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ON A LAMÉ'S PROBLEM IN THE MICROMORPHIC THEORY OF ANISOTROPIC DAMAGE

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ABSTRACT—The behaviour of a hollow damaging tube under pressure is discussed in the paper. A model, proposed by one of the authors of anisotropic brittle damage coupled with elastic deformation, is employed. The model represents a natural development of Eringen and Şuhubi's micromorphic theory [1] provided the microdistortion tensor of the latter is identified with that resulting from microcracking. A closed form solution of the problem is obtained using Laplace transform with respect to time.

1. INTRODUCTION

The aim of this paper is a further investigation of a microstructural model of damage. The model was introduced in a very brief form in [2]; in more details it was discussed and compared to some of the existing damage theories in the recent paper [3], but in the particular case of scalar damage only. In general, the model is a natural extension of Eringen and Şuhubi's theory microstructural theory [1], called by these authors micromorphic. The basic assumption of the latter is that the solid consists of "points" each of which represents a microvolume undergoing a microdistortion described by a second-rank tensor $\boldsymbol{\alpha}$. The tensor $\boldsymbol{\alpha}$ is kinematically independent of the macro-distortion $\mathbf{A} = \nabla\mathbf{u}$, generated by the macrodisplacement field $\mathbf{u}(\mathbf{x})$, obtained through an appropriate macroscopical averaging of the microdeformation. The premises of the theory and its mathematical development are given in a very clear and thorough form in the already mentioned well-known paper of Eringen and Şuhubi's [1]. As a matter of fact, an equivalent model was independently developed about the same time by Mindlin [4], using a bit more intuitive arguments.

A number of particular cases of the micromorphic theory are of special interest. First, this is the case when the tensor $\boldsymbol{\alpha}$ is skew-symmetric, so that the "points" of the body can rotate freely while their mass-centers move under straining. This assumption corresponds to the well-known micropolar theory of elasticity. Even a simpler model corresponds to the assumption that $\boldsymbol{\alpha}$ is spherical, $\boldsymbol{\alpha} = \theta\mathbf{I}$, which means that the body "points" undergo a free dilatation independent on the displacement of their mass-centers; as usual $\mathbf{I}$ stands hereafter for the unit 2nd rank tensor. The micromorphic theory and its generalizations were discussed in this particular case independently by Markov [5,6,7] (under the name dilatation theory of elasticity) and by Nunziato and Cowin [8] and Cowin *et al.* (under the name theory of materials with voids), see [9,10] and references therein, in the context of elastic materials with voids. Problems of Lamé's type, concerned with stress concentration and the influence of the microstructure over the latter were investigated in detail in [11] for the dilatation theory and in [12] for the more general micropolar-dilatation theory (in which the microdistortion is a sum of a skew-symmetric and spherical tensors).

In general, damage in solids, especially in the brittle case, appears in the form of microcracks that locally follow the main stress axes. A convenient characteristics of such an anisotropic damage is the so-called Vakulenko-Kachanov tensor $\boldsymbol{\alpha}$ [13,14]. Recall that the tensor $\boldsymbol{\alpha}$ is defined as

$$\boldsymbol{\alpha} = \frac{1}{V}\sum_{k=1}^{N}\int_{S_k} \mathbf{n}_k(\mathbf{x})\mathbf{b}_k(\mathbf{x})\,\mathrm{d}S_k, \tag{1.1}$$

where S_k is the surface of the kth crack S_k in the microvolume V; $\mathbf{n}_k(\mathbf{x})$ is the unit normal to S_k at the point $\mathbf{x} \in S_k$, $\mathbf{b}_k(\mathbf{x})$ is the crack opening at the same point $\mathbf{x}$; $\mathbf{n}_k\mathbf{b}_k = \mathbf{n}_k \otimes \mathbf{b}_k$ is the dyadic (tensor) product, N is the total number of cracks in the volume V. Let us point out immediately that normal opening is of central interest for us, which means that the vectors $\mathbf{n}_k$ and $\mathbf{b}_k$ are collinear and thus the tensor $\boldsymbol{\alpha}$ is symmetric—something to be assumed hereafter.

