Recall that the volume fraction of the spheres is $\eta_1 = n V_a$. Thus

$$\boldsymbol{W} = \frac{\eta_2}{\eta_1} \overline{\boldsymbol{U}}_0. \tag{2.16}$$

Let

$$\mathcal{A} = \left\{ \widetilde{\boldsymbol{u}}(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathbb{R}^3, \ \widetilde{\boldsymbol{u}} \text{ satisfies (2.13), (2.14), (2.15)} \right\} \tag{2.17}$$

be the class of 'trial' fields we just introduced. We shall show that

$$\left\langle \boldsymbol{T}_\sigma : \widetilde{\boldsymbol{T}}_e \right\rangle = -\overline{U}_0^2, \quad \forall\, \widetilde{\boldsymbol{u}} \in \mathcal{A}, \tag{2.18}$$

i.e. the numerator in (2.4) is a constant, *independent* of the choice of $\widetilde{\boldsymbol{T}}_e$. This fact represents obviously the key point in the present study.

Indeed, observe that

$$\left.\begin{aligned} &\left\langle \widetilde{\boldsymbol{T}}_\sigma : \boldsymbol{T}_e \right\rangle = \left\langle I_2(\boldsymbol{x}) \widetilde{\boldsymbol{T}}_\sigma : \nabla \boldsymbol{u} \right\rangle \\ &= \left\langle \nabla \cdot \left[I_2(\boldsymbol{x}) \widetilde{\boldsymbol{T}}_\sigma \cdot \boldsymbol{u} \right] \right\rangle - \left\langle \boldsymbol{u} \cdot \left(\nabla \cdot \left[I_2(\boldsymbol{x}) \widetilde{\boldsymbol{T}}_\sigma \right] \right) \right\rangle, \end{aligned}\right\} \tag{2.19}$$

see (2.12).

Consider the first term in the right-hand side of (2.19). For a given realization of the random dispersion one has

$$\begin{aligned} &\left\langle \nabla \cdot \left(\widetilde{\boldsymbol{T}}_\sigma \cdot \boldsymbol{u} \right) \right\rangle = \left\langle \nabla \cdot \left[(I_1(\boldsymbol{x}) + I_2(\boldsymbol{x})) \, \widetilde{\boldsymbol{T}}_\sigma \cdot \boldsymbol{u} \right] \right\rangle \\ &= \frac{1}{\mathrm{Vol}\,(\mathcal{V})} \int_{\mathcal{V}} \nabla \cdot \left(\widetilde{\boldsymbol{T}}_\sigma \cdot \boldsymbol{u} \right) \mathrm{d}\boldsymbol{x} = \frac{1}{\mathrm{Vol}\,(\mathcal{V})} \int_{\partial\mathcal{V}} \widetilde{\boldsymbol{\sigma}}_n \cdot \boldsymbol{u} \, \mathrm{d}s = 0, \end{aligned}$$

see (1.1). That is why

$$\left\langle \nabla \cdot \left[I_2(\boldsymbol{x}) \widetilde{\boldsymbol{T}}_\sigma \cdot \boldsymbol{u} \right] \right\rangle = -\left\langle \nabla \cdot \left[I_1(\boldsymbol{x}) \widetilde{\boldsymbol{T}}_\sigma \cdot \boldsymbol{u} \right] \right\rangle. \tag{2.20}$$

But

$$\left.\begin{aligned} &\left\langle \nabla \cdot \left[I_1(\boldsymbol{x}) \widetilde{\boldsymbol{T}}_\sigma \cdot \boldsymbol{u} \right] \right\rangle = \frac{1}{\mathrm{Vol}\,(\mathcal{V})} \sum_\alpha \int_{\mathcal{V}_\alpha} \left(\nabla \cdot \widetilde{\boldsymbol{T}}_\sigma \right) \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \\ &= \frac{1}{\mathrm{Vol}\,(\mathcal{V})} \sum_\alpha \boldsymbol{U}_\alpha \cdot \int_{\mathcal{V}_\alpha} \left(\nabla \cdot \widetilde{\boldsymbol{T}}_\sigma \right) \mathrm{d}\boldsymbol{x} = \frac{N V_a}{\mathrm{Vol}\,(\mathcal{V})} \overline{\boldsymbol{U}}_0 \cdot \boldsymbol{W} \\ &= \eta_1 \overline{\boldsymbol{U}}_0 \cdot \boldsymbol{W} = \eta_2 \overline{U}_0^2, \end{aligned}\right\} \tag{2.21}$$

see (1.5) and (2.16).

