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A CONDITION FOR NONDEGENERACY FOR A SECOND–ORDER SYMMETRIC TENSOR

KONSTANTIN MARKOV

Константин Марков. УСЛОВИЕ НЕВЫРОЖДЕННОСТИ ДВУХВАЛЕНТНОГО СИММЕТРИЧНОГО ТЕНЗОРА

Двухвалентный симметричный тензор над трехмерным евклидовым пространством называется невырожденным, если его собственные числа попарно различны. В работе дается простой критерий невырожденности. Именно для этого необходимо и достаточно существование декартовой координатной системы, в которой одна и только одна недиагональная компонента тензора нулевая. Как следствие отсюда получается, что собственные числа симметричной 3×3 матрицы попарно различны тогда и только тогда, когда она подобна матрице с только одним нулевым недиагональным элементом.

Konstantin Markov. A CONDITION FOR NONDEGENERACY FOR A SECOND–ORDER SYMMETRIC TENSOR

A second–order symmetric tensor over a three–dimensional Euclidean space is called nondegenerate if its eigenvalues differ one from another. In the paper a simple criterion for nondegeneracy is given, namely, it appears that the tensor is nondegenerate iff there exists a Cartesian system in which only one of the nondiagonal components of the tensor vanishes. As a consequence it is noticed that the eigenvalues of a symmetric 3×3 matrix are mutually different iff it is similar to a matrix which has only one vanishing nondiagonal element.

Let $\boldsymbol{T}$ be a second-order symmetric tensor over a three-dimensional real Euclidean space. As is well-known, the tensor $\boldsymbol{T}$ has three real eigenvalues λ_i to which correspond three unit eigenvectors $\boldsymbol{e}_i$; the directions of the vectors $\boldsymbol{e}_i$ are uniquely defined provided $\lambda_i \neq \lambda_j$, $i \neq j$, $i, j = 1,\ 2,\ 3$, see, e. g. [1, p. 112]. In the latter case we call the tensor $\boldsymbol{T}$, after [2], nondegenerate.

In a recent paper [3], in the context of inertia tensor for a solid, it was shown that the condition

$$(1) \qquad t_{12} = 0, \qquad t_{23}\, t_{31} \neq 0, \qquad t_{22} > t_{11}, \qquad t_{22} > t_{33},$$

where t_{ij}, $i, j = 1, 2, 3$, are the components of the tensor $\boldsymbol{T}$ in a certain Cartesian system, suffices for $\boldsymbol{T}$ to be nondegenerate.

In the theorem below we propose a condition which is weaker than (1) and at the same time it is not only sufficient but necessary as well for nondegeneracy.

T h e o r e m. The tensor $\boldsymbol{T}$ is nondegenerate iff there exists a Cartesian coordinate system in which only one nondiagonal component of $\boldsymbol{T}$ vanishes.

Proof. a) Let $\boldsymbol{T}$ be nondegenerate and $\boldsymbol{e}_i$ denote the unit eigenvectors for $\boldsymbol{T}$. We choose an unit vector $\boldsymbol{e}_{1'}$, which neither coincides with $\boldsymbol{e}_i$ nor is perpendicular to $\boldsymbol{e}_i$, $i = 1, 2, 3$. Thus $\boldsymbol{e}_{1'}$ cannot be an eigenvector for $\boldsymbol{T}$. Consider the vector $\boldsymbol{a} = \boldsymbol{T}\cdot \boldsymbol{e}_{1'}$; hereafter the dot means contraction with respect to one pair of indexes, e. g. $(\boldsymbol{T}\cdot \boldsymbol{x})^i = t^{ij}\, x_j$. Obviously $\boldsymbol{a} \neq 0$; otherwise $\boldsymbol{a}$ would be an eigenvector, corresponding to the eigenvalue $\lambda = 0$. We introduce next the unit vector $\boldsymbol{e}_{2'}$ which is perpendicular both to $\boldsymbol{e}_{1'}$ and to $\boldsymbol{a}$ so that, in particular,

$$(2) \qquad \boldsymbol{a} \cdot \boldsymbol{e}_{2'} = \boldsymbol{e}_{1'} \cdot \boldsymbol{T}\cdot \boldsymbol{e}_{2'} = 0,$$

since $\boldsymbol{T}$ is symmetric. Note that $\boldsymbol{e}_{2'}$ cannot coincide with some of the eigenvectors $\boldsymbol{e}_i$ for $\boldsymbol{T}$, because we have chosen $\boldsymbol{e}_{1'}$ not to be perpendicular to anyone of $\boldsymbol{e}_i$, $i = 1, 2, 3$.

