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On the inhomogeneity problem in micropolar elasticity[1]

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Abstract. The problem of determining the stress and couple-stress fields within a single micropolar inhomogeneity, immersed into an unbounded micropolar matrix loaded at infinity, is discussed. Using Eshelby's method, an approximate solution of the problem is found for the case of spherical inhomogeneity. In order to analyse the applicability of this solution, a system of integral equations is delivered which describes the strain fields in the micropolar body with a micropolar inhomogeneity in it, and an iterative process, solving the system, is proposed. The first step of this process leads to the approximate solution, found by Eshelby's method. It is shown, too, that the approximate solution is the linear part of the asymptotic expansion in the case of vanishing difference between the micropolar moduli of the matrix and the inhomogeneity.

1. Inclusion Problem for a Micropolar Body

At first, we would like to remind of some well known facts from the micropolar elasticity [1, 2].

Let $\boldsymbol{u}$ and $\boldsymbol{\varphi}$ be the independent vector fields of displacements and rotations of the points of the micropolar body and

$$\boldsymbol{T}_\gamma = \nabla \boldsymbol{u} - \boldsymbol{E}\cdot\boldsymbol{\varphi}, \quad \boldsymbol{T}_\kappa = \nabla\boldsymbol{\varphi}, \tag{1.1}$$

be the strain tensors; here $\boldsymbol{E} = \|\varepsilon_{ijk}\|$ is Ricci's alternator, the dot means a contraction with respect to one of the indices, so $(\boldsymbol{E}\cdot\boldsymbol{\varphi})_{ij} = \varepsilon_{ijk}\varphi_k$. Tensors (1.1) are connected with the stress tensor $\boldsymbol{T}_\sigma$ To and the couple-stress tensor $\boldsymbol{T}_\mu$ by the following linear tensor relations:

$$\begin{aligned} \boldsymbol{T}_\sigma &= \boldsymbol{L} : \boldsymbol{T}_\gamma = (\mu+\alpha)\boldsymbol{T}_\gamma + (\mu+\alpha)\boldsymbol{T}^*_\gamma + \boldsymbol{I}\,\mathrm{sp}\,\boldsymbol{T}_\gamma\,, \\ T_\mu &= \boldsymbol{M} : \boldsymbol{T}_\kappa = (\gamma+\varepsilon)\boldsymbol{T}_\kappa + (\gamma-\varepsilon)\boldsymbol{T}^*_\kappa + \boldsymbol{I}\,\mathrm{sp}\,T_\kappa\,, \end{aligned} \tag{1.2}$$

here $\boldsymbol{I}$ is the unit tensor, $\boldsymbol{T}^*$ is the tensor conjugated with T; the colon denotes a contraction with respect to two pair of indices.

Hooke's law (1.2) together with the equations of motion yields to the known system of Lamé's type for the micropolar body, namely

$$\begin{aligned} (\mu+\alpha)\Delta\boldsymbol{u} + (\lambda+\mu-\alpha)\nabla\nabla\cdot\boldsymbol{u} + 2\alpha\nabla\times\boldsymbol{\varphi} + \boldsymbol{f} = 0\,, \\ (\gamma+\varepsilon)\Delta\boldsymbol{\varphi} - 4\alpha\boldsymbol{\varphi} + (\beta+\gamma-\varepsilon)\nabla\nabla\cdot\boldsymbol{\varphi} + 2\alpha\nabla\times\boldsymbol{u} + \boldsymbol{m} = 0\,, \end{aligned} \tag{1.3}$$

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where Δ is Laplace's operator, $\nabla \cdot \boldsymbol{a} = \operatorname{div} \boldsymbol{a}$, $\nabla \times \boldsymbol{a} = \operatorname{rot} \boldsymbol{a} = \operatorname{curl} \boldsymbol{a}$.

Green's tensor for the system (1.3) can be constructed by several methods. We give only the final-expressions for the components of this-tensor (in the case of an unbounded body)

$$\boldsymbol{G}_{11} = \frac{1}{4\pi\mu}\left(\boldsymbol{I}\left(\frac{1}{R} - \frac{\alpha}{\mu+\alpha}\,\frac{\mathrm{e}^{-R/\ell}}{R}\right) - \nabla\nabla\left(\frac{\lambda+\mu}{2(\lambda+2\mu)}\,R + \frac{\gamma+\varepsilon}{4\mu}\,\frac{1-\mathrm{e}^{-R/\ell}}{R}\right)\right),$$

$$\boldsymbol{G}_{12} = \boldsymbol{G}_{21} = -\frac{1}{8\pi\mu}\,\boldsymbol{E}\cdot\nabla\frac{1-\mathrm{e}^{-R/\ell}}{R}, \tag{1.4}$$

$$\boldsymbol{G}_{22} = \frac{1}{4\pi(\gamma+\varepsilon)}\,\frac{\mathrm{e}^{-R/\ell}}{R}\,\boldsymbol{I} + \frac{1}{16\pi\mu}\nabla\nabla\left(\frac{1-\mathrm{e}^{R/\ell}}{R} + \frac{\mu}{\alpha}\,\frac{\mathrm{e}^{-R/h} - \mathrm{e}^{-R/\ell}}{R}\right),$$

$R = |\boldsymbol{x} - \boldsymbol{x}'$ — see, for example, [1 – 4]. It is to note that Green's tensor is written here in a matrix form: the vector fields

$$\begin{bmatrix} \boldsymbol{u}(\boldsymbol{x}) \\ \boldsymbol{\varphi}(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_{11}(\boldsymbol{x}-\boldsymbol{x}') & \boldsymbol{G}_{12}(\boldsymbol{x}-\boldsymbol{x}') \\ \boldsymbol{G}_{21}(\boldsymbol{x}-\boldsymbol{x}') & \boldsymbol{G}_{22}(\boldsymbol{x}-\boldsymbol{x}') \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{f}_{x'} \\ \boldsymbol{m}_{x'} \end{bmatrix} \tag{1.5}$$

present the displacement and rotation in the point $\boldsymbol{x}$, generated by concentrated force $\boldsymbol{f}_{x'}$ and moment $\boldsymbol{m}_{x'}$ in the point $\boldsymbol{x}'$. In (1.4) the length-dimensional parameters ℓ and h are introduced as usual

$$\ell^2 = \frac{(\mu+\alpha)(\gamma+\varepsilon)}{4\alpha\mu}, \quad h^2 = \frac{\beta+\gamma}{4\alpha}. \tag{1.6}$$