The second term in the right-hand side in (2.19) is evaluated as follows

$$\left\langle \boldsymbol{u} \cdot \left(\nabla \cdot \left[I_2(\boldsymbol{x}) \widetilde{\boldsymbol{T}}_\sigma \right] \right) \right\rangle = - \langle I_2(\boldsymbol{x}) \boldsymbol{u} \rangle \cdot \overline{\boldsymbol{U}}_0,$$

due to (2.14). In turn,

$$\langle \boldsymbol{u} \rangle = \langle \boldsymbol{u} \left[I_1(\boldsymbol{x}) + I_2(\boldsymbol{x}) \right] \rangle = \eta_1 \overline{\boldsymbol{U}}_0 + \langle \boldsymbol{u} I_2(\boldsymbol{x}) \rangle = 0,$$

see (1.2), i.e. $\langle \boldsymbol{u} I_2(\boldsymbol{x}) \rangle = -\eta_1 \overline{\boldsymbol{U}}_0$ and thus

$$\left\langle \boldsymbol{u} \cdot \left(\nabla \cdot \left[I_2(\boldsymbol{x}) \widetilde{\boldsymbol{T}}_\sigma \right] \right) \right\rangle = \eta_1 \overline{U}_0^2. \tag{2.22}$$

The validity of (2.18) now follows from (2.19)—(2.22).

Inserting (2.11) and (2.18) into (2.4) yields

$$n\alpha^* \geq \frac{\overline{U}_0^2}{\left\langle \widetilde{\boldsymbol{T}}_\sigma : \widetilde{\boldsymbol{T}}_e \right\rangle}, \quad \forall\, \widetilde{\boldsymbol{u}} \in \mathcal{A}. \tag{2.23}$$

Hence the ‘mixed’ term disappears in the right-hand side of the inequality (2.4), if $\widetilde{\boldsymbol{u}} \in \mathcal{A}$, so that the latter, in the form (2.23), can already produce estimates on the effective constant α^*.

It is noted that (2.23) resembles at a first glance a variational statement. However, the true field $\boldsymbol{u}(\boldsymbol{x})$ does not belong to the class $\mathcal{A}$ since there are no body forces in the matrix, $L[\boldsymbol{u}] = 0$, cf. (2.14). The equality sign in (2.23) is thus *never* achieved there.

3. The energy of the ‘trial’ fields

It remains to specify the denominator of the left-hand side of (2.23), i.e. the elastic energy $\left\langle \widetilde{\boldsymbol{T}}_\sigma : \widetilde{\boldsymbol{T}}_e \right\rangle$ of the ‘trial’ fields $\widetilde{\boldsymbol{u}} \in \mathcal{A}$.

Let

$$\nabla \cdot \widetilde{\boldsymbol{T}}_\sigma + \chi(r_\alpha) \boldsymbol{W} = 0, \quad r_\alpha = |\boldsymbol{x} - \boldsymbol{x}_\alpha| \leq a, \quad \alpha = 1, 2, \ldots \tag{3.1}$$

The vector $\boldsymbol{W}$ is given in (2.16), so that the function f, assumed for simplicity spherically-symmetric, is subject to the constraint

$$\frac{1}{V_a} \int_{|\boldsymbol{x}| \leq a} \chi(r)\, \mathrm{d}\boldsymbol{x} = 1. \tag{3.2}$$