As argued in [2], the tensor $\boldsymbol{\alpha}$ can be identified with the microdistortion in the micromorphic theory of Eringen and Şuhubi (which explains why the same notation $\boldsymbol{\alpha}$ was used for both quantities). This is quite natural, since macrocracking generates a certain additional microdistortion which is obviously kinematically independent of the local deformation. Thus the basic kinematical premises of the micromorphic theory apply directly to microcracked solids.[1] Moreover, the foregoing remark concerning symmetricity of the Vakulenko-Kachanov tensor explains why the microdistortion will be assumed hereafter symmetric.

The laws of the damage evolution are very sensitive however to the type of fracture behaviour in the sense of whether the solid fails in a brittle or plastic manner. In the latter case it is reasonable to couple the micromorphic kinematics with the thermodynamic time of Vakulenko [16] thus building up models of "endochronic" type in Valanis' meaning, see again [2] for a bit more details and arguments. In the former case—brittle failure, that is—there is no irreversible macrodeformation (at a fixed damage level $\boldsymbol{\alpha}$), so that only the small strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla)$ should be taken into account. The damage evolution is described then through introducing the rate of damage tensor $\dot{\boldsymbol{\alpha}}$ into the constitutive equations. Details of the thermodynamical analysis of the appropriate such equations are given in [2,3]. (Note that the analysis is a straightforward generalization of Nunziato and Cowin's arguments [8] for including the rate of the volume change $\dot{\theta}$ in their theory of materials with voids.)

2. BASIC EQUATIONS

Recall first the basic balance equations of the micromorphic theory in the static case (under the assumption of small strain), at the absence of body sources:

$$\nabla \cdot \boldsymbol{\sigma} = 0, \quad \nabla \cdot \boldsymbol{\Lambda} + \mathbf{s} - \boldsymbol{\sigma} = 0. \tag{2.1}$$

Here $\mathbf{s}$ is the microstress (a 2nd rank symmetric tensor field), $\boldsymbol{\sigma}$ is the Cauchy macrostress (a 2nd rank tensor field, nonsymmetric in general) and $\boldsymbol{\Lambda}$ is the hyperstress (a third-rank tensor field).

The general constitutive equations of the micromorphic model of anisotropic damage read

$$\boldsymbol{\sigma} = \rho\frac{\partial W}{\partial \boldsymbol{\varepsilon}}, \quad \mathbf{s} - \boldsymbol{\sigma} = \rho\frac{\partial W}{\partial \boldsymbol{\alpha}} + F(\dot{\boldsymbol{\alpha}}), \quad \boldsymbol{\Lambda} = \rho\frac{\partial W}{\partial \nabla\boldsymbol{\alpha}}\,, \tag{2.2}$$

ρ is the density of the solid. Neglecting the thermal effects, the potential function has the form $W = W(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}, \nabla\boldsymbol{\alpha})$ and it should be nonnegative. Also

$$F(\dot{\boldsymbol{\alpha}}) : \dot{\boldsymbol{\alpha}} \geq 0, \tag{2.3}$$

[1] This possibility was clear, in principle, to Eringen and Şuhubi [15]; one could only regret that they have not pursue it in their subsequent work.

which is the dissipation inequality of the model; the colon denotes contraction with respect to two pairs of indices.

In the linearized version of the model W is a quadratic function of its arguments. The most general such a version contains 20 material parameters [2]. However, the fact that the microdistortion $\boldsymbol{\alpha}$ is symmetric reduces this number to 11. Without trying to write down the most general equations in this case, let us assume

$$\rho W = \frac{1}{2}\lambda \mathrm{tr}^{\,2}\boldsymbol{\varepsilon} + \mu \mathrm{tr}\,(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}) - \xi_1 \mathrm{tr}\,(\boldsymbol{\varepsilon}\cdot\boldsymbol{\alpha}) + \frac{1}{2}\xi_2 \mathrm{tr}\,(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}) + \boldsymbol{\Lambda}.\nabla\boldsymbol{\alpha}, \tag{2.4}$$

the dot in the product $\boldsymbol{\Lambda}.\nabla\boldsymbol{\alpha}$ standing for full contraction ($\Lambda_{kml}\alpha_{lm,k}$) with respect to the three pairs of indices;

$$F(\dot{\boldsymbol{\alpha}}) = f\dot{\boldsymbol{\alpha}} \tag{2.5}$$

and, finally,

$$\Lambda_{kml} = \eta_1\alpha_{lm,k} + \eta_2(\alpha_{kl,m} + \alpha_{km,l}) \tag{2.6}$$

with respect to a Cartesian coordinate system. Without analyzing in more detail the consequences of the positive definiteness of $\boldsymbol{\Lambda}.\nabla\boldsymbol{\alpha}$, we shall simply assume η_1, $\eta_2 > 0$.