Let (V) be an arbitrary region in a micropolar body $\mathcal{B}$. We consider (V) as an "inclusion" in the sense of Eshelby [5], so that (V) undergoes a change of shape and size, described by the constant strain tensors ("distortion") tensors $\boldsymbol{T}^0_\gamma$ and $\boldsymbol{T}^0_\kappa$.

In order to determine the constrained fields of displacements $\boldsymbol{u}^c$ and rotations $\boldsymbol{\varphi}^c$, due to the inclusion (V), we apply the same simple set of imaginary cutting, straining and welding operations as those used by Eshelby [5]. These operations yield the fact that the fields $\boldsymbol{u}^c$ and $\boldsymbol{\varphi}^c$ are generated in the micropolar case considered by the layers of surface tractions $\boldsymbol{\sigma}^0_n = \boldsymbol{n}\cdot\boldsymbol{T}^0_\sigma$ and moments $\boldsymbol{\mu}^0_n = \boldsymbol{n}\cdot\boldsymbol{T}^\mu$, acting on (S), as well as by the body moment $\boldsymbol{\sigma}^0_a = \boldsymbol{E} : \boldsymbol{T}^0_\sigma$, acting into (V); here $\boldsymbol{n}$ denotes the outward normal unit vector to the surface (S) of the region (V) and $\boldsymbol{T}^0_\sigma = \boldsymbol{L} : \boldsymbol{T}^0_\gamma$, $\boldsymbol{T}^0_\mu = \boldsymbol{M} : \boldsymbol{T}^0_\kappa$ are the stresses, connected formally with the distortion tensors $\boldsymbol{T}^0_\gamma$ and bT^0_κ by Hooke's law (1.2). Then

$$\begin{aligned} \begin{bmatrix} \boldsymbol{u}^c(\boldsymbol{x}) \\ \boldsymbol{\varphi}^c(\boldsymbol{x}) \end{bmatrix} &= \int_S \begin{bmatrix} \boldsymbol{G}_{11}(\boldsymbol{x}-\boldsymbol{x}') & \boldsymbol{G}_{12}(\boldsymbol{x}-\boldsymbol{x}') \\ \boldsymbol{G}_{21}(\boldsymbol{x}-\boldsymbol{x}') & \boldsymbol{G}_{22}(\boldsymbol{x}-\boldsymbol{x}') \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\sigma}^0_n \\ \boldsymbol{\mu}^0_n \end{bmatrix} ds' \\ &+ \int_S \begin{bmatrix} \boldsymbol{G}_{12}(\boldsymbol{x}-\boldsymbol{x}') \\ \boldsymbol{G}_{22}(\boldsymbol{x}-\boldsymbol{x}') \end{bmatrix} \cdot \boldsymbol{\sigma}^0_a \, dv', \end{aligned} \tag{1.7}$$

where $\boldsymbol{G} = \|\boldsymbol{G}_{ij}(\boldsymbol{x}, \boldsymbol{x}')\|, i, j = l, 2$, is Green's tensor for the micropolar body $\mathcal{B}$. It is worth noting that the same result (1.7) could be also found using the consideration of W. Nowacki [6, 7].

With the aid of Gauss' theorem we rewrite (1.7) in the following form:

$$\begin{aligned}\boldsymbol{u}^c = & -\boldsymbol{T}_\sigma^{0*} : \int_V \left(\nabla \boldsymbol{G}_{11}(\boldsymbol{x}-\boldsymbol{x}') - \boldsymbol{E}\cdot\boldsymbol{G}_{12}(\boldsymbol{x}-\boldsymbol{x}')\right) dv' \\ & + \boldsymbol{T}_\mu^{0*} : \int_V \nabla \boldsymbol{G}_{12}(\boldsymbol{x}-\boldsymbol{x}')\, dv' , \\ \boldsymbol{\varphi}^c = & \, \boldsymbol{T}_\sigma^{0*} : \int_V \left(\nabla \boldsymbol{G}_{21}(\boldsymbol{x}-\boldsymbol{x}') - \boldsymbol{E}\cdot\boldsymbol{G}_{22}(\boldsymbol{x}-\boldsymbol{x}')\right) dv' \\ & - \boldsymbol{T}_\mu^{0*} : \int_V \nabla \boldsymbol{G}_{22}(\boldsymbol{x}-\boldsymbol{x}')\, dv' ;\end{aligned} \tag{1.8}$$

here nabla operator relates to the point $\boldsymbol{x}$, so it could be taken out in front of the integrals.