In virtue of (2.14) to (2.16), the trial fields $\widetilde{\boldsymbol{u}} \in \mathcal{A}$ satisfy the inhomogeneous Lamé equation

$$L[\widetilde{u}] + q(\boldsymbol{x}) \overline{\boldsymbol{U}}_0 = 0, \tag{3.3}$$

$$q(\boldsymbol{x}) = I_2(\boldsymbol{x}) - \frac{\eta_2}{\eta_1} \sum_\alpha \chi(\boldsymbol{x} - \boldsymbol{x}_\alpha) h(\boldsymbol{x} - \boldsymbol{x}_\alpha), \tag{3.4}$$

where $\{\boldsymbol{x}_\alpha\}$ is the set of sphere's centers and $h(\boldsymbol{x})$ is the characteristic function of a single sphere located at the origin.

To represent the source term in (3.4) in a concise form, let us introduce after Stratonovich (1963) the random density field

$$\psi(\boldsymbol{x}) = \sum_{\alpha} \delta(\boldsymbol{x} - \boldsymbol{x}_\alpha) \tag{3.5}$$

that corresponds to the random set $\{\boldsymbol{x}_\alpha\}$. This field allows one to recast (3.4) as

$$q(\boldsymbol{x}) = -\frac{1}{\eta_1}\, q_0(\boldsymbol{x}), \quad q_0(\boldsymbol{x}) = \int H(\boldsymbol{x}-\boldsymbol{y})\psi'(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \tag{3.6}$$

$\psi'(\boldsymbol{y}) = \psi(\boldsymbol{y}) - n$, where

$$H(\boldsymbol{x}) = (\eta_1 + \eta_2\chi(r))\, h(\boldsymbol{x}). \tag{3.7}$$

The function $H(\boldsymbol{x})$ is subject to the same constraint (3.2)

$$\frac{1}{V_a}\int H(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1, \tag{3.8}$$

as it follows from (3.7).

Recall that $\langle\psi(\boldsymbol{x})\rangle = n$, so that $\psi'(\boldsymbol{x}) = \psi(\boldsymbol{x}) - n$ is the fluctuating part of $\psi(\boldsymbol{x})$. Hence $q_0(\boldsymbol{x})$ is fluctuation as well, $\langle q_0(\boldsymbol{x})\rangle = 0$, as it should be, cf. (2.13) and (3.3). Also

$$\langle\psi(\boldsymbol{y}_1)\psi(\boldsymbol{y}_2)\rangle = n\delta(\boldsymbol{y}_1 - \boldsymbol{y}_2) + n^2 R(\boldsymbol{y}_1 - \boldsymbol{y}_2), \tag{3.9}$$

where $g(\boldsymbol{x}) = g(r)$ is the radial distribution function for the random set $\{\boldsymbol{x}_\alpha\}$ and

$$R(\boldsymbol{x}) = g(\boldsymbol{x}) - 1 = -h_{2a}(\boldsymbol{x}) + \nu(\boldsymbol{x}), \quad \nu(\boldsymbol{x}) = 0, \ \ |\boldsymbol{x}| \le 2a, \tag{3.10}$$

The term $h_{2a}(\boldsymbol{x})$ here corresponds to the well-stirred (or hard spheres) approximation for which $g(\boldsymbol{x}) = 0$ at $|\boldsymbol{x}| \ge 2a$. The function $\nu(\boldsymbol{x})$ is the binary (or total) correlation function for set $\{\boldsymbol{x}_\alpha\}$; it signifies the deviation of $g(\boldsymbol{x})$ from the hard spheres approximation.

The solution of (3.3) has the form

$$\widetilde{\boldsymbol{u}}(\boldsymbol{x}) = \overline{\boldsymbol{U}}_0 \cdot \int \boldsymbol{G}(\boldsymbol{x}-\boldsymbol{y})q(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \tag{3.11}$$

where integration is spread over the whole $\mathbb{R}^3$, and

$$\boldsymbol{G}(\boldsymbol{x}) = \frac{1}{16\pi\mu(1-\nu)r}\left((3-4\nu)\boldsymbol{I} + \frac{\boldsymbol{x}\boldsymbol{x}}{r^2}\right), \quad r = |\boldsymbol{x}|, \tag{3.12}$$

is the Green (more precisely, the Kelvin-Somigliana) tensor for Lamé's operator L, $\boldsymbol{I}$ is the unit second-rank tensor.

