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STRESS CONCENTRATION AROUND A HOLE IN A MICROPOLAR SOLID WITH VOIDS

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Валентина Механджиева, Константин Марков. **КОНЦЕНТРАЦИЯ НАПРЯЖЕНИЙ ОКОЛО ОТВЕРСТИЯ В МИКРОПОЛЯРНОМ ТЕЛЕ С ПОРАМИ**

Предложено общее решение плоской задачи микрополярно-дилатационной теории упругости. Эта теория описывает поведение тела, в котором поля перемещения, объемного расширения (дилатации) и вращения точек среды кинематически независимы. Полученное решение применяеця к классической задаче о концентрации напряжений около кругового отверстия в неограниченном микрополярном теле с порами. Получено явное выражение для коэффициента концентрации напряжений, которое обобщает известные результаты Пальмова (для микрополярного тела) и Кауена (для дилатационно–упругого тела).

Valentina Mehandjieva, Konstantin Markov. **STRESS CONCENTRATION AROUND A HOLE IN A MICROPOLAR SOLID WITH VOIDS**

The general solution of the plane problem in the micropolar–dilatation theory of elasticity is proposed. Such a theory describes a solid in which the fields of displacement, change of the volume (dilatation) and rotation are kinematically independent. The obtained solution is applied to the classical problem of stress concentration around a circular hole in an unbounded micropolar solid with continuously distributed set of voids which is modelled as micropolar–dilatation. An explicit formula for the stress concentration factor is derived which generalizes the earlier results of V. A. Palmov (in the micropolar case) and of S. Cowin (in the dilatation case).

1. INTRODUCTION

In the classical continuum mechanics there is a fundamental assumption that the basic characteristic of the motion is the field $\boldsymbol{u}$ of the body points displacement.

This field is basic in the sense that all the rest deformation characteristics can be found provided the field $\boldsymbol{u}$ is known. For example, under the supposition of small strain the simple relations hold

$$\boldsymbol{\varphi} = \frac{1}{2}\nabla \times \boldsymbol{u}, \quad \theta = \nabla \cdot \boldsymbol{u} \tag{1.1}$$

which describe the rotation $\boldsymbol{\varphi}$ and dilatation θ for a small volume of the body. If, however, the microstructure of the body is to be taken into account, we should reject the relations (1.1) and consider the fields $\boldsymbol{\varphi}$ and θ as certain kinematically independent characteristics of deformation.

The assumption that the rotation field $\boldsymbol{\varphi}$ does not depend on the displacement field $\boldsymbol{u}$ has been first proposed by Voigt in 1887. Such an assumption leads to the well–known now micropolar theory of elasticity which has been widely investigated during the last two decades, cf., e. g. [1, 2].

Theories of solids in which displacement and dilatation are kinematically independent have drawn considerably less interest than the micropolar elasticity. Such theories have been proposed independently by S. Cowin et al. [3, 4, 5] and the author [6, 7, 8]. It is natural to call theories of this type dilatation–elastic; we shall recall their premises and basic equations in §2. The theories of S. Cowin and the author are compared in [8], the basic difference being that the former introduces not only the dilatation θ, but its time–derivative $\dot{\theta}$ as well in the respective constitutive equations. As a result certain relaxation effects, known from experiments in granular media, are adequately described theoretically. The influence of the time–effects upon the strain and stress fields in a dilatation–elastic solid are investigated in detail in [9] in the context of the classical problem of uni– and bi–axial tension of an unbounded plane containing a circular hole; the stress concentration factor K was calculated, in particular, and in uniaxial tension it appeared to be higher that the classical one (which equals 3, cf., e.g., [10]). This is opposite to the effect of the micropolar stresses taken into account for which K becomes smaller than 3, as it was shown by V. Palmov [11].

The present work is devoted to a detailed study of the above mentioned problem of uniaxial tension of a solid with a circular hole taking however into account both dilatation and micropolar effects. In other words, the said problem is solved here within the frame of the so–called micropolar–dilatation elasticity, introduced in [6, 7], which is characterized by the independence of dilatation and rotation from displacement. Besides, the rate of dilatation is accounted for similarly to the theory of S. Cowin et al. [4, 9]. The aim of this work is thus the investigation of the combined effect of micropolar and dilatation stress upon the stress concentration factor K in the simplest stress concentrator — the circular one. It is to be expected that K could attain already values both smaller and bigger than 3. It is curious to see, in particular, whether K could approach values close to zero or it will have a certain lower bound as it is the case of the micropolar solid for which $\frac{5}{3} < K < 3$, cf. [11].