In the case of an unbounded body $\mathcal{B}$, inserting Green's tensor (1.4) into (1.8), we get that the constraint fields $\boldsymbol{u}^c$ and $\boldsymbol{\varphi}^c$ are expressed by the three potentials of the region (V), namely,

$$\varphi(\boldsymbol{x}) = \frac{1}{4\pi}\int_V \frac{dv'}{R}\,, \quad \psi(\boldsymbol{x}) = \frac{1}{4\pi}\int_V R\, dv'\,, \quad \chi_\ell(\boldsymbol{x}) = \frac{1}{4\pi}\int_V \frac{\mathrm{e}^{-R/\ell}}{R}\, dv'\,, \tag{1.9}$$

$R = |\boldsymbol{x}-\boldsymbol{x}'|$, in the following form:

$$\begin{aligned}\boldsymbol{u}^c = & \frac{1}{\mu}\Bigg(-\left(\nabla\varphi - \frac{\alpha}{\mu+\alpha}\nabla\chi_\ell\right)\cdot\boldsymbol{T}_\sigma^0 \\ & + \boldsymbol{S}_\sigma^0 : \nabla\nabla\nabla\left(\frac{\lambda+\mu}{2(\lambda+2\mu)}\psi + \frac{\gamma+\varepsilon}{4\mu}(\varphi-\chi_\ell)\right) \\ & - \boldsymbol{A}_\sigma^0\cdot\nabla(\varphi-\chi_\ell) + \frac{1}{2}\boldsymbol{E} : \nabla\nabla(\varphi-\chi_\ell)\cdot\boldsymbol{T}_\mu^0\Bigg)\,, \\ \boldsymbol{\varphi}^c = & -\frac{1}{\gamma+\varepsilon}\left(\nabla\chi_\ell\cdot\boldsymbol{T}_\mu^0 + \ell^2\boldsymbol{S}_\mu^0 : \nabla\nabla\nabla\left(\frac{\mu}{\mu+\alpha}\chi_h - \chi_\ell + \frac{\alpha}{\mu+\alpha}\varphi\right)\right) \\ & + \frac{1}{2\mu}\boldsymbol{E} : \nabla\nabla(\varphi-\chi_\ell)\cdot\boldsymbol{T}_\sigma^0 \\ & + \frac{1}{\gamma+\varepsilon}\boldsymbol{A}_\sigma^0 : \boldsymbol{E}\cdot\left(\chi_\ell\boldsymbol{I} + \ell^2\nabla\nabla\left(\frac{\mu}{\mu+\alpha}\chi_h - \chi_\ell + \frac{\alpha}{\mu+\alpha}\varphi\right)\right)\,.\end{aligned} \tag{1.10}$$

Here the distortion tensors are decomposed into symmetric and skew-symmetrical parts: $\boldsymbol{T}_\sigma^0 = \boldsymbol{S}_\sigma^0 + \boldsymbol{A}_\sigma^0$, $\boldsymbol{T}_\mu^0 = \boldsymbol{S}_\mu^0 + \boldsymbol{A}_\mu^0$ and $\boldsymbol{S}^* = \boldsymbol{S}$, $\boldsymbol{A}^* = -\boldsymbol{A}$.

2. Case of a Spherical Inclusion

Having in mind the possible applications of the inhomogeneity problem to the theory of micropolar composite material with spherical particles in it, we shall consider in details the average and couple-stress fields within a spherical inclusion of the unbounded micropolar body.

The potentials (1.9) for the sphere (V_a) with a radius a are well known to be

$$\begin{aligned}\varphi &= \frac{1}{6}\left(3a^2 - R^2\right)\,, \quad \psi = \frac{1}{12}\left(3a^4 + 2a^2R^2 - \frac{1}{5}R^4\right)\,, \\ \chi_\ell &= \ell^2\left(1 - \left(1+\frac{a}{\ell}\right)\mathrm{e}^{-a/\ell}\,\frac{\sinh(R/\ell)}{R/\ell}\right)\,, \quad R = |x| \le a\,.\end{aligned} \tag{2.1}$$

They are central-symmetric functions, obviously, which solve the equations [5, 8]:

$$\begin{aligned} &\Delta\varphi + 1 = 0\,, \quad \Delta\Delta\psi + 2 = 0\,, \\ &\Delta\chi_\ell - \frac{1}{\ell^2}\,\chi_\ell + 1 = 0\,, \quad R = |\boldsymbol{x}| \le a\,. \end{aligned} \tag{2.2}$$

As a simple consequence of the equations (2.2) and the central-symmetry of the potentials (2.1) we find the following relations for the average gradients:

$$\begin{aligned} &\nabla\nabla\varphi = \langle\nabla\nabla\varphi\rangle = -\frac{1}{3}\boldsymbol{I}\,, \quad \langle\nabla\nabla\chi_\ell\rangle = -\frac{1}{3}\,f(\delta_\ell)\boldsymbol{I}\,, \\ &\langle\nabla\nabla\nabla\nabla\psi\rangle = -\frac{2}{15}\boldsymbol{H}\,, \quad \langle\nabla\nabla\nabla\nabla\chi_\ell\rangle = -\frac{1}{15\ell^2}\,f(\delta_\ell)\boldsymbol{H}\,, \end{aligned} \tag{2.3}$$

where $\langle\cdot\rangle = \dfrac{1}{V_a}\displaystyle\int_{V_a}\cdot\, dv$ denotes, as usual, an averaging over the sphere (V_a); $\boldsymbol{H} = \|H_{ijkl}\|$ is the isotropic, fourth-order and fully-symmetric tensor with components $H_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}$ and

$$f(\delta_\ell) = -3\frac{\chi'_\ell(a)}{a} = 3\frac{1+\delta_\ell}{\delta_\ell^3}\,\mathrm{e}^{-\delta_\ell}\,(\delta_\ell\cosh\delta_\ell - \sinh\delta_\ell) \tag{2.4}$$

with the dimensionless parameter $\delta_\ell = a/\ell$.

It is worth to point out that the constrained field $\nabla\boldsymbol{u}^c$ is not constant into the sphere in the micropolar case considered, unlike the elastic case. The reason is the presence of the "Helmholtz potential" χ_ℓ in (1.10).