After the above preliminaries, consider the denominator of (2.23) for the trial field $\widetilde{\boldsymbol{u}} \in \mathcal{A}$, represented in the integral form (3.11), with $q(\boldsymbol{x})$ defined in (3.6).

In a Cartesian system one has

$$\left\langle \widetilde{\boldsymbol{T}}_\sigma : \widetilde{\boldsymbol{T}}_e \right\rangle = \frac{1}{\eta_1^2}\,\overline{U}^0_\alpha \overline{U}^0_\beta \iint L_{ijpq} G_{p\beta,q}(\boldsymbol{z}_1) G_{i\alpha,j}(\boldsymbol{z}_2) Q(\boldsymbol{z}_1 - \boldsymbol{z}_2)\,\mathrm{d}\boldsymbol{z}_1 \mathrm{d}\boldsymbol{z}_2, \tag{3.13}$$

where

$$Q(\boldsymbol{x}) = \langle q_0(\boldsymbol{x}) q_0(0) \rangle \,, \tag{3.14}$$

is the two-point correlation function for the field $q_0(\boldsymbol{x})$. Repeating indices hereafter imply summation. Integration by parts in (3.13) yields

$$\left\langle \widetilde{\boldsymbol{T}}_\sigma : \widetilde{\boldsymbol{T}}_e \right\rangle = \frac{1}{\eta_1^2} \overline{U}^0_\alpha \overline{U}^0_\beta \iint \partial_j \left[L_{ijpq} G_{p\beta,q}(\boldsymbol{z}_1) \right] G_{i\alpha}(\boldsymbol{z}_2) Q(\boldsymbol{z}_1 - \boldsymbol{z}_2) \, \mathrm{d}\boldsymbol{z}_1 \mathrm{d}\boldsymbol{z}_2,$$

assuming that the correlation (3.14) tends fast enough to zero as $|\boldsymbol{z}_1 - \boldsymbol{z}_2| \to \infty$

Therefore

$$\left\langle \widetilde{\boldsymbol{T}}_\sigma : \widetilde{\boldsymbol{T}}_e \right\rangle = \frac{1}{\eta_1^2} \overline{U}^0_\alpha \overline{U}^0_\beta \int G_{\alpha\beta}(\boldsymbol{z}) Q(\boldsymbol{z}) \, \mathrm{d}\boldsymbol{z}, \tag{3.15}$$

since

$$\partial_j \left[L_{ijpq} G_{p\beta,q}(\boldsymbol{z}) \right] + \delta_{i\beta} \delta(\boldsymbol{z}) = 0.$$

The latter is just the definition of Green's tensor for the Lamé operator.

The integral in (3.15) is an isotropic second-rank tensor, so that

$$\left\langle \widetilde{\boldsymbol{T}}_\sigma : \widetilde{\boldsymbol{T}}_e \right\rangle = \frac{4\pi a^3}{\eta_1^2 A_0} \, \theta \, \overline{U}_0^2, \tag{3.16}$$

having taken the trace of the Green tensor (3.12); the constant A_0 is defined in (1.9), and

$$\theta = \int_0^\infty \rho Q(a\rho) \, \mathrm{d}\rho, \quad \rho = r/a, \tag{3.17}$$

is a certain parameter that depends on the trial field $\widetilde{\boldsymbol{u}}$ and on the statistical properties of the dispersion under study. It is positive, $\theta > 0$, being proportional to the elastic energy $\left\langle \widetilde{\boldsymbol{T}}_\sigma : \widetilde{\boldsymbol{T}}_e \right\rangle$, see (3.16). The specific form of the function $Q(\boldsymbol{x})$, and thus the value of the parameter θ, can be found by means of (3.6), (3.9) and (3.14).