Besides, the conditions

$$\mu > 0, \quad k = \lambda + \frac{2}{3}\mu > 0, \quad \xi_2 > 0, \quad k\xi_2^2 - \xi_1^2 > 0 \tag{2.7}$$

are necessary in order to assure positive definiteness of W. Also $f > 0$ due to the dissipation inequality (2.3).

As it follows from the definition (1.1) of the tensor $\boldsymbol{\alpha}$, its trace represents the volume change (porosity) due to damage. That is why the density ρ of the solid has now the form

$$\rho = \rho_0(1 - \mathrm{tr}\,\boldsymbol{\alpha}), \tag{2.8}$$

where ρ_0 is the density of the undamaged material.

The dependence of ρ upon damage makes the constitutive equations of the model nonlinear. This important fact was pointed out in [2] in the scalar case. The situation here, when anisotropic damage in the form of microcracking is taken into account, is even more complicated. The reason is that the elastic moduli λ, μ and, more generally, the material parameters, depend on damage. In the isotropic case, when only microvoids appear and develop under loading, the solid remains macroscopically isotropic and there exist comparatively simple and accurate formulae for calculating λ and μ for a solid with voids, see, e.g., the book [17]. When damage is anisotropic, the body becomes macroscopically anisotropic as well, with symmetry properties defined locally by the tensor $\boldsymbol{\alpha}$. To a certain extent this fact is reflected in the form (2.4) of the potential W, through the joint invariant $\mathrm{tr}\,(\boldsymbol{\varepsilon}\cdot\boldsymbol{\alpha})$ of the strain and damage tensors. However, the dependence of the elastic moduli upon microcracking should be accounted for here; the appropriate formulae are discussed in the survey [14]. Thus an attempt to build up a more rigorous micromorphic theory of brittle damage is necessarily quite complicated. Here, however, we shall neglect the dependence of the density and material properties upon damage (assuming, in particular, $\mathrm{tr}\,\boldsymbol{\alpha}$ small enough). In this way we are able, hopefully, to concentrate on the essential effects of damage anisotropy on the behaviour of a brittle microcracked solid during loading, within the frame of the above model.

The constitutive equations (2.2) and (2.4)–(2.6) yield

$$\begin{aligned}\boldsymbol{\sigma} &= \lambda \mathbf{I}\,\mathrm{tr}\,\boldsymbol{\varepsilon} + 2\mu\boldsymbol{\varepsilon} - \xi_1\boldsymbol{\alpha},\\ \mathbf{s} - \boldsymbol{\sigma} &= -\xi_1\boldsymbol{\varepsilon} + \xi_2\boldsymbol{\alpha} + f\dot{\boldsymbol{\alpha}}\end{aligned} \tag{2.9}$$

which, when inserted into the balance laws (2.1) gives now the system

$$\begin{aligned}&\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} - \xi_1\nabla\boldsymbol{\alpha} = 0,\\ &\eta_1\Delta\boldsymbol{\alpha} + \eta_2\Big[\nabla(\nabla\cdot\boldsymbol{\alpha}) + (\nabla\cdot\boldsymbol{\alpha})\nabla\Big] - \xi_2\boldsymbol{\alpha} + \xi_1\boldsymbol{\varepsilon} - f\dot{\boldsymbol{\alpha}} = 0\end{aligned} \tag{2.10}$$

for the unknown displacement, $\mathbf{u}(\mathbf{x},t)$, and damage tensor, $\boldsymbol{\alpha}(\mathbf{x},t)$, fields. Provided $\eta_2 = 0$, the system (2.10) reduces to that of the dilatation theory [2,7] if the tensor $\boldsymbol{\alpha}$ is spherical and the dependence of the density ρ upon damage is neglected.