The present paper has however a certain aim that is broader than the investigation of the stress concentration in microstructural theories of elasticity. To elucidate this aim let us recall that one of the interpretations of the dilatation elasticity is connected with the behaviour of microdamaged solids [7, 8]. The kinematical independence of the dilatation θ is then a result of the presence of defects like microvoids. The presence of the rate of dilatation change $\dot{\theta}$ among the state variables reflects the process of accumulation and growth of these microvoids, i. e. the developing microfracture process in the solid. In reality, however, defects in solids have much more complicated nature representing a certain assemblage of microcracks and dislocations. Geometrically the defects in a "point" of the damaged solid, i. e. in a representative volume, can be described by means of the so–called damage tensor $\boldsymbol{T}_\alpha$ introduced by A. Vakulenko and M. Kachanov [12]. This is a second–order tensor which for spherical defects (microvoids) is spherical as well and whose trace, $\mathrm{tr}\,\boldsymbol{T}_\alpha$, in the latter case equals the volume fraction of the voids. In general, the introduction of the time–derivative $\dot{\boldsymbol{T}}_\alpha$ of the damage tensor into the energy of deformation — $W = W(\boldsymbol{T}_e,\, \boldsymbol{T}_\alpha,\, \dot{\boldsymbol{T}}_\alpha)$ — together with the principle of virtual work leads to a theory which is able to model, at least qualitatively, the inhomogeneous distribution and evolution of defects like microcracks during the s-training of a damaged brittle solid. (As usual, the tensor $\boldsymbol{T}_e$ among the arguments of W denotes the small strain tensor.) This theory from a formal point of view represents a certain generalization of the micromorphic theory of Eringen [2] in which $\boldsymbol{T}_\alpha$ is an independent kinematical characteristics — the so–called microdistortion tensor. The premises of such a theory have been recently discussed by one of the authors [13], where it has been noticed that the tensor $\boldsymbol{T}_\alpha$, i. e. the damage field, should be more conspicuous in the vicinity of stress concentrators like notches, macrocracks, holes, etc. Thus the present work may be also considered as a certain preliminary attempt to employ microstructural theories of elasticity when analyzing damage process in solids and, especially, the concentration of damage around stress concentrators — an effect well–known from experiments but rather difficult to be put into a theoretical framework, see [13] for brief details and references. The particular case herein considered refers to a damage tensor which is a linear combination of a skew–symmetric and spherical tensors and, moreover, only the spherical part is assumed to change in time. It goes without saying that such an assumption looks artificial and hardly can be realized in practice. The problem considered in what follows should be viewed first of all as a model one, in the course of solution of which it is possible to introduce and check the performance of certain methods that could appear useful in a further more realistic analysis of damage processes in solids.

The outline of the paper is as follows. In §2 we recall very briefly the basic equations of the micropolar, dilatation and micropolar–dilatation theories. In §3 the plane problem for a micropolar–dilatation solid is considered; the basic potential functions are introduced and the respective governing equations for them are derived. In §4 we employ, after S. Cowin [9], the Laplace transformation technique. In §5, which is central for the paper, the plane strain problem of uniaxial tension of a micropolar–dilatation solid, containing a circular cylindrical hole is solved. In order to inverse the obtained Laplace transformants a simple approximate method is proposed in §6. In §7, making use of the results of §5, the stress concentration factor is found explicitly. The formula for K generalizes the respective formula of V. Palmov [11] for micropolar solids and that of S. Cowin [9] for dilatation ones. A detailed numerical and asymptotical analysis of our formula for K is performed as well for various ratios of the micropolar and dilatation constants.

2. BASIC EQUATIONS FOR A MICROPOLAR SOLID WITH VOIDS

2.1. **Basic equations of the micropolar elasticity**. We shall first recall the basic assumptions and equations of the micropolar elasticity, cf., e. g. [1, 2].

As already mentioned, the basic assumption in this theory is the kinematical independence of the displacement and rotation fields. As a consequence the Cauchy stress tensor $\boldsymbol{T}_\sigma$ becomes nonsymmetric in general and the tensor of moment stresses $\boldsymbol{T}_\mu$ appears, the latter being defined similarly to $\boldsymbol{T}_\sigma$, namely, $\boldsymbol{\mu}_n = \boldsymbol{n} \cdot \boldsymbol{T}_\mu$ is the moment density which acts on a surface element with unit normal $\boldsymbol{n}$. The equilibrium equations in the static case are

$$\nabla \cdot \boldsymbol{T}_\sigma + \boldsymbol{F} = 0, \quad \nabla \cdot \boldsymbol{T}_\mu - \boldsymbol{E} : \boldsymbol{T}_\sigma + \boldsymbol{M} = 0, \tag{2.1}$$

where $\boldsymbol{E} = \|\varepsilon_{ijk}\|$ is the alternating tensor, the colon denotes contraction with respect to two pair of indices, $\boldsymbol{F}$ and $\boldsymbol{M}$ are the densities of the body forces and moments respectively.

The constitutive equations of the micropolar theory can be derived by means, e. g., of the principle of virtual work. In the case of small deformation and rotation they have the form

$$\begin{aligned} \boldsymbol{T}_\sigma &= \lambda \boldsymbol{I} \ \mathrm{tr}\, \boldsymbol{T}_\gamma + 2\mu \boldsymbol{T}_\gamma^s + 2\alpha \boldsymbol{T}_\gamma^a, \\ \boldsymbol{T}_\mu &= \beta \boldsymbol{I} \ \mathrm{tr}\, \boldsymbol{T}_\kappa + 2\gamma \boldsymbol{T}_\kappa^s + 2\varepsilon \boldsymbol{T}_\kappa^a, \end{aligned} \tag{2.2}$$

where $\boldsymbol{I}$ is the unit second–order tensor and

$$\boldsymbol{T}_\gamma = \nabla \boldsymbol{u} - \boldsymbol{E} \cdot \boldsymbol{\varphi}, \quad \boldsymbol{T}_\kappa = \nabla \boldsymbol{\varphi} \tag{2.3}$$

are the strain characteristics of the micropolar body. The superscripts "s" and "a" denote, respectively, the symmetric and skew–symmetric parts of the tensors, e. g.

$$\boldsymbol{T}_\gamma^s = \frac{1}{2}\left(\boldsymbol{T}_\gamma + \boldsymbol{T}_\gamma^*\right), \quad \boldsymbol{T}_\gamma^a = \frac{1}{2}\left(\boldsymbol{T}_\gamma - \boldsymbol{T}_\gamma^*\right),$$

etc.; here $\boldsymbol{T}^*$ is the tensor conjugate to $\boldsymbol{T}$.

2.2. **Basic equations of the dilatation elasticity**. Another very simple microstructural theory is based upon the assumption that the volume change (dilatation) θ is kinematically independent of the displacement field $\boldsymbol{u}$. Theories of this type can be called, as suggested in [8], dilatation–elastic; they were introduced and considered at some length by S. Cowin et al. [3, 4, 5] and the author [6, 7, 8].