4. Estimates for the effective constant α^*

Let

$$\mathcal{H} = \left\{ H(\boldsymbol{x}) \;\middle|\; H(\boldsymbol{x}) \text{ satisfies (3.8)}, \; |\boldsymbol{x}| \le a \right\} \tag{4.1}$$

be the (convex) set of functions which generate admissible fields $\widetilde{\boldsymbol{u}} \in \mathcal{A}$ by means of (3.3) and (3.6). The parameter (3.17) is a functional over this set, $\theta = \theta[H(\cdot)]$. To get the optimal bound on α^* one should minimize the latter

$$\theta[H(\cdot)] \to \min, \quad H \in \mathcal{H}. \tag{4.2}$$

If

$$T = \min_{H \in \mathcal{H}} \theta[H(\cdot)], \tag{4.3}$$

then

$$\alpha^* \ge \frac{\eta_1 A_0}{3T} \tag{4.4}$$

is the best lower bound on α^* derivable from our procedure, as it follows from (2.23), (3.16) and (4.3).

The variational problem (4.2) has an interest of its own, not connected with any Robin's or sedimentation context. Though it will not be discussed here, it is worth formulating it in purely elastic terms. Namely, assume that body forces with a constant density $\boldsymbol{P}$ act in the matrix of an elastic dispersion. These forces are balanced by means of forces with nonhomogeneous density $\chi(r_\alpha)\boldsymbol{Q}$, $r_\alpha = |\boldsymbol{x} - \boldsymbol{x}_\alpha|$, acting within the spheres. One then looks for the force distribution $\chi(r)$ that minimizes the resulting elastic energy of the dispersion.

Instead of trying to solve the problem (4.2) we shall simply illustrate the performance of the proposed procedure by choosing particular examples of functions $H \in \mathcal{H}$ and observing that each such function yields the lower bound

$$\alpha^* \geq \frac{\eta_1 A_0}{3\theta[H(\cdot)]} \tag{4.5}$$

on the effective constant α^*, cf. (2.23) and (3.16). Moreover, it will turn out by chance that the parameters θ, corresponding to the functions $H(\boldsymbol{x})$ to be chosen here, have already been evaluated by the author in different context (Markov, 1998*a*).

The simplest choice is obviously is

$$\chi(r) \equiv 1, \quad \text{i.e.} \quad H(\boldsymbol{x}) = h(\boldsymbol{x}), \tag{4.6}$$

which means that the force is homogeneously distributed within the spheres. Then

$$q_0(\boldsymbol{x}) = \int h(\boldsymbol{x} - \boldsymbol{y})\psi'(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = I_1'(\boldsymbol{x}), \quad q(\boldsymbol{x}) = -\frac{1}{\eta_1}\,q_0(\boldsymbol{x}), \tag{4.7}$$

see (3.6), where $I_1'(\boldsymbol{x}) = I_1(\boldsymbol{x}) - \eta_1$ is the fluctuating part of the characteristic function for the region $\mathcal{K}_1$, occupied by the spheres. In turn

$$Q(\boldsymbol{x}) = \langle I_1'(\boldsymbol{x}) I_1'(0)\rangle = F^{pp}(\boldsymbol{x}) \tag{4.8}$$

is the 'particle-particle' correlation function for the dispersion. The parameter θ in this case is known to be

$$\theta = \theta^{\,\mathrm{pp}} = \eta_1\left(\frac{2 - 9\eta_1}{5} + m\eta_1\right), \tag{4.9}$$

see Torquato & Rubinstein (1989), Markov & Willis (1998) or Markov (1998*a*). Here

$$m = \int_2^\infty \rho\nu(\rho)\,\mathrm{d}\rho, \quad \nu(\rho) = g(\rho) - 1, \quad \rho = r/a \geq 2, \tag{4.10}$$

i.e. $\nu(r) = g(r) - 1$, $r \geq 2a$, is the total correlation function, associated with the radial distribution function $g(r)$ of the sphere's centers, see (3.10). Recall that $\nu(r) = 0$ for the hard-sphere approximation so that $m = 0$ in this case. The statistical quantity m is familiar—it has already appeared in the theory of random particulate media in different contexts: we shall point out the papers of Talbot & Willis (1980), dealing with absorption by a system of spherical sinks and Brady & Durlofsky (1982) in sedimentation context. .