The classical initial and boundary conditions in stress and/or displacement are then imposed. As far as the initial conditions in damage are concerned, we require

$$\boldsymbol{\alpha}(\mathbf{x},0) = 0 \tag{2.11}$$

—no damage in the initial state ($t=0$) of the body. The only specific thing is the boundary condition in hyperstresses; a natural choice is the requirement of no normal hyperstresses on S:

$$\mathbf{n}\cdot\boldsymbol{\Lambda}\Big|_S = 0, \tag{2.12}$$

where $\mathbf{n}$ is the outward unit normal vector to the boundary S of the solid.

Note that a rigorous proof of the existence and uniqueness theorem for such an initial-boundary-value problem for the system (2.10) (and for the more general micromorphic model of [2]) would be of interest. Such a theorem has been recently proved in [18] for the particular case of the system (2.10), corresponding to the scalar case, when $\boldsymbol{\alpha}$ is spherical and $\dot{\boldsymbol{\alpha}}$ is neglected.

3. THE LAMÉ PROBLEM

Consider a hollow circular tube, unbounded along its axis, which occupies the region $R_1 \leq r \leq R_2$; hereafter polar coordinates r, φ in the cross-section of the tube are used. The tube is subject to the boundary conditions

$$\sigma_r \Big|_{r=R_i} = P_i, \quad i = 1, 2\,, \tag{3.1}$$

i.e., known pressure on the internal and external tube's surfaces. Due to the axial symmetry, the problem is plane and the solution should depend on the radial coordinate r solely. In particular, the displacement field is

$$\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = u(r, t)\mathbf{e}_r, \quad r = |\mathbf{x}|\,, \tag{3.2}$$

where $u = u_r$ is the radial displacement; $\mathbf{e}_r = \mathbf{r}/r$ is the unit radial vector, the prime denotes differentiation with respect to the radial coordinate r. Hence the strain tensor is

$$\boldsymbol{\varepsilon} = \nabla\mathbf{u} = \left(u' - \frac{u}{r}\right)\mathbf{e}_r\mathbf{e}_r + \frac{u}{r}\mathbf{I}, \quad \nabla\cdot\mathbf{u} = u' + \frac{u}{r}\,. \tag{3.3}$$

Also,

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}(r, t) = a(r, t)\mathbf{e}_r\mathbf{e}_r + b(r, t)\mathbf{I}\,, \tag{3.4}$$

where $\mathbf{I}$ is the unit second rank tensor in R^2, so that $\operatorname{tr}\mathbf{I} = 2$. Therefore

$$\begin{aligned} &\nabla\cdot\boldsymbol{\alpha} = \chi(r, t), \quad \chi(r, t) = a'(r, t) + b'(r, t) + \frac{a(r, t)}{r}, \\ &\Delta\boldsymbol{\alpha} = \left(a'' + \frac{a'}{r} - \frac{4a}{r^2}\right)\mathbf{e}_r\mathbf{e}_r + \left(b'' + \frac{b'}{r} + \frac{2a}{r^2}\right)\mathbf{I}\,. \end{aligned} \tag{3.5}$$

Introducing (3.2) and (3.4) into (2.9) and taking into account (3.3) and (3.5) yields after some algebra

$$(\lambda + 2\mu)\left(u' + \frac{u}{r}\right)' - \xi_1\chi = 0, \tag{3.6a}$$

$$\eta_1\left(a'' + \frac{a'}{r} - \frac{4a}{r^2}\right) + \eta_2\left(\chi' - \frac{\chi}{r}\right) + \xi_1\left(u' - \frac{u}{r}\right) - \xi_2 a - f\dot{a} = 0, \tag{3.6b}$$

$$\eta_1\left(b''+\frac{b'}{r}+\frac{2a}{r^2}\right)+2\eta_2\frac{\chi}{r}+\xi_1\frac{u}{r}-\xi_2 b-f\dot{b}=0, \tag{3.6c}$$

which is the basic system for the unknown functions $u=u(r,t)$, $a=a(r,t)$ and $b=b(r,t)$; the function $\chi=\chi(r,t)$ is defined in (3.5).