The basic consequence of the dilatation independence is the appearance of the so–called hydrostatic stress, characterized by the vector field $\boldsymbol{h}$ which is defined as follows: the integral

$$-\int_S \boldsymbol{h} \cdot \boldsymbol{dS}$$

taken over an arbitrary closed surface S in the body equals the full hydrostatic stress which acts on the part of the body inside S as a result of its interaction with the part outside S. Also, the so–called free dilatation $\theta^c = \theta - \nabla \cdot \boldsymbol{u}$ is introduced [8], where $\nabla \cdot \boldsymbol{u}$ is the well–known volume change due to elastic deformation.

The equilibrium equations of the dilatation–elastic solid read

$$\nabla \cdot \boldsymbol{T}_\sigma + \boldsymbol{F} = 0, \quad \nabla \cdot \boldsymbol{h} + g = 0. \tag{2.4}$$

The first equation here is the classical one; moreover, the stress tensor $\boldsymbol{T}_\sigma$ remains symmetric. The second equation (2.4) reflects the balance of hydrostatic stress so that

$$g = s - p; \tag{2.5}$$

here s is the full hydrostatic stress at a point and $p = \frac{1}{3}\sigma_{ii} = \frac{1}{3}\operatorname{tr}\boldsymbol{T}_\sigma$ is the hydrostatic stress due to elastic deformation only.

The easiest way to derive the constitutive equations for a dilatation solid is through the principle of virtual work [6, 7]. We assume the existence of the energy of deformation $W = W(\boldsymbol{u},\ \theta^c)$. Simple arguments, employing the above definitions of the stress characteristics $\boldsymbol{T}_\sigma$, g and $\boldsymbol{h}$, yield

$$W = W(\boldsymbol{T}_e,\ \theta^c,\ \nabla\theta^c) \tag{2.6}$$

as well as the constitutive equations

$$\boldsymbol{T}_\sigma = \frac{\partial W}{\partial \boldsymbol{T}_e}, \quad g = \frac{\partial W}{\partial \theta^c}, \quad -\boldsymbol{h} = \frac{\partial W}{\partial \nabla\theta^c} \tag{2.7}$$

(for the case of small deformation which is only considered hereafter). For an isotropic solid the linear version of (2.7) is

$$\boldsymbol{T}_\sigma = \lambda e \boldsymbol{I} + 2\mu \boldsymbol{T}_e - \sigma\theta^c \boldsymbol{I}, \quad g = \xi\theta^c - \sigma e, \quad \boldsymbol{h} = -\delta\nabla\theta^c, \tag{2.8}$$

where $\boldsymbol{T}_e = \frac{1}{2}(\nabla\boldsymbol{u} + \boldsymbol{u}\nabla)$ is the small strain tensor and $e = \operatorname{tr}\boldsymbol{T}_e = \nabla\cdot\boldsymbol{u}$. The equations (2.8) represent the dilatation–elastic version of the Hooke law.

It is to be noted that S. Cowin et al. [3, 4, 5, 9] employ, instead of θ^c, the so–called change–of–volume field Φ which is introduced through the change of density of the solid due to porosity. Simple analysis shows that

$$\theta^c = -\Phi. \tag{2.9}$$

Hereafter, we shall employ the field Φ instead of θ^c, in order to be closer to Cowin's notations [9] but keeping in mind (2.9).

The authors of [3, 4, 5, 9] introduced also $\dot{\Phi}$, i. e. $\dot{\theta}^c$, as a state parameter in order to account for certain experimentally observed relaxation effects in granular and porous media. The thermodynamical analysis performed in [3, 9] yields the following modification of the Hooke law for the case when $\dot{\Phi}$ is included into the analysis

$$\boldsymbol{T}_\sigma = \lambda e \boldsymbol{I} + 2\mu \boldsymbol{T}_e + \sigma\Phi \boldsymbol{I}, \quad g = -\xi\Phi - \xi_1\dot{\Phi} - \sigma e, \quad \boldsymbol{h} = \delta\nabla\Phi. \tag{2.10}$$

Note that S. Cowin [9] employs different notations for some of the material constants. The correspondence with our constants is given in Table 1.

Table 1

Cowin [9]	β	ω	α
our notations	σ	ξ_1	δ

Cowin's notations are changed to a certain degree in order to keep the classical notations (α, β, γ, ε) for the micropolar constants, cf. (2.2).

The basic equations (2.4) to (2.9) can be also considered as pertaining to a model of an elastic solid containing voids, cf. [3, 4, 7, 8]. In turn, voids in many cases are a result of a damaging process in the solid under load. That is why the model represented by the equations (2.4) to (2.9) can be also considered as a dilatation–elastic model of a brittle damaged solid in which damage is isotropic, i. e. appears in the form of spherical microvoids, see [13] for more details and comments.

Certain boundary–value problems for the equations (2.4) to (2.9) are solved by S. Cowin et al. [4, 9]. For instance, in [9] the classical plane problem of uniaxial tension for a solid with a circular hole is treated and time– and dilatation effects upon the stress concentration factor K are investigated in detail. The same problem will be considered below within the frame of a more general model.

2.3. Basic equations of the micropolar–dilatation elasticity. A natural and straightforward generalization of both micropolar and dilatation–elastic models is based upon the assumption that rotation and dilatation fields are both kinematically independent characteristics of the body motion [7]. Such a model, which is called micropolar–dilatation, could serve also as a model of a micropolar solid with voids due to reasons briefly explained in the foregoing §2.2.