Inserting (4.9) into (4.5) yields the lower bound

$$\alpha^* \geq \frac{5A_0}{3(2 - 9\eta_1 + 5\eta_1 m)} \tag{4.11}$$

on the effective parameter α^*.

It is seen that the bound (4.11), though nontrivial, does not produce the exact value A_0 of the parameter α^*, yielding instead

$$\alpha^* \geq \tfrac{5}{6} A_0 + o(\eta_1),$$

cf. (1.9).

Note that the bound (4.11) is the counterpart of the 'particle-particle' (or 'void-void') estimate of Torquato & Rubinstein (1989) for the effective absorption constant of a dispersion of spheres. In the permeability context a bound of this type was first derived by Prager (1961), see the survey of Torquato (2000) for details.

Consider next the δ-sequence

$$\chi_\varepsilon(r) = \tfrac{1}{3} a \begin{cases} 0, & 0 \leq r < a(1-\varepsilon), \\ \dfrac{1}{a\varepsilon}, & a(1-\varepsilon) \leq r \leq a, \end{cases}$$

so that

$$\chi(r) = \lim_{\varepsilon \to 0} \chi_\varepsilon(r) = \tfrac{1}{3} a \delta(r-a), \tag{4.12}$$

which means that the force is homogeneously concentrated over the spheres' surfaces only.

Obviously, both $\chi_\varepsilon(r)$ and its limit $\chi(r)$ satisfy (3.2). The function $q_0(\boldsymbol{x})$, generated by $\chi(r)$ according to (3.6), reads

$$q_0(\boldsymbol{x}) = \int H(\boldsymbol{x}-\boldsymbol{y}) \psi'(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} = \eta_1 I_1'(\boldsymbol{x}) + \eta_1 \eta_2 \left(\frac{a}{3\eta_1} \sum_\alpha \delta(|\boldsymbol{x}-\boldsymbol{x}_\alpha| - a) - 1 \right).$$

But

$$\frac{a}{3\eta_1} = \frac{1}{S}, \quad S = 4\pi a^2 n,$$

where S is easily recognized as the specific surface of the dispersion. Observe also that

$$\sum_\alpha \delta(|\boldsymbol{x}-\boldsymbol{x}_\alpha| - a) = |\nabla I_1(\boldsymbol{x})|.$$

Hence

$$q_0(\boldsymbol{x}) = \eta_1 I_1'(\boldsymbol{x}) + \eta_1 \eta_2 \left(\frac{1}{S} |\nabla I_1(\boldsymbol{x})| - 1 \right), \tag{4.13}$$

is the 'surface' counterpart of the 'particle' function (4.7). Then

$$Q(\boldsymbol{x}) = \eta_1^2 \left(F^{\mathrm{pp}}(\boldsymbol{x}) + 2\eta_2 F^{\mathrm{ps}}(\boldsymbol{x}) + \eta_2^2 F^{\mathrm{ss}}(\boldsymbol{x}) \right), \tag{4.14}$$

where

$$\left. \begin{aligned} F^{\mathrm{ps}}(\boldsymbol{x}) &= \frac{1}{S^2} \left\langle I_1'(\boldsymbol{x}) \left(|\nabla I_1(0)| - S \right) \right\rangle, \\ F^{\mathrm{ss}}(\boldsymbol{x}) &= \frac{1}{S^2} \left\langle \left(|\nabla I_1(\boldsymbol{x})| - S \right) \left(|\nabla I_1(0)| - S \right) \right\rangle, \end{aligned} \right\} \tag{4.15}$$

are, respectively, the 'surface-particle' and the 'surface-surface' correlation functions for the dispersion. In turn

$$\theta = \eta_1^2 \left(\theta^{\rm pp} + 2\eta_2 \theta^{\rm ps} + \eta_2^2 \theta^{\rm ss} \right) \tag{4.16}$$

where

$$\theta^{\rm ps} = \int_0^\infty \rho F^{\rm ps}(a\rho)\, {\rm d}\rho, \quad \theta^{\rm ss} = \int_0^\infty \rho F^{\rm ss}(a\rho)\, {\rm d}\rho \tag{4.17}$$