The boundary conditions in stress (3.1), with (2.9), (3.3) and (3.4) taken into account, read

$$(\lambda+2\mu)u'(r,t)+\lambda\frac{u(r,t)}{r}-\xi_1[a(r,t)+b(r,t)]=P_i \quad \text{at} \quad r=R_i, \quad i=1,2. \tag{3.7}$$

In turn, keeping in mind (2.6) and the form (3.4) of the tensor $\boldsymbol{\alpha}$, one gets

$$\mathbf{e}_r\cdot\boldsymbol{\Lambda}=\left[\eta_1 a'+2\eta_2\left(a'+b'-\frac{a}{r}\right)\right]\mathbf{e}_r\mathbf{e}_r+\left(\eta_1 b'+2\eta_2\frac{a}{r}\right)\mathbf{I}\,.$$

The condition (2.12) in hyperstresses then implies the following boundary conditions for the unknown functions $a(r,t)$ and $b(r,t)$:

$$\begin{aligned} a'(r,t)+b'(r,t)&=0,\\ \eta_1 b'(r,t)+2\eta_2\frac{a(r,t)}{r}&=0 \quad \text{at} \quad r=R_1,R_2. \end{aligned} \tag{3.8}$$

The initial conditions are

$$u(r,0)=0, \quad a(r,0)=0, \quad b(r,0)=0. \tag{3.9}$$

Hence the Lamé problem under study consists mathematically in solving the initial-boundary-value problem (3.6), (3.7)–(3.9).

The "key" moment of the solution of this problem is the following remark. Note that $\varphi'-\dfrac{\varphi}{r}=r\left(\dfrac{\varphi}{r}\right)'$, for any φ, and rewrite (3.6b) in the form

$$\eta_1\left(a''+\frac{a'}{r}-\frac{4a}{r^2}\right)+2\eta_2 r\left(\frac{\chi}{r}\right)'+\xi_1 r\left(\frac{u}{r}\right)'-\xi_2 a-f\dot{a}=0\,. \tag{3.10}$$

Differentiate (3.6c) with respect to r and multiply the result by r

$$\eta_1 r\left(b''+\frac{b'}{r}+\frac{2a}{r^2}\right)'+2\eta_2 r\left(\frac{\chi}{r}\right)'+\xi_1 r\left(\frac{u}{r}\right)'-\xi_2 r b-f r\dot{b}=0;$$

extract the latter from (3.10)

$$\eta_1 r\left[\left(a''+\frac{a'}{r}-\frac{4a}{r^2}\right)-r\left(b''+\frac{b'}{r}+\frac{2a}{r^2}\right)'\right]-\xi_2 r(a-rb')-f(\dot{a}-r\dot{b}')=0. \tag{3.11}$$

The very structure of the left side of (3.11) suggests to introduce the new function

$$F = F(r,t) = a(r,t) - rb(r,t) \tag{3.12}$$

with respect to which (3.11) considerably simplifies

$$\eta_1 \left(F'' - \frac{1}{r} F' \right) - \xi_2 F - f\dot{F} = 0. \tag{3.13}$$

To solve (3.13), let us apply the Laplace transform $\varphi(t) \to \overline{\varphi}(s)$ with respect to the time variable t. Note that the idea of utilizing the Laplace transform is fully natural here, due to the very structure of the system (3.6). Moreover, this transform was already successfully applied to similar problems in the theory of materials with voids [11] and in the micropolar-dilatation model [12].

Since

$$\overline{\dot{F}} = s\overline{F} - F_0$$

and, due to the initial conditions (3.9), $F_0 = F(r,0) = a(r,0) - rb(r,0) = 0$, (3.13) yields

$$\overline{F}'' - \frac{1}{r}\overline{F}' - \kappa_1^2 \overline{F} = 0, \quad \kappa_1^2 = \frac{\xi_2 + fs}{\eta_1} > 0\,. \tag{3.14}$$

The solution of (3.14) is obvious

$$\overline{F}(r,s) = r[C_1 I_1(\kappa_1 r) + C_2 K_1(\kappa_1 r)] \tag{3.15}$$

where I_1 and K_1 stand for the respective modified Bessel functions and C_1, C_2 are functions of s only whose determination will be discussed later on.