Thus, having taken the stress characteristics for both micropolar and dilatation theories (see §2.1 and §2.2), we introduce the three stress fields in the so–called micropolar–dilatation elastic model (for brevity, MDE–model): the classical Cauchy stress tensor $\boldsymbol{T}_\sigma$, the micropolar–stress tensor $\boldsymbol{T}_\mu$ and the vector $\boldsymbol{h}$ of hydrostatic stress. The balance equations are also three

$$\nabla \cdot \boldsymbol{T}_\sigma + \boldsymbol{F} = 0, \quad \nabla \cdot \boldsymbol{T}_\mu - \boldsymbol{E} : \boldsymbol{T}_\sigma + \boldsymbol{M} = 0, \quad \nabla \cdot \boldsymbol{h} + g = 0. \tag{2.11}$$

The application of the principle of virtual work [7], yields for the potential energy of deformation

$$W = W(\boldsymbol{T}_\gamma,\, \boldsymbol{T}_\kappa,\, \Phi,\, \nabla\Phi), \tag{2.12}$$

where $\boldsymbol{T}_\gamma$ and $\boldsymbol{T}_\kappa$ are the strain characteristics (2.3) of a micropolar solid. In the case of small strain we get from the same principle the constitutive equations

$$\begin{aligned} \boldsymbol{T}_\sigma &= \frac{\partial W}{\partial T_\gamma}, \qquad & \boldsymbol{T}_\mu &= \frac{\partial W}{\partial \boldsymbol{T}_\kappa}, \\ g &= \frac{\partial W}{\partial \Phi}, \qquad & \boldsymbol{h} &= \frac{\partial W}{\partial \nabla\Phi}. \end{aligned} \tag{2.13}$$

For an isotropic solid, assuming W to be quadratic in its arguments, we get from (2.13)

$$\begin{aligned} \boldsymbol{T}_\sigma &= \lambda \boldsymbol{I}\ \mathrm{tr}\, \boldsymbol{T}_\gamma + 2\mu \boldsymbol{T}_\gamma^s + 2\alpha \boldsymbol{T}_\gamma^a + \sigma\Phi\boldsymbol{I}, \\ \boldsymbol{T}_\mu &= \beta \boldsymbol{I}\ \mathrm{tr}\, \boldsymbol{T}_\kappa + 2\gamma \boldsymbol{T}_\kappa^s + 2\varepsilon \boldsymbol{T}_\kappa^a, \\ \boldsymbol{h} &= \delta\nabla\Phi, \quad g = -\xi_1\dot{\Phi} - \xi\Phi - \sigma\ \mathrm{tr}\, \boldsymbol{T}_\gamma, \end{aligned} \tag{2.14}$$

where, similarly to S. Cowin et al. [4, 9], we have taken into account the time–derivative $\dot{\Phi}$ as well.

Inserting (2.14) into the equilibrium equations (2.11) we obtain the equations of Lamé type for the MDE–model

$$\begin{aligned} &(\mu + \alpha)\Delta\boldsymbol{u} + (\lambda + \mu - \alpha)\nabla\nabla \cdot \boldsymbol{u} + 2\alpha\nabla \times \boldsymbol{\varphi} + \sigma\nabla\Phi = 0, \\ &(\gamma + \varepsilon)\Delta\boldsymbol{\varphi} + (\beta + \gamma - \varepsilon)\nabla\nabla \cdot \boldsymbol{\varphi} - 4\alpha\boldsymbol{\varphi} + 2\alpha\nabla \times \boldsymbol{u} = 0, \\ &\delta\Delta\Phi - \xi_1\dot{\Phi} - \xi\Phi - \sigma\nabla \cdot \boldsymbol{u} = 0 \end{aligned} \tag{2.15}$$

(no body sources). On the free boundary S of the solid, the following standard boundary conditions for the system (2.15) are required

$$\sigma_n\big|_S = 0, \quad \boldsymbol{\mu}_n\big|_S = 0, \quad \frac{\partial\Phi}{\partial n}\big|_S = 0, \tag{2.16}$$

where $\sigma_n = n \cdot \boldsymbol{T}_\sigma$ and $\boldsymbol{\mu}_n = n \cdot \boldsymbol{T}_\mu$ are the stress and moment vectors on S, $\boldsymbol{n}$ is the unit outward normal to S. Note that the last boundary condition (2.16) is not so obvious; it is discussed in more detail, e. g. in [3].

3. PLANE–STRAIN PROBLEM FOR THE MDE–MODEL

Suppose a solid, governed by the equations (2.11), (2.14), (2.16) of the MDE–model, is subject to plane strain so that

$$\begin{aligned} &\boldsymbol{u} = u_1(x_1, x_2, t)\,\boldsymbol{i}_1 + u_2(x_1, x_2, t)\,\boldsymbol{i}_2, \\ &\boldsymbol{\varphi} = \boldsymbol{\varphi}(x_1, x_2, t)\,\boldsymbol{i}_3, \quad \Phi = \Phi(x_1, x_2, t)\,; \end{aligned} \tag{3.1}$$

here $\boldsymbol{i}_k$ are the unit base vectors of the Cartesian system x_k, $k = 1, 2, 3$, in the frame of which we shall operate hereafter.

The definition (2.3) of the strain tensors $\boldsymbol{T}_\gamma$ and $\boldsymbol{T}_\kappa$, when written in the said Cartesian system x_k, yields

$$\gamma_{21,1} - \gamma_{11,2} = \kappa_{13}, \quad \gamma_{22,1} - \gamma_{12,2} = \kappa_{23}, \quad \kappa_{23,1} = \kappa_{13,2}, \tag{3.2}$$

or

$$\begin{aligned} &\gamma_{22,11} + \gamma_{11,22} = (\gamma_{12} + \gamma_{21})_{,12}\,, \\ &\gamma_{12,22} - \gamma_{21,11} = (\gamma_{22} - \gamma_{11})_{,12} - (\kappa_{13,1} + \kappa_{23,2})\,; \end{aligned} \tag{3.3}$$

here, e. g. $\gamma_{11,2} = \dfrac{\partial \gamma_{11}}{\partial x_2}$, etc. The relations (3.3) express the conditions of geometrical compatibility for plane strain in our model.