The parameters $\theta^{\rm ps}$ and $\theta^{\rm ss}$ has been evaluated by the author (Markov, 1998*a*) as

$$\theta^{\rm ps} = \frac{5 - 26\eta_1}{15} + \eta_1 m, \quad \theta^{\rm ss} = \frac{1 - 5\eta_1}{3\eta_1} + m, \tag{4.18}$$

where m is the statistical quantity, defined in (4.10).

Utilizing (4.9), (4.16), (4.18) in the bound (4.5) yields eventually

$$\alpha^* \geq \frac{A_0}{1 - 5\eta_1 - \eta_1^2/5 + 3\eta_1 m}\,. \tag{4.19}$$

Obviously the bound (4.19) already reproduces the exact dilute value $\alpha^* = A_0 + o(\eta_1)$ of the coefficient α^*. Moreover, (4.19) is the counterpart of the Doi-Talbot-Willis bound on the absorption coefficient of dispersion, see Doi (1976), Talbot & Willis (1980), and also Markov (1998*b*).

It seems that the singular force distribution (4.12) solves the variational problem (4.2), so that the bound (4.19) is the best one within the frame of our procedure. However, we shall not discuss this point here.

To recast the foregoing bounds to a more tangible form, denote by

$$\boldsymbol{U}_s = \frac{\boldsymbol{F}_0}{A_0} \tag{4.20}$$

the displacement of a single sphere, cemented into the matrix, due to the force F_0, see (1.9). By means of (1.8) and (4.20), (4.19) is recast as

$$\overline{U}_0 \leq U_s \left(1 - 5\eta_1 - \eta_1^2/5 + 3\eta_1 m \right) \tag{4.21}$$

which bounds the mean displacement of the spheres when the multiparticle interactions are taken into account. In sedimentation context ($\nu = \frac{1}{2}$, U interpreted as the mean creeping velocity), (4.21) reads

$$\overline{V}_0 \leq V_s \left(1 - 5\eta_1 - \eta_1^2/5 + 3\eta_1 m \right), \tag{4.22}$$

where $V_s = F_0/(6\pi\mu a)$ is the well-known Stokes velocity of a single sphere, due to the force F_0. Hence, if we introduce the hindered settling function f through the relation $\overline{V}_0 = fV_s$, then (4.22) yields an upper bound for f, namely,

$$f \leq 1 - 5\eta_1 - \eta_1^2/5 + 3\eta_1 m, \tag{4.23}$$

Curiously enough, the r. h. side of (4.23) coincides with the denominator of the Talbot-Willis bound on the effective absorption coefficient of a random array of nonoverlapping spheres, see Talbot and Willis (1980). More important, it coincides with the far-field approximation f^∞ for the function f, as proposed by Brady & Durlofsky (1982). We shall comment on this fact in §5.

Recall that the sedimentation speed is usually represented as

$$\overline{V}_0 = V_s\,(1 - S\eta_1)\,, \tag{4.24}$$

where S is called the coefficient of sedimentation. Inserting (4.24) into (4.22) yields

$$S \geq 5 + \tfrac{1}{5}\eta_1 - 3m. \tag{4.25}$$

and this inequality, together with (4.23), are our main results.

It is to be specially emphasized that the derivation of (4.25) and (4.23) involves here *no* specific assumptions, all the appearing integrals are absolutely convergent and hence *no* 'renormalization' procedures are needed.

5. Discussion

As already pointed out, the r. h. side of (4.23) coincides with the far-field approximation f^∞ for the hindered settling function f, as proposed by Brady & Durlofsky (1982). This approximation, discussed in depth by the authors, captures correctly the far-field interactions in a settling dispersion. The inequality (4.23) thus means that the approximation of Brady & Durlofsky provides a rigorous upper bound on the function f, and this fact perhaps is the central finding of the present study.