Let us now multiply (3.6c) by 2 and add the result to (3.6b). In this way we get an equation, containing $u' + \frac{u}{r}$, i.e., the same expression that enters (3.6a); this fact will allow us to exclude the latter and get an equation for another combination of the functions a and b. The procedure is equivalent to taking the trace of the second equation (2.10):

$$\eta_1 \Delta\psi + 2\eta_2 \nabla \cdot \boldsymbol{\alpha} \cdot \nabla + \xi_1 \left(u' + \frac{u}{r} \right) - \xi_2 \psi - f\dot{\psi} = 0, \tag{3.16}$$

since $\operatorname{tr} \boldsymbol{\varepsilon} = \nabla \cdot \mathbf{u} = u' + \frac{u}{r}$, and $\psi = \psi(r,t) = a(r,t) + 2b(r,t)$. But $\nabla \cdot \boldsymbol{\alpha} \cdot \nabla = \chi' + \frac{\chi}{r}$ as it follows from (3.4) and the definition of χ, see (3.5). Thus (3.16) becomes

$$\eta_1 \left(\psi'' + \frac{\psi'}{r} \right) + 2\eta_2 \left(\chi' + \frac{\chi}{r} \right) + \xi_1 \left(u' + \frac{u}{r} \right) - \xi_2 \psi - f\dot{\psi} = 0. \tag{3.17}$$

Note that

$$\chi = \psi' + \frac{1}{r}F \tag{3.18}$$

which allows to recast (3.17) as

$$(\eta_1 + 2\eta_2)\left(\psi'' + \frac{\psi'}{r}\right) + \xi_1\left(u' + \frac{u}{r}\right) - \xi_2\psi - f\dot{\psi} + 2\eta_2\frac{1}{r}F' = 0. \tag{3.19}$$

Differentiating (3.19) with respect to r allows to exclude $u' + \dfrac{u}{r}$ from (3.6a) and thus yields an equation for the function

$$\Psi = \Psi(r,t) = \psi'(r,t) = a'(r,t) + 2b'(r,t), \tag{3.20}$$

namely,

$$(\eta_1 + 2\eta_2)\left(\Psi' + \frac{\Psi}{r}\right)' - \left(\xi_2 - \frac{\xi_1^2}{\lambda + 2\mu}\right)\Psi - f\dot{\Psi} + \frac{\xi_1^2}{\lambda + 2\mu}\,\frac{1}{r}F + 2\eta_2\left(\frac{1}{r}F'\right)' = 0\,.$$

Apply again the Laplace transform to the last equation and take into account the equation (3.14) for the function $\overline{F}$:

$$\left(\overline{\Psi}' + \frac{\overline{\Psi}}{r}\right)' - \kappa_2^2\overline{\Psi} + \kappa_3^2\frac{1}{r}\overline{F} = 0\,, \tag{3.21}$$

with the constants

$$\kappa_2^2 = \frac{(\xi_2 + fs)(\lambda + 2\mu) - \xi_1^2}{(\eta_1 + 2\eta_2)(\lambda + 2\mu)} > 0, \quad \kappa_3^2 = \frac{\xi_1^2 + 2\eta_2(\lambda + 2\mu)\kappa_1^2}{(\eta_1 + 2\eta_2)(\lambda + 2\mu)} > 0,$$

κ_1 is defined in (3.14). The general solution of the homogeneous equation (3.21) is

$$\overline{\Psi}(r,t) = D_3 I_1(\kappa_2 r) + D_4 K_1(\kappa_2 r)\,. \tag{3.22}$$

Since $\dfrac{1}{r}\overline{F}$ is a linear combination of the functions $I_1(\kappa_1 r)$ and $K_1(\kappa_1 r)$, see (3.15), both of which solve the equation

$$\left(G' + \frac{G}{r}\right)' = \kappa^2 G, \quad \kappa = \kappa_1, \tag{3.23}$$

it is easily seen that a particular solution of (3.21) is

$$\frac{\kappa_3^2}{\kappa_2^2 - \kappa_1^2}K_1(\kappa_1 r) \;\text{ or }\; \frac{\kappa_3^2}{\kappa_2^2 - \kappa_1^2}I_1(\kappa_1 r). \tag{3.24}$$