In order to construct the average strain fields in the spherical inclusion we insert first (2.3) into the expression for $\langle\nabla\boldsymbol{u}^c\rangle$, calculated by (1.10). Finally we get

$$\begin{aligned} \langle\nabla\boldsymbol{u}^c\rangle = \frac{1}{\mu}\Bigg(&\frac{1-2\nu}{18(1-\nu)}\,\boldsymbol{I}\,\mathrm{sp}\,\boldsymbol{T}^0_\sigma \\ &+ \left(\frac{4-5\nu}{15(1-\nu)} - \frac{1}{5}\,\frac{\alpha}{\alpha+\mu}\,f(\delta_\ell)\right)\boldsymbol{D}^0_\sigma + \frac{1}{3}\,\frac{\mu}{\alpha+\mu}\,f(\delta_\ell)\boldsymbol{A}^0_\sigma\Bigg) \end{aligned} \tag{2.5}$$

The distortion stress tensor $\boldsymbol{T}^0_\sigma$ is decomposed in (2.5) into spherical, deviatoric and skew-symmetrical parts, so

$$\boldsymbol{T}^0_\sigma = \frac{1}{3}\boldsymbol{I}\,\mathrm{sp}\,\boldsymbol{T}^0_\sigma + \boldsymbol{D}^0_\sigma + \boldsymbol{A}^0_\sigma\,; \quad \boldsymbol{D}^0_\sigma = \boldsymbol{D}^{0*}_\sigma\,, \boldsymbol{A}^0_\sigma = -\boldsymbol{A}^{0*}_\sigma\,. \tag{2.6}$$

With the aid of similar calculations we obtain from (1.10) and (2.3) that

$$\langle\boldsymbol{\varphi}^c\rangle = \frac{1}{12\alpha}\,(3 - 2f(\delta_\ell) - f(\delta_h))\,\boldsymbol{\sigma}^0_a\,, \quad \boldsymbol{\sigma}^0_a = \boldsymbol{E} : \boldsymbol{T}^0_\sigma\,, \tag{2.7}$$

$$\begin{aligned} \langle\boldsymbol{T}^c_\kappa\rangle = \langle\nabla\boldsymbol{\varphi}^c\rangle &= \frac{1}{9}\,\frac{1}{\beta+2\gamma}\,f(\delta_h)\boldsymbol{I}\,\mathrm{sp}\,\boldsymbol{T}^0_\mu \\ &+ \frac{1}{5}\left(\frac{1}{\gamma+\varepsilon}\,f(\delta_\ell) + \frac{2}{3}\,\frac{1}{\beta+2\gamma}\,f(\delta_h)\right)\boldsymbol{D}^0_\mu + \frac{1}{3}\,\frac{1}{\gamma+\varepsilon}\,f(\delta_\ell)\boldsymbol{A}^0_\mu\,, \end{aligned} \tag{2.8}$$

where $\boldsymbol{T}^0_\mu = \frac{1}{3}\boldsymbol{I}\mathrm{sp}\,\boldsymbol{T}^0_\mu + \boldsymbol{D}^0_\mu + \boldsymbol{A}^0_\mu$ is the decomposition of the type (2.6) for the couple-stress distortion tensor $\boldsymbol{T}^0_\mu$ and $\delta_h = a/h$.

From the relations (2.5) and (2.7) we obtain now the average value of the constrained field $\boldsymbol{T}^0_\gamma$:

$$\left\langle \boldsymbol{T}^c_\gamma \right\rangle = \langle \nabla \boldsymbol{u}^c \rangle - \boldsymbol{E} \cdot \langle \boldsymbol{\varphi}^c \rangle$$

$$= \frac{1}{\mu} \left(\frac{1-2\nu}{18(1-\nu)} \boldsymbol{I} \,\mathrm{sp}\, \boldsymbol{T}^0_\sigma + \left(\frac{4-5\nu}{15(1-\nu)} - \frac{1}{5} \frac{\mu}{\mu+\alpha} f(\delta_\ell) \right) \boldsymbol{D}^0_\sigma \right. \tag{2.9}$$

$$\left. + \frac{1}{2} \frac{\mu}{\alpha} \left(1 - \frac{1}{3} f(\delta_h) - \frac{2}{3} \frac{\mu}{\mu+\alpha} f(\delta_\ell) \right) \boldsymbol{A}^0_\sigma \right) .$$

It is to note that the average strain tensor $\left\langle \boldsymbol{T}^c_\gamma \right\rangle$ depends only on the distortion stress tensor $\boldsymbol{T}^0_\sigma$ and on the radius of the inclusion through the quantities $\delta_\ell = a/\ell$ and $\delta_h = a/h$. Similarly, according to (2.8), the average tensor $\boldsymbol{T}^c_\kappa$ depends only on the distortion couple-stress tensor $\boldsymbol{T}^0_\mu$ and on radius a through the same quantities δ_ℓ and δ_h.

Taking into account Hooke's law (1.2), we obtain by (2.8) and (2.9) the following formulae for the average constrained stress and couple-stress tensors within the spherical inclusion:

$$\begin{aligned}
\mathrm{sp}\, \boldsymbol{T}^c_\sigma &= B_1 \mathrm{sp}\, \boldsymbol{T}^0_\sigma & & \left(k = \lambda + \tfrac{2}{3}\mu\right) \\
\langle \boldsymbol{D}^c_\sigma \rangle &= B_2 \boldsymbol{D}^0_\sigma & & (\mu) \\
\langle \boldsymbol{A}^c_\sigma \rangle &= B_2 \boldsymbol{A}^0_\sigma & & (\alpha) \\
\mathrm{sp}\, \boldsymbol{T}^c_\mu &= B_4 \mathrm{sp}\, \boldsymbol{T}^0_\mu & & \left(\beta + \tfrac{2}{3}\gamma\right) \\
\boldsymbol{D}^c_\mu &= B_5 \boldsymbol{D}^0_\mu & & (\gamma) \\
\boldsymbol{A}^c_\mu &= B_6 \boldsymbol{A}^0_\mu & & (\varepsilon)
\end{aligned} \tag{2.10}$$

with the coefficients

$$B_1 = \frac{3k}{3k+4\mu} , \quad B_2 = \frac{6}{5} \frac{k+2\mu}{3k+4\mu} - \frac{2}{5} \frac{\alpha}{\mu+\alpha} f(\delta_\ell) ,$$

$$B_3 = 1 - \frac{1}{3} f(\delta_h) - \frac{\mu}{\mu+\alpha} f(\delta_\ell) , \quad B_4 = \frac{1}{3} \frac{2\gamma+3\beta}{\beta+2\gamma} f(\delta_h) , \tag{2.11}$$

$$B_5 = \frac{2}{5} \gamma \left(\frac{1}{\gamma+\varepsilon} f(\delta_\ell) + \frac{2}{3} \frac{1}{\beta+2\gamma} f(\delta_h) \right) , \quad B_6 = \frac{2}{3} \frac{\varepsilon}{\gamma+\varepsilon} f(\delta_\ell) .$$

The relations (2.10) are given for spherical, deviatoric and skew-symmetric parts of the corresponding stress and couple-stress tensors.