The stress and micropolar–stress tensors, $\boldsymbol{T}_\sigma$ and $\boldsymbol{T}_\mu$, have the following matrices in the coordinates x_k

$$\boldsymbol{T}_\sigma = \begin{Vmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{Vmatrix}, \quad \boldsymbol{T}_\mu = \begin{Vmatrix} 0 & 0 & \mu_{13} \\ 0 & 0 & \mu_{23} \\ \mu_{31} & \mu_{32} & 0 \end{Vmatrix},$$

so that the Hooke law, when written in components with respect to the system x_k, becomes

$$\begin{aligned} &\sigma_{11} = \lambda\gamma_{kk} + 2\mu\gamma_{11} + \sigma\Phi, \quad \sigma_{12} = (\mu + \alpha)\gamma_{12} + (\mu - \alpha)\gamma_{21}, \\ &\sigma_{22} = \lambda\gamma_{kk} + 2\mu\gamma_{22} + \sigma\Phi, \quad \sigma_{21} = (\mu + \alpha)\gamma_{21} + (\mu - \alpha)\gamma_{12}, \\ &\sigma_{33} = \lambda\gamma_{kk} + \sigma\Phi; \quad \mu_{13} = (\gamma + \varepsilon)\kappa_{13}, \quad \mu_{23} = (\gamma + \varepsilon)\kappa_{23}, \\ &\mu_{31} = (\gamma - \varepsilon)\kappa_{13}, \quad \mu_{23} = (\gamma - \varepsilon)\kappa_{23}, \quad \gamma_{kk} = \gamma_{11} + \gamma_{22}. \end{aligned} \tag{3.4}$$

Solving the equations (3.4) with respect to the strain components γ_{ij} we get

$$
\begin{aligned}
\gamma_{11} &= \frac{1}{2\mu}\left[\sigma_{11} - \frac{\lambda}{2(\lambda+\mu)}(\sigma_{11}+\sigma_{22}) - \frac{\sigma\mu}{\lambda+\mu}\Phi\right],\\
\gamma_{22} &= \frac{1}{2\mu}\left[\sigma_{22} - \frac{\lambda}{2(\lambda+\mu)}(\sigma_{11}+\sigma_{22}) - \frac{\sigma\mu}{\lambda+\mu}\Phi\right],\\
\gamma_{12} &= \frac{1}{4\mu}(\sigma_{12}+\sigma_{21}) + \frac{1}{4\alpha}(\sigma_{12}-\sigma_{21}),\\
\gamma_{21} &= \frac{1}{4\mu}(\sigma_{12}+\sigma_{21}) - \frac{1}{4\alpha}(\sigma_{12}-\sigma_{21}).
\end{aligned}
\tag{3.5}
$$

In plane strain the equilibrium equations (2.11) have the following form in components

$$
\begin{aligned}
&\sigma_{11,1}+\sigma_{12,2}=0, && \sigma_{12,1}+\sigma_{22,2}=0,\\
&\mu_{13,1}+\mu_{23,2}+\sigma_{12}-\sigma_{21}=0, && \boldsymbol{h}_{k,k}+g=0.
\end{aligned}
\tag{3.6}
$$

Note that the first three equilibrium equations (3.6) coincide with those for a micropolar solid in plane strain. We can therefore employ Palmov's considerations and introduce after him [11] the stress functions (or potentials) $F(x_1,x_2,t)$ and $\Psi(x_1,x_2,t)$ as follows

$$
\begin{aligned}
&\sigma_{11}=F_{,22}-\Psi_{,12}, \quad \sigma_{22}=F_{,11}+\Psi_{,12},\\
&\sigma_{12}=-F_{,12}-\Psi_{,22}, \quad \sigma_{21}=-F_{,12}+\Psi_{,11},\\
&\mu_{13}=\Psi_{,1}, \quad \mu_{23}=\Psi_{,2}.
\end{aligned}
\tag{3.7}
$$

A direct check demonstrates that if (3.7) is adopted, the first three equilibrium equations (3.6) are identically satisfied.

In order to derive the differential equations for the potentials F and Ψ, we express the strain components γ_{ij} by means of F and Ψ, i. e. we insert (3.7) into (3.5), and then employ the compatibility conditions (3.3). This procedure yields the needed equations for F and Ψ, namely,

$$
\Delta\Delta F - \frac{2\mu\sigma}{\lambda+2\mu}\Delta\Phi = 0, \quad \Delta\left(l^2\Delta - 1\right)\Psi = 0,
\tag{3.8}
$$

where

$$
l = \left[\frac{(\alpha+\mu)(\gamma+\varepsilon)}{4\alpha\mu}\right]^{1/2}
\tag{3.9}
$$

is the well–known combination of the micropolar constants with dimension of length.