Choosing, say, the first one and using the well-known properties of the modified Bessel functions, we get

$$
\begin{aligned}
\overline{a}(r,s) + 2\overline{b}(r,s) &= \int_{R_1}^{r} \overline{\Psi}(\tau,s)\,\mathrm{d}\tau \\
&= C_3 I_0(\kappa_2 r) + C_4 K_0(\kappa_2 r) - \frac{\kappa_3^2}{\kappa_1(\kappa_2^2-\kappa_1^2)} K_0(\kappa_1 r) + C_5,
\end{aligned}
\tag{3.25}
$$

see (3.20), where $C_3 = D_3/\kappa_2$, $C_4 = -D_4/\kappa_2$ and C_5 are functions of s.

To complete the solution, the displacement field u remains to be found. To this end we apply again the Laplace transform to (3.6a), using (3.18) and (3.20)

$$
\left(\overline{u}' + \frac{\overline{u}}{r}\right)' = \frac{\xi_1}{\lambda+2\mu}\left(\overline{\Psi} + \frac{1}{r}\overline{F}\right). \tag{3.26}
$$

The general solution of the homogeneous equation (3.26) is

$$
U(r,s) = C_6 r + \frac{C_7}{r}\cdot \tag{3.27}
$$

As far as a particular solution of the inhomogeneous equation (3.26) is concerned we can repeat the arguments that yielded (3.24)—the particular solution of (3.21). Indeed, the right side of (3.26) is a linear combination of the functions $I_1(\kappa_2 r)$, $K_1(\kappa_2 r)$ and, say, $K_1(\kappa_1 r)$, as it follows from (3.15), (3.22) and (3.24):

$$
\left(\overline{u}' + \frac{\overline{u}}{r}\right)' = \frac{\xi_1}{\lambda+2\mu}\left[(C_1 + C_3\kappa_2) I_1(\kappa_2 r) + (C_2 - C_4\kappa_2) K_1(\kappa_2 r) + \frac{\kappa_3^2}{\kappa_2^2-\kappa_1^2} K_1(\kappa_1 r)\right].
$$

These functions satisfy the equation (3.23) at $\kappa = \kappa_1$ or $\kappa = \kappa_2$; hence it is easy to see that

$$
\begin{aligned}
V(r,s) = \frac{\xi_1}{\lambda+2\mu}\Big[&\frac{C_1 + C_3\kappa_2}{\kappa_2^2} I_1(\kappa_2 r) + \frac{C_2 - C_4\kappa_2}{\kappa_2^2} K_1(\kappa_2 r) \\
&+ \frac{\kappa_3^2}{\kappa_1^2(\kappa_2^2-\kappa_1^2)} K_1(\kappa_1 r)\Big]
\end{aligned}
\tag{3.28}
$$

is a needed particular solution of (3.26). Thus the general form of the Laplace transform of the displacement field is

$$
\overline{u}(r,s) = U(r,s) + V(r,s), \tag{3.29}
$$

where U and V are given in (3.27) and (3.28) respectively.

From (3.15) and (3.25) we can easily find $\overline{a}(r,s)$ and $\overline{b}(r,s)$. Thus the Laplace transforms (with respect to time) of the unknown functions—$u(r,t)$, $a(r,t)$ and $b(r,t)$—in the

Lamé problem (3.6) are found in a closed form, containing the seven functions $C_i(s)$, $i = 1, \dots, 7$. To determine the latter, the Laplace transforms of the six boundary conditions (3.7) and (3.8) of the problem are invoked. The additional interconnection between $C_i(s)$ is supplied by the Laplace transform of (3.19). (Recall that we have differentiated it in order to exclude $u' + \dfrac{u}{r}$.)

6. CONCLUDING REMARKS

We have shown that the Lamé problem in the micromorphic theory of brittle anisotropic damage admits a closed form analytical solution, provided Laplace transform with respect to time is utilized. The numerical implementation of the obtained solution is not straightforward however and should be addressed separately in a further study. The numerical results would allow to investigate the microcracks orientation which obviously is nonhomogeneous spatially and possess a strong anisotropy. Moreover, it is expected that a well pronounced damage localization should be observed near the tube's surfaces.

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