It should be noted that the first relation of (2.10), connecting the spherical parts of the distortion and constrained stress tensors, is the same one as in the elastic case [5]. The reason is the absence of the potential χ_ℓ in (1.10), when $\boldsymbol{T}^0_\sigma$, is a spherical tensor. The first relation of (2.10) is valid, therefore, not only for a sphere; it does not depend on the geometry of the inclusion.

3. Micropolar Spherical Inhomogeneity — an Approximate Solution

The relations (2.10) allow solving approximately the spherical inhomogeneity problem for an unbounded micropolar body.

Let us imagine that an unbounded micropolar body (matrix) contains a micropolar spherical inhomogeneity in it. If we assume that the constant stress $\boldsymbol{S}^\infty$ acts to the matrix at infinity, then the stress field $\boldsymbol{S}^v$ appears within the inhomogeneity. From now on $\boldsymbol{S}$ denotes one of the six stresses or couple-stresses, cited in (2.10); the index '0' refers to the matrix and the index '1' — to the inhomogeneity.

For the determination of stress fields $\boldsymbol{S}^v$ we apply Eshelby's method, at which the inhomogeneity is replaced by an inclusion with unknown distortion stress tensor $\boldsymbol{S}^0$ [5]. Let $\boldsymbol{S}^c$ be the stress into the inclusion, due to the distortion $\boldsymbol{S}^0$. According to (2.10), we have $\langle \boldsymbol{S}^c \rangle = B^{(0)} \boldsymbol{S}^0$, with the coefficient $B^{(0)}$ from (2.11), corresponding to the stress $\boldsymbol{S}$ under consideration; the index '0' means that $B^{(0)}$ is calculated using the matrix micropolar moduli (the inhomogeneity is replaced by a "homogeneity" with the same moduli as those of the matrix).

Firstly, it is necessary to make the stress $\boldsymbol{S}^v$ equal to the same stress, generated by the inclusion:

$$\boldsymbol{S}^v = \boldsymbol{S}^c + \boldsymbol{S}^\infty - \boldsymbol{S}^0 \,. \tag{3.1}$$

Since the tensor fields in (3.1) are nonhomogeneous within the spherical inclusion (Sect. 2), we shall require the validity of (3.1) only for the average values of the fields:

$$\langle \boldsymbol{S}^v \rangle = \langle \boldsymbol{S}^c \rangle + \boldsymbol{S}^\infty - \boldsymbol{S}^0 \,. \tag{3.1}$$

Secondly, it is necessary to make the average strains of the inhomogeneity $L_1^{-1} \langle \boldsymbol{S}^v \rangle$ equal to the same strains $L_0^{-1} (\langle \boldsymbol{S}^c \rangle + \boldsymbol{S}^\infty)$, due to the inclusion:

$$\frac{1}{L_1} \langle \boldsymbol{S}^v \rangle = \frac{1}{L_0} \left(\langle \boldsymbol{S}^c \rangle + \boldsymbol{S}^\infty \right) , \tag{3.3}$$

where L are the elastic moduli, corresponding to the stresses $\boldsymbol{S}$ and given in the brackets in (2.10).

Employing (3.2) and (3.3) we arrive at the following six relations:

$$\langle \boldsymbol{S}^v \rangle = \frac{L_1}{L_0 + B^{(0)}(L_1 - L_0)} \boldsymbol{S}^\infty \tag{3.4}$$

for the average stress fields $\langle \boldsymbol{S}^v \rangle$ within the spherical inhomogeneity, immersed into an unbounded micropolar body loaded at infinity with constant $\boldsymbol{S}^\infty$.

The relations (3.4) present, naturally, an approximate solution for the micropolar spherical inhomogeneity problem, because the nonhomogeneity of the stress fields within the sphere (V_a) is not taken into account. Merely the first relation of (3.4), connecting the spherical parts $\mathrm{sp}\, \boldsymbol{T}_\sigma^v$ and $\mathrm{sp}\, \boldsymbol{T}_\sigma^\infty$ is exact, moreover, it is the same one as in the elastic body and does not depend on the geometry of the inhomogeneity (see the end of Sect. 2).

However, these considerations give no information about the "distance" between the exact values of the average stress fields and those, calculated by the approximate solution (3.4). That is why we consider below the micropolar inhomogeneity problem in a rigorous way.

4. Micropolar Inhomogeneity Problem — Integral Equations

Let us consider again an unbounded micropolar body (matrix), containing a micropolar inhomogeneity (V) in it. For such a nonhomogeneous material Hooke's law (1.2) micropolar fourth-order tensors $\boldsymbol{L}(\boldsymbol{x})$ and $\boldsymbol{M}(\boldsymbol{x})$ are step-constant functions of the following form:

$$\boldsymbol{L}(\boldsymbol{x}) = \boldsymbol{L}_0 + \delta \boldsymbol{L}(\boldsymbol{x}) \,, \quad \boldsymbol{M}(\boldsymbol{x}) = \boldsymbol{M}_0 + \delta \boldsymbol{M}(\boldsymbol{x}) \,, \tag{4.1}$$

where $\delta\boldsymbol{L}(\boldsymbol{x}) = (\boldsymbol{L}_1 - \boldsymbol{L}_0)h_V(\boldsymbol{x})$, $\delta\boldsymbol{M}(\boldsymbol{x}) = (\boldsymbol{M}_1 - \boldsymbol{M}_0)h_V(\boldsymbol{x})$ and $h_V(\boldsymbol{x})$ is the characteristic function of the region (V); as in Sect. 3 the index '0' refers to the matrix and '1' — to the inhomogeneity.