Let $\boldsymbol{e}_{3'}$ be the unit vector perpendicular to $\boldsymbol{e}_{1'}$ and $\boldsymbol{e}_{2'}$. We introduce now the Cartesian system whose axes $x_{i'}$ are along the vectors $\boldsymbol{e}_{i'}$, $i' = 1, 2, 3$. The components of $\boldsymbol{T}$ in the latter system are $t_{i'j'} = \boldsymbol{e}_{i'} \cdot \boldsymbol{T}\cdot \boldsymbol{e}_{j'}$, $i', j' = 1, 2, 3$, so that (2) means that the nondiagonal component $t_{1'2'}$ vanishes, $t_{1'2'} = 0$, and it remains to show that $t_{2'3'}t_{3'1'} \neq 0$.

Indeed, let say, $t_{2'3'} = 0$. In this case $\boldsymbol{T}\cdot \boldsymbol{e}_{2'}$ is perpendicular to both $\boldsymbol{e}_{1'}$ and $\boldsymbol{e}_{3'}$, so that $\boldsymbol{T}\cdot \boldsymbol{e}_{2'} \parallel \boldsymbol{e}_{2'}$ and thus $\boldsymbol{e}_{2'}$ is eigenvector for $\boldsymbol{T}$ which cannot happen by the very construction of $\boldsymbol{e}_{2'}$. Thus we have found a Cartesian system in which only one nondiagonal component, $t_{1'2'}$, of the nondegenerate tensor $\boldsymbol{T}$ vanishes so that the necessity of our condition is proved.

b) Suppose there exists a Cartesian system $x_{i'}$ with unit vectors $\boldsymbol{e}_{i'}$, $i' = 1, 2, 3$, such that, say,

$$(3) \qquad t_{1'2'} = 0, \qquad t_{2'3'}\, t_{3'1'} \neq 0.$$

Assume that the tensor $\boldsymbol{T}$ is degenerate so that two of its eigenvalues, say λ_1 and λ_2, coincide, $\lambda_1 = \lambda_2 \neq \lambda_3$. Then the tensors $\boldsymbol{I}$, $\boldsymbol{T}$, $\boldsymbol{T}\cdot \boldsymbol{T}$ are linearly dependent (because we can find a quadratic function $f(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2$ in this case, for which $f(\lambda_i) = 0$, $i = 1, 2, 3$, see, e. g. [1, p. 105]) so that

$$(4) \qquad \alpha_0\boldsymbol{I} + \alpha_1\boldsymbol{T} + \alpha_2\boldsymbol{T}\cdot \boldsymbol{T} = 0,$$

where $\alpha_0^2 + \alpha_1^2 + \alpha_2^2 \neq 0$. We write the tensor relation (4) with respect to the Cartesian system $x_{1'}x_{2'}x_{3'}$

$$(5) \quad \alpha_0 \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} + \alpha_1 \begin{Vmatrix} t_{1'1'} & 0 & t_{1'3'} \\ 0 & t_{2'2'} & t_{2'3'} \\ t_{1'3'} & t_{2'3'} & t_{3'3'} \end{Vmatrix} + \alpha_2 \begin{Vmatrix} \cdot & t_{1'3'}t_{2'3'} & \cdot \\ t_{1'3'}t_{2'3'} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{Vmatrix} = 0;$$

the dots replace the respective components of the tensor $\boldsymbol{T}\cdot \boldsymbol{T}$ whose obvious expressions are of no significance for us. Due to (3), we have from (5) $\alpha_2 = 0$ so that $\boldsymbol{T}$ should be spherical and therefore all its nondiagonal components vanish.

The latter contradicts however (3) and thus the sufficiency of our condition (3) for nondegeneracy of the tensor $\boldsymbol{T}$ is proved.

Two obvious corollaries of the foregoing theorem seem worth mentioning.

C o r o l l a r y 1. Let $\boldsymbol{A} = \|a_{ij}\|$ be a 3×3 symmetric real matrix. The eigenvalues λ_i of $\boldsymbol{T}$ are mutually different, $\lambda_i \neq \lambda_j$, $i \neq j$, iff the matrix $\boldsymbol{T}$ is similar to a matrix which has only one nondiagonal element that vanishes.

C o r o l l a r y 2. Let $\boldsymbol{I}$ be the inertia tensor for a solid whose main inertia moments I_i are mutually different, $I_i \neq I_j$, $i \neq j$. Then there exist infinitely many Cartesian coordinate systems for which only one of the nondiagonal inertia components vanishes. Moreover, some of these Cartesian systems are explicitly described in the first part a) of the proof of the Theorem.

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