It is important to point out that the potentials F and Ψ are not independent because they are interrelated through the equation (3.2). Indeed, upon expressing the components γ_{ij} by means of σ_{ij}, cf. (3.5), and σ_{ij} by the potentials F and Ψ according to (3.7), we find as a consequence of (3.2)

$$
\begin{aligned}
\left(\Psi - l^2\Delta\Psi\right)_{,1} &= -\left(c^2\Delta F - d\Phi\right)_{,2},\\
\left(\Psi - l^2\Delta\Psi\right)_{,2} &= \left(c^2\Delta F - d\Phi\right)_{,1},
\end{aligned}
\tag{3.10}
$$

where

$$c = \left[\frac{(\lambda+2\mu)(\gamma+\varepsilon)}{4\mu(\lambda+\mu)}\right]^{1/2}, \quad d = \frac{\sigma(\gamma+\varepsilon)}{2(\lambda+\mu)} \cdot \tag{3.11}$$

If we introduce the last two equations (2.14), i. e. these for $\boldsymbol{h}$ and g, into the last of the equilibrium equations (3.6) and make use of the formula for $\operatorname{tr} \boldsymbol{T}_\gamma = \gamma_{11} + \gamma_{22}$ which follows from (3.5), we get eventually

$$\delta\Delta\Phi - \xi_1\dot{\Phi} - \frac{\delta}{h^2}\Phi = \frac{\sigma}{2(\lambda+\mu)}\left[\Delta F - \frac{2\mu\sigma}{\lambda+2\mu}\Phi\right]; \tag{3.12}$$

here h is the characteristic "dilatation" length, introduced by S. Cowin [9]

$$\frac{\delta}{h^2} = \xi - \frac{\sigma^2}{\lambda+2\mu} \cdot \tag{3.13}$$

The equations (3.8) and (3.12) are the basic ones for the plane problem in the MDE–model of the solid. If the rate $\dot{\Phi}$ of the change–of–volume field is ignored, i. e. if $\xi_1 = 0$, these equations simplify

$$\begin{aligned} &\Delta\Delta F_\infty - \frac{2\mu\sigma}{\lambda+2\mu}\Delta\Phi_\infty = 0, \\ &\delta\Delta\Phi_\infty - \frac{\delta\Phi_\infty}{h^2} = \frac{\sigma}{2(\lambda+\mu)}\left(\Delta F_\infty - \frac{2\mu\sigma}{\lambda+2\mu}\Phi_\infty\right), \\ &\Delta(l^2\Delta - 1)\Psi_\infty = 0. \end{aligned} \tag{3.14}$$

The subscript "∞" is used here, similarly to S. Cowin [9, p. 447], because it will be seen in §4 that the solutions F_∞, Ψ_∞, Φ_∞ of (3.14) represent the limiting values of the functions F, Ψ, Φ, respectively, at $t \to \infty$.

4. APPLICATION OF THE LAPLACE TRANSFORMATION

Here we shall employ the Laplace transformation

$$\overline{f}(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{4.1}$$

in order to simplify the basic system (3.8) and (3.12) of the plane problem for the micropolar–dilatation solid under sduty.

Suppose that the boundary conditions, to which the solid is subject, have been held steady in the unbounded time–interval $(-\infty, 0)$, so that all rate processes have ceased. Then $\dot{\Phi} = 0$ at $t < 0$ which is equivalent to the assumption $\xi_1 = 0$. Consequently, the functions F, Ψ and Φ satisfy the system (3.14) at $t < 0$ and we denote them by F^0_∞, Ψ^0_∞ and Φ^0_∞ respectively.

In order to find the functions F, Ψ and Φ at $t \geqq 0$, we employ the Laplace transformation (4.1) to the equations (3.8) and (3.12)

$$\begin{aligned} &\Delta\Delta\overline{F} - \frac{2\mu\sigma}{\lambda+2\mu}\Delta\overline{\Phi} = 0, \quad l^2\Delta\Delta\overline{\Psi} - \Delta\overline{\Psi} = 0, \\ &\delta\Delta\overline{\Phi} - \frac{\delta}{h^2}\overline{\Phi} + \xi_1\Phi_0 = \frac{\sigma}{2(\lambda+\mu)}\left(\Delta\overline{F} - \frac{2\mu\sigma}{\lambda+2\mu}\overline{\Phi}\right), \end{aligned} \tag{4.2}$$

where $\overline{h}$ is defined as

$$\frac{\delta}{\overline{h}^2} = \frac{\delta}{h^2} + \xi_1 s. \tag{4.3}$$

In derivation of (4.2) we have used the well–known property of the Laplace transformation [14, p. 42]

$$\overline{\dot{\Phi}} = s\overline{\Phi} - \Phi_0, \quad \Phi_0 = \Phi_0(x_1, x_2) = \Phi(x_1, x_2, 0). \tag{4.4}$$

We shall employ now the final– and initial–value theorems for the Laplace transformation (4.1) which state, respectively,

$$\lim_{s\to 0} s\overline{f}(s) = \lim_{t\to\infty} f(t), \tag{4.5}$$

$$\lim_{s\to\infty} s\overline{f}(s) = \lim_{t\to 0} f(t), \tag{4.6}$$

see, e. g. [14, p. 183].

Let us multiply (4.2) by s and employ (4.5) for the functions $s\overline{\Phi}(s)$, $s\overline{\Psi}(s)$, $s\overline{F}(s)$; as a result we get the system (3.14). This means that at $t \to \infty$ the solution of the system (3.8) and (3.12) becomes stationary and thus, as already mentioned, it satisfies the system (3.14), corresponding to the stationary case $\xi_1 = 0$ or $\dot{\Phi} = 0$.

Let us multiply (4.2) by s but employ now (4.6) for the same functions $s\overline{\Phi}(s)$, $s\overline{\Psi}(s)$ and $s\overline{F}(s)$. Then

$$\Delta\Delta F_0 - \frac{2\mu\sigma}{\lambda + 2\mu}\Delta\Phi_0 = 0, \quad l^2\Delta\Delta\Psi_0 - \Delta\Psi_0 = 0, \tag{4.7}$$

and the third equation (4.2) is satisfied identically. The subscript "zero" in (4.7) means that the respective functions are taken at $t = 0$, e. g. $F_0(x_1, x_2) = F(x_1, x_2, 0)$. This result shows that if the initial change–in–volume field $\Phi_0 = 0$, i. e. the free dilatation $\theta^c = 0$, then the stress function F_0 is biharmonic just as it is in the micropolar elasticity. Consequently, the solution of the plane problem for a micropolar solid with voids (i. e. in the MDE–model) at $t = 0$ coincides with the solution of the same micropolar problem, provided $\Phi_0(x_1, x_2) = 0$. This fact generalizes the respective conclusion of S. Cowin [9, p. 449] who dealt with elastic materials with voids.