Let $\boldsymbol{T}_\gamma(\boldsymbol{x})$ and $\boldsymbol{T}_\kappa(\boldsymbol{x})$ be the strain fields (1.1), which appear in the body with the inhomogeneity in it, when stresses $\boldsymbol{T}_\sigma^\infty$ and $\boldsymbol{T}_\mu^\infty$ act at infinity. According to Hooke's law (1.2),

$$\boldsymbol{T}_\sigma(\boldsymbol{x}) = \boldsymbol{L}(\boldsymbol{x}) : \boldsymbol{T}_\gamma(\boldsymbol{x})\,, \quad \boldsymbol{T}_\mu(\boldsymbol{x}) = \boldsymbol{M}(\boldsymbol{x}) : \boldsymbol{T}_\kappa(\boldsymbol{x}) \tag{4.2}$$

are the stress fields in the body. The fields (4.2) satisfy the known balance equations (1.2), so that we have

$$\begin{gathered}\nabla\cdot(\boldsymbol{L}_0 : \boldsymbol{T}_\gamma(\boldsymbol{x})) + \nabla\cdot(\delta\boldsymbol{L}(\boldsymbol{x}) : \boldsymbol{T}_\gamma(\boldsymbol{x})) = 0\,,\\ \nabla\cdot(\boldsymbol{M}_0 : \boldsymbol{T}_\kappa(\boldsymbol{x})) - \boldsymbol{E} : \boldsymbol{L}_0 : \boldsymbol{T}_\gamma(\boldsymbol{x})\\ +\nabla\cdot(\delta\boldsymbol{M}(\boldsymbol{x}) : \boldsymbol{T}_\kappa(\boldsymbol{x}) - \boldsymbol{E} : \delta\boldsymbol{L}(\boldsymbol{x}) : \boldsymbol{T}_\gamma(\boldsymbol{x})) = 0\end{gathered} \tag{4.3}$$

with (4.1) taken into account. The interpretation of (4.3) is quite simple: the existence of an inhomogeneity in the micropolar body is equivalent to the action of fictitious body forces and moments.

Inserting (1.1) into (4.3) we can rewrite the system (4.3) in the following integral form by using Green's tensor (1.4):

$$\begin{gathered}\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}^\infty(\boldsymbol{x}) + \int \boldsymbol{G}_{11}(\boldsymbol{x}-\boldsymbol{x}')\cdot\nabla'\cdot\left(\delta\boldsymbol{L}(\boldsymbol{x}') : \boldsymbol{T}_\gamma(\boldsymbol{x}')\right)\,dv'\\ +\int \boldsymbol{G}_{12}\,(\boldsymbol{x}-\boldsymbol{x}')\cdot\nabla'\cdot\left(\delta\boldsymbol{M}(\boldsymbol{x}') : \boldsymbol{T}_\kappa(\boldsymbol{x}') - \boldsymbol{E} : \delta\boldsymbol{L} : \boldsymbol{T}_\gamma(\boldsymbol{x})\right)\,dv'\,,\\ \boldsymbol{\varphi}^c(\boldsymbol{x}) = \boldsymbol{\varphi}^\infty(\boldsymbol{x}) + \int \boldsymbol{G}_{12}(\boldsymbol{x}-\boldsymbol{x}')\cdot\nabla'\cdot\left(\delta\boldsymbol{L}(\boldsymbol{x}') : \boldsymbol{T}_\gamma(\boldsymbol{x}')\right)\\ +\int \boldsymbol{G}_{22}(\boldsymbol{x}-\boldsymbol{x}')\cdot\left(\nabla'\cdot\left(\delta\boldsymbol{M}(\boldsymbol{x}') : \boldsymbol{T}_\kappa(\boldsymbol{x}')\right) - \boldsymbol{E} : \delta\boldsymbol{L} : \boldsymbol{T}_\gamma(\boldsymbol{x})\right)\,dv'\,.\end{gathered} \tag{4.4}$$

With the aid of Gauss theorem, we get from (4.4) the basic system of integral equations:

$$\begin{aligned}\boldsymbol{T}_\gamma(\boldsymbol{x}) &= \boldsymbol{T}_\gamma^\infty + \int_V \boldsymbol{G}_{11}(\boldsymbol{x}-\boldsymbol{x}') : \delta\boldsymbol{L} : \boldsymbol{T}_\gamma(\boldsymbol{x}')\,dv' + \int_V \boldsymbol{G}_{12}(\boldsymbol{x}-\boldsymbol{x}') : \delta\boldsymbol{M} : \boldsymbol{T}_\kappa(\boldsymbol{x}')\\ \boldsymbol{T}_\kappa(\boldsymbol{x}) &= \boldsymbol{T}_\kappa^\infty + \int_V \boldsymbol{G}_{21}(\boldsymbol{x}-\boldsymbol{x}') : \delta\boldsymbol{L} : \boldsymbol{T}_\gamma(\boldsymbol{x}')\,dv' + \int_V \boldsymbol{G}_{22}(\boldsymbol{x}-\boldsymbol{x}') : \delta\boldsymbol{M} : \boldsymbol{T}_\kappa(\boldsymbol{x}')\,,\end{aligned} \tag{4.5}$$

$\delta\boldsymbol{L} = \boldsymbol{L}_1 - \boldsymbol{L}_0$, $\delta\boldsymbol{M} = \boldsymbol{M}_1 - \boldsymbol{M}_0$; here the fourth-order tensor functions $\boldsymbol{\Gamma}_{i,j}$, $i, j = l, 2$, are expressed by the gradients of Green's tensor (1.4), viz.