The most important feature of the estimates (4.23) and (4.25) is the presence of the term m which accounts, in an integral form, for the binary correlation function of the array of spheres. For the simplest hard-sphere approximation $m = 0$ the bound (4.23) becomes negative at $\eta_1^* \approx 0.2$, and hence the approximation becomes unrealistic at sphere fractions $\eta_1 \geq \eta_1^*$. This fact was pointed out by Talbot & Willis (1980). The hard-sphere approximation, however, fails 'earlier', namely at $\eta_1^* = 0.125$, since the appropriate two-point correlation functions loses its positive-definiteness at this value of η_1, see Markov (1996) and Markov & Willis (1998). Presumably this is the main reason why the approximation of Glendinning–Russel (1982) for the function f, based on the hard-sphere distribution, fails at $\eta_1 > 0.27$.

For the more realistic Percus-Yevick distribution of the hard spheres the parameter m reads

$$m = \frac{\eta_1(22 - \eta_1)}{5(1 + 2\eta_1)}\,, \tag{5.1}$$

according to Talbot & Willis (1980). The bound (4.23) on the hindered settling function f in this case is extremely simple

$$f \leq f^{\mathrm{PY}}, \quad f^{\mathrm{PY}} = \frac{(1-\eta_1)^3}{1 + 2\eta_1}\,, \tag{5.2}$$

see Brady & Durlofsky (1982) and Hayakawa & Ichiki (1995). The extensive experimental results, cited in these two references, agree with (5.2). Moreover, at $\eta_1 \geq 0.3$ these results are very close to the curve f^{PY}, falling a bit lower, as it should be. This fact indicates that the far-field contribution is predominant in sedimenting dispersions in the non-dilute case.

As far as the sedimentation coefficient is concerned, the bound (4.25) in the hard-sphere approximation becomes

$$S \geq 5 + \tfrac{1}{5}\eta_1. \tag{5.3}$$

This agrees with the result of Batchelor (1972) $S \approx 6.55 + o(\eta_1)$ in the dilute limit. The bound (5.3) is also in good agreement with the experimental data, analyzed by Feuillebois et al. (1999) and Bruneau et al. (1990), for which it turned out that $S = 5 \div 5.6$. Such values of S indicate that either the parameter m is close to zero or it is nonnegative for the dispersions under study.

In a number of cases, however, the sedimentation coefficient was found to be considerably less than 5. An example is provided by the experiments of Ham & Homsy (1988), who fitted their findings with the law $S = 4 - 8\eta_1$, $\eta_1 \in (0.025, 0.10)$. Since (4.25) is a rigorous result, one can claim that there is a strong attractive trend in the studied dispersions, i.e., $g(r) - 1 > 0$, and thus $m > 0$, see (4.10). Moreover, this trend increases with increasing sphere fraction, in order to explain the observed decrease of S.

There were successful attempts to gather experimental information about the binary correlation function in a settling dispersion, using the 'N. M. R.-scattering' experimental technique (Bruneau et al., 1998). The experimental findings of these authors suggested again an attractive trend, $g(r) - 1 > 0$ which, curiously enough, increases for smaller volume fractions, according to them. More precisely, they fitted well their experimental data by the binary correlation

$$\nu(r) = \begin{cases} \dfrac{0.02}{\eta_1}, & 2 \leq \rho \leq 4, \\ 0, & \rho > 4, \end{cases}$$

valid for sphere fractions η_1 between 0.05 and 0.3. The parameter m for such a correlation is easily calculated as

$$m = \frac{0.12}{\eta_1},$$

see (4.10). For $\eta_1 = 0.05$ the right-hand side of (4.25) us negative, and hence it brings forth no useful information about sedimentation speed. With increasing η_1 it increases monotonically, reaching the value 3.86 at $\eta_1 = 0.3$.

The foregoing analysis suggests in passing that (i) the steady-state internal structure of the settling dispersion may greatly differ from the simplest hard-sphere approximation and (ii) this structure may drastically influences the value of the sedimentation coefficient.

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