In the problems considered below we shall assume that F^0_∞, Ψ^0_∞ and Φ^0_∞ all vanish, i. e. in the time–interval $(-\infty, 0)$ there are no strain and stress in the solid. We shall also note that the basic system (4.2) can be recast in the form

$$\overline{h}^2\Delta\Delta\overline{\Phi} = \Delta\overline{\Phi}, \quad \Delta\Delta\overline{\Sigma} = 0, \quad l^2\Delta\Delta\overline{\Psi} = \Delta\overline{\Psi}, \tag{4.8}$$

where the new potential $\overline{\Sigma}$ is defined as

$$\overline{\Sigma} = \overline{F} - \frac{2\mu\sigma}{\lambda + 2\mu}\,\overline{h}^2\,\overline{\Phi}. \tag{4.9}$$

The form (4.8) of the system (4.2) is very convenient when solving concrete boundary-value problems, as it will be seen in §5. Besides, since both Φ and Φ_∞ satisfy the boundary condition $(2.16)_3$, we should have

$$\boldsymbol{n}\cdot\nabla\overline{\Phi} = 0 \tag{4.10}$$

on the boundary of the solid.

5. UNIAXIAL TENSION OF A SOLID CONTAINING A CIRCULAR CYLINDRICAL HOLE

To illustrate the usefulness of the results of §4, we shall consider here the classical plane problem concerning uniaxial tension of a solid with a cylindrical hole within the frame, however, of the MDE–model, introduced in §2.3. The solution of this problem will allow, in particular, to investigate in §7 the influence of micro-structural effects — both dilatation and micropolar in our case — upon the stress concentration factor.

We assume that the boundary of the cylinder is free of stresses, i. e.

$$\overline{\sigma}_{rr} = 0, \quad \overline{\sigma}_{r\theta} = 0, \quad \overline{\mu}_{rz} = 0 \tag{5.1}$$

at $r = a$; here and in what follows we shall employ polar coordinates r, θ in the cross–section perpendicular to the axis $x_3 = z$ of the cylinder.

By supposition, uniaxial tension takes place at infinity

$$\overline{\sigma}_{11} = \overline{p}^{\infty}, \quad \overline{\sigma}_{22} = \overline{\sigma}_{12} = \overline{\sigma}_{21} = 0, \quad \overline{\mu}_{13} = \overline{\mu}_{23} = 0 \tag{5.2}$$

at $r \to \infty$. These conditions, when written in the polar coordinates, become

$$\begin{aligned} &\overline{\sigma}_{rr} = \frac{1}{2}\overline{p}^{\infty}(1+\cos 2\theta), \quad \overline{\sigma}_{\theta\theta} = \frac{1}{2}\overline{p}^{\infty}(1-\cos 2\theta), \\ &\overline{\sigma}_{r\theta} = \overline{\sigma}_{\theta r} = -\frac{1}{2}\overline{p}^{\infty}\sin 2\theta, \quad \overline{\mu}_{rz} = \overline{\mu}_{\theta z} = 0. \end{aligned} \tag{5.3}$$

Besides, we should also have, due to (4.10),

$$\frac{\partial \overline{\Phi}}{\partial r}\Big|_{r=a} = 0. \tag{5.4}$$

We seek the functions $\overline{\Phi}$ and $\overline{\Sigma}$ in the form $f(r) + g(r)\cos 2\theta$, suggested by the solution of the respective elastic problem and we take $\overline{\Psi}$ in the form $f(r)+g(r)\sin 2\theta$. Inserting these forms in (4.8) we get the biharmonic equation for f's and Bessel's for g's. Thus

$$\begin{aligned} \overline{\Sigma} &= C_1 r^2 \ln r + C_2 r^2 + C_3 \ln r + C_4 + \left(A_1 r^{-2} + A_2 r^2 + B_1 r^4 + B_2\right)\cos 2\theta, \\ \overline{\Psi} &= E_1 K_0(r/l) + E_2 I_0(r/l) + D_1 \ln r + D_2 \\ &\qquad + \left[G_1 K_2(r/l) + G_2 I_2(r/l) + F_1 r^2 + F_2 r^{-2}\right]\sin 2\theta, \\ \overline{\Phi} &= M_1 K_0(r/\overline{h}) + M_2 I_0(r/\overline{h}) + N_1 \ln r + N_2 \\ &\qquad + \left[P_1 K_2(r/\overline{h}) + P_2 I_2(r/\overline{h}) + Q_1 r^2 + Q_2 r^{-2}\right]\cos 2\theta, \end{aligned} \tag{5.5}$$

where I_0, K_0, I_2, K_2 are the modified Bessel functions of the respective order, and $C_1, \ldots, Q_2$ are functions of the transform variable s only.

From the boundary condition (5.3) it follows that

$$\begin{aligned} &F_1 = C_1 = B_1 = E_2 = G_2 = M_2 = P_2 = D_2 = 0, \\ &C_2 = \frac{1}{4}\overline{p}^{\infty}, \quad \overline{L}Q_1 + A_2 = -\frac{1}{4}\overline{p}^{\infty}, \end{aligned} \tag{5.6}$$

where

$$\overline{L} = \frac{2\mu\sigma\overline{h}^2}{\lambda + 2\mu}. \tag{5.7}$$

Inserting the functions $\overline{F}$ and $\overline{\Psi}$ from (5.5) into the last equation (4.2) we also get

$$(5.8)\qquad \begin{aligned} N_1 &= 0, \quad N_2 = \frac{\xi \overline{p}^{\infty}}{(2-m)\sigma\xi - m\sigma\xi_1 s}, \\ Q_1 &= 0, \quad Q_2 = -\frac{4\xi B_2}{(2-m)\sigma\xi - m\sigma\xi_1 s}, \end{aligned}$$

where

$$(5.9)\qquad m = \frac{2(\lambda+\mu)\xi}{\sigma^2}.$$

Note that $m > 2$, see §7.