$$\begin{gathered}\boldsymbol{\Gamma}_{11} = \nabla\boldsymbol{G}_{11}\nabla - \nabla\boldsymbol{G}_{12}\cdot\boldsymbol{E} - \boldsymbol{E}\cdot\boldsymbol{G}_{12}\nabla + \boldsymbol{E}\cdot\boldsymbol{G}_{22}\cdot\boldsymbol{E}\,,\\ \boldsymbol{\Gamma}_{12} = \nabla\boldsymbol{G}_{12}\nabla - \boldsymbol{E}\cdot\boldsymbol{G}_{22}\nabla\,, \quad \boldsymbol{\Gamma}_{21} = \nabla\boldsymbol{G}_{12}\nabla - \nabla\boldsymbol{G}_{22}\cdot\boldsymbol{E}\,,\\ \boldsymbol{\Gamma}_{22} = \nabla\boldsymbol{G}_{22}\nabla\,.\end{gathered} \tag{4.6}$$

The system (4.5) determines the strain state of an unbounded micropolar body, containing a micropolar inhomogeneity in it.

5. Iterative Solution of the System of Integral Equations

Let us consider the case of constancy of the field $\boldsymbol{T}_\gamma^\infty$ and $\boldsymbol{T}_\kappa^\infty$. Then we can assume first that the solution $\boldsymbol{T}_\gamma(\boldsymbol{x})$ and $\boldsymbol{T}_\kappa(\boldsymbol{x})$ of (4.5) is also constant within the inhomogeneity (V), so that $\boldsymbol{T}_\gamma = \langle \boldsymbol{T}_\gamma \rangle$ and $\boldsymbol{T}_\kappa = \langle \boldsymbol{T}_\kappa \rangle$. When we insert these values of the strain fields in the right-hand side of (4.5), we get

$$\begin{aligned} \boldsymbol{T}_\gamma(\boldsymbol{x}) &\approx \boldsymbol{T}_\gamma^\infty + \widetilde{\boldsymbol{\Gamma}}_{11} : \delta \boldsymbol{L} : \langle \boldsymbol{T}_\gamma \rangle + \widetilde{\boldsymbol{\Gamma}}_{12} : \delta \boldsymbol{M} : \langle \boldsymbol{T}_\kappa \rangle \ , \\ \boldsymbol{T}_\kappa(\boldsymbol{x}) &\approx \boldsymbol{T}_\kappa^\infty + \widetilde{\boldsymbol{\Gamma}}_{21} : \delta \boldsymbol{L} : \langle \boldsymbol{T}_\gamma \rangle + \widetilde{\boldsymbol{\Gamma}}_{22} : \delta \boldsymbol{M} : \langle \boldsymbol{T}_\kappa \rangle \ , \end{aligned} \tag{5.1}$$

where

$$\widetilde{\boldsymbol{\Gamma}}_{ij} = \int_V \boldsymbol{\Gamma}_{ij}(\boldsymbol{x} - \boldsymbol{x}')\, dv' \ , \tag{5.2}$$

$i, j = 1, 2$. According to (1.4) and (4.6), the tensor fields $\widetilde{\boldsymbol{\Gamma}}_{ij}$ can be expressed by the potentials (1.9) of the region (V).

Let us suppose the inhomogeneity (V) to be a centrosymmetric one. If we take the average values over (V) on both sides of (5.1), we get

$$\begin{aligned} \langle \boldsymbol{T}_\gamma \rangle_1 &= \left(\boldsymbol{J} - \left\langle \widetilde{\boldsymbol{\Gamma}}_{11} \right\rangle : \delta \boldsymbol{L} \right)^{-1} : \boldsymbol{T}_\gamma^\infty \ , \\ \langle \boldsymbol{T}_\kappa \rangle_1 &= \left(\boldsymbol{J} - \left\langle \widetilde{\boldsymbol{\Gamma}}_{22} \right\rangle : \delta \boldsymbol{M} \right)^{-1} : \boldsymbol{T}_\kappa^\infty \ , \end{aligned} \tag{5.3}$$

where $\boldsymbol{J}$ is the "unit" fourth-order tensor, so that $\boldsymbol{J} : \boldsymbol{T} = \boldsymbol{T}$ for each second-order tensor $\boldsymbol{T}$. Note that the average values $\left\langle \widetilde{\boldsymbol{\Gamma}}_{12} \right\rangle$ and $\left\langle \widetilde{\boldsymbol{\Gamma}}_{21} \right\rangle$ are absent in (5.3) because the functions $\widetilde{\boldsymbol{\Gamma}}_{12}$ and $\widetilde{\boldsymbol{\Gamma}}_{21}$ are expressed in the case under consideration by odd gradients of centrosymmetric potentials (1.9). The average strain values (5.3) are denoted by the index '1' in view of our interpretation for them as the first approximation for the average fields within the inhomogeneity (V).

We find the second approximation $\langle \boldsymbol{T}_\gamma \rangle_2$, $\langle \boldsymbol{T}_\kappa \rangle_2$ for the average strain tensors by introducing (5.1) in the right side of (4.5) and averaging over (V) of the so obtained relations. Then we get

$$\begin{aligned} \langle \boldsymbol{T}_\gamma \rangle_2 &= \boldsymbol{T}_\gamma^\infty + \left\langle \widetilde{\boldsymbol{\Gamma}}_{11} \right\rangle : \delta \boldsymbol{L} : \boldsymbol{T}_\gamma^\infty + \left(\left\langle \widetilde{\boldsymbol{\Gamma}}_{11} : \delta \boldsymbol{L} : \widetilde{\boldsymbol{\Gamma}}_{11} \right\rangle + \left\langle \widetilde{\boldsymbol{\Gamma}}_{12} : \delta \boldsymbol{M} : \widetilde{\boldsymbol{\Gamma}}_{21} \right\rangle \right) : \delta \boldsymbol{L} : \langle \boldsymbol{T}_\gamma \rangle_2 \ , \\ \langle \boldsymbol{T}_\kappa \rangle_2 &= \boldsymbol{T}_\kappa^\infty + \left\langle \widetilde{\boldsymbol{\Gamma}}_{22} \right\rangle : \delta \boldsymbol{M} : \boldsymbol{T}_\kappa^\infty + \left(\left\langle \widetilde{\boldsymbol{\Gamma}}_{21} : \delta \boldsymbol{L} : \widetilde{\boldsymbol{\Gamma}}_{12} \right\rangle + \left\langle \widetilde{\boldsymbol{\Gamma}}_{22} : \delta \boldsymbol{M} : \widetilde{\boldsymbol{\Gamma}}_{22} \right\rangle \right) : \delta \boldsymbol{M} : \langle \boldsymbol{T}_\kappa \rangle_2 \ , \end{aligned} \tag{5.4}$$

the central symmetry of the inhomogeneity (V) is taken into account once again. The third approximation can be similarly found, and so on.

Let us estimate the "distance" between the first and the second approximations. After simple calculations, we obtain from (5.3) and (5.4) that

$$\begin{aligned} \boldsymbol{F}_\gamma : \left(\langle \boldsymbol{T}_\gamma \rangle_2 - \langle \boldsymbol{T}_\gamma \rangle_1 \right) &= \boldsymbol{C}_\gamma + \boldsymbol{B}_\gamma : \langle \boldsymbol{T}_\gamma \rangle_1 \ , \\ \boldsymbol{F}_\kappa : \left(\langle \boldsymbol{T}_\kappa \rangle_2 - \langle \boldsymbol{T}_\kappa \rangle_1 \right) &= \boldsymbol{C}_\kappa + \boldsymbol{B}_\kappa : \langle \boldsymbol{T}_\gamma \rangle_1 \ , \end{aligned} \tag{5.5}$$

where

$$\boldsymbol{B}_\gamma = \left\langle \widetilde{\boldsymbol{\Gamma}}_{11} : \delta \boldsymbol{L} : \widetilde{\boldsymbol{\Gamma}}_{11} \right\rangle - \left\langle \widetilde{\boldsymbol{\Gamma}}_{11} \right\rangle : \delta \boldsymbol{L} : \left\langle \widetilde{\boldsymbol{\Gamma}}_{11} \right\rangle \ ,$$

$$\boldsymbol{B}_\kappa = \left\langle \widetilde{\boldsymbol{\Gamma}}_{22} : \delta \boldsymbol{L} : \widetilde{\boldsymbol{\Gamma}}_{22} \right\rangle - \left\langle \widetilde{\boldsymbol{\Gamma}}_{22} \right\rangle : \delta \boldsymbol{M} : \left\langle \widetilde{\boldsymbol{\Gamma}}_{22} \right\rangle \ ,$$

$$\boldsymbol{F}_\gamma = \boldsymbol{J} - \left\langle \widetilde{\mathbf{\Gamma}}_{11} : \delta\boldsymbol{L} : \widetilde{\mathbf{\Gamma}}_{11} \right\rangle : \delta\boldsymbol{L}\,, \quad \boldsymbol{F}_\kappa = \boldsymbol{J} - \left\langle \widetilde{\mathbf{\Gamma}}_{22} : \delta\boldsymbol{M} : \widetilde{\mathbf{\Gamma}}_{22} \right\rangle : \delta\boldsymbol{M}\,, \tag{5.6}$$

$$\boldsymbol{C}_\gamma = \left\langle \widetilde{\mathbf{\Gamma}}_{12} : \delta\boldsymbol{M} : \widetilde{\mathbf{\Gamma}}_{21} \right\rangle : \delta\boldsymbol{L} : \langle \boldsymbol{T}_\gamma \rangle_2\,,$$

$$\boldsymbol{C}_\kappa = \left\langle \widetilde{\mathbf{\Gamma}}_{21} : \delta\boldsymbol{M} : \widetilde{\mathbf{\Gamma}}_{12} \right\rangle : \delta\boldsymbol{M} : \langle \boldsymbol{T}_\kappa \rangle_2\,.$$

According to (5.6), only the contractions $\delta\boldsymbol{L} : \delta\boldsymbol{L}$, $\delta\boldsymbol{L} : \delta\boldsymbol{M}$, $\delta\boldsymbol{M} : \delta\boldsymbol{M}$ enter into the right side of (5.5). That is why, if the differences between micropolar properties of the matrix and the inhomogeneity vanish, i. e. $\delta\boldsymbol{L}$, $\delta\boldsymbol{M} \to 0$, then we have

$$\begin{aligned} \langle \boldsymbol{T}_\gamma \rangle &= \langle \boldsymbol{T}_\gamma \rangle_1 + o(\delta\boldsymbol{L}, \delta\boldsymbol{M})\,, \\ \langle \boldsymbol{T}_\kappa \rangle &= \langle \boldsymbol{T}_\kappa \rangle_1 + o(\delta\boldsymbol{L}, \delta\boldsymbol{M})\,. \end{aligned} \tag{5.7}$$

Thus, for the case of centrosymmetric inhomogeneity, the approximation (5.3) gives us the correct linear parts of the power series expansions of the average fields $\langle \boldsymbol{T}_\gamma \rangle$ and $\langle \boldsymbol{T}_\kappa \rangle$ with respect to $\delta\boldsymbol{L}$ and $\delta\boldsymbol{M}$.

In conclusion, returning to the case of spherical inhomogeneity, we note that, according to (1.4), (2.1), (4.6) and (5.1), the solution (3.4) coincides with (5.3). In this way, the average strain fields (3.4), found by Eshelby's method, present the first step of the iterative process proposed. Therefore, they possess correct linear terms with respect to the small parameters $\delta\boldsymbol{L}$, $\delta\boldsymbol{M} \to 0$.

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