In turn, the boundary condition (5.4) yields

$$(5.10)\qquad M_1 = 0, \quad P_1 = -\frac{8B_2\overline{h}\xi}{a^3 K_2'(a/\overline{h})\left[(2-m)\sigma\xi - m\sigma\xi_1 s\right]}\cdot$$

The equation (3.10) allows to obtain two more relations between the introduced parameters, namely,

$$(5.11)\qquad D_1 = 0, \quad F_2 = 4B_2\overline{W}; \quad \overline{W} = c^2 - \frac{d\xi}{(2-m)\sigma\xi - m\sigma\xi_1 s}\cdot$$

Taking into account all obtained relations (5.6) to (5.11), we get for the needed functions

$$(5.12)\qquad \begin{aligned} \overline{\Sigma} &= \frac{1}{4}\overline{p}^{\infty}r^2 + C_3\ln r + \left[A_1 r^{-2} + B_2 - \frac{1}{4}\overline{p}^{\infty}r^2\right]\cos 2\theta, \\ \overline{\Psi} &= E_1 K_0(r/l) + \left[G_1 K_2(r/l) + 4B_2\overline{W}r^{-2}\right]\sin 2\theta, \\ \overline{\Phi} &= \frac{\overline{p}^{\infty}\xi}{(2-m)\sigma\xi - m\sigma\xi_1 s} - \frac{(\lambda+2\mu)B_2}{\mu\overline{h}^2\sigma}\left[\overline{R}(r) - 1\right]\cos 2\theta, \end{aligned}$$

where

$$(5.13)\qquad \begin{aligned} \overline{R}(r) &= 1 - \frac{4\mu\xi\overline{h}^2}{8\mu N\xi + (\lambda+2\mu)m\xi_1 s}\left[\frac{1}{r^2} + \frac{2\overline{h}}{a^3}\,\frac{K_2(r/\overline{h})}{K_2'(a/\overline{h})}\right], \\ N &= \frac{\lambda+2\mu}{8\mu}(m-2). \end{aligned}$$

The function $\overline{R}(r)$, introduced in (5.13), is very important in the solution of the problem under consideration. At the absence of micropolar effects, let us note, it coincides with Cowin's function $\overline{F}(r)$, see [9, p. 454]. Similarly to the latter function, the overbar indicates that $\overline{R}(r)$ depends on the transform parameter s directly and through $\overline{h}$. The notation $R(r)$ corresponds to the function $\overline{R}(r)$ at $s = 0$, i. e. when $\overline{h}$ is replaced by h. The primes on $\overline{R}(r)$ mean derivatives with respect to r. It is easily seen from (5.4) and (5.12) that

$$(5.14)\qquad \overline{R}'(a) = 0.$$

In this way only the parameters C_3, A_1, B_2, E_1 and G_1 remain to be specified, cf. (5.12). It is done by means of the boundary condition (5.1). The final result is

$$E_1 = 0, \quad G_1 = \frac{8lB_2}{a^3 K_2'(a/l)}\overline{W}, \quad C_3 = -\frac{1}{2}\overline{p}^{\infty}a^2, \tag{5.15}$$

$$A_1 = \frac{2\,\overline{p}^{\infty}a^4 G(a)\overline{W}}{3 - 2\overline{R}(a) + 4\overline{W}G(a)} - \frac{1}{4}\overline{p}^{\infty}a^4, \quad B_2 = \frac{\overline{p}^{\infty}a^2}{2[3 - 2\overline{R}(a) + 4\overline{W}G(a)]},$$

where

$$G(r) = \frac{1}{r^2} + \frac{2l}{a^3}\,\frac{K_2(r/l)}{K_2'(a/l)}, \tag{5.16}$$

so that

$$G'(a) = 0. \tag{5.17}$$

Upon introducing (5.15) into (5.12) we find eventually

$$\begin{aligned}
\overline{\Sigma} =& \frac{1}{4}\overline{p}^{\infty}r^2 - \frac{1}{2}\overline{p}^{\infty}a^2\ln r \\
&+ \overline{p}^{\infty}\left[\frac{2\overline{W}a^4G(a)}{\overline{X}r^2} - \frac{a^4}{4r^2} + \frac{a^2}{2\overline{X}} - \frac{r^2}{4}\right]\cos 2\theta, \\
\overline{\Psi} =& \frac{2\overline{W}a^2}{\overline{X}}G(r)\overline{p}^{\infty}\sin 2\theta, \\
\overline{\Phi} =& \frac{\xi\overline{p}^{\infty}}{(2-m)\sigma\xi - m\sigma\xi_1 s} - \frac{(\lambda+2\mu)a^2}{2\overline{X}\mu\overline{h}^2\sigma}[\overline{R}(r) - 1]\overline{p}^{\infty}\cos 2\theta,
\end{aligned} \tag{5.18}$$

where

$$\overline{X} = 3 - 2\overline{R}(a) + 4\overline{W}G(a). \tag{5.19}$$

The formulae (5.18) and (5.19) terminate the solution of the problem under study. The tansformed stresses $\overline{\sigma}_{rr}$, $\overline{\sigma}_{\theta\theta}$, $\overline{\sigma}_{r\theta}$, $\overline{\sigma}_{\theta r}$, $\overline{\mu}_{rz}$, $\overline{\mu}_{\theta z}$ are to be obtained by inserting (5.18) into the equations (3.7) written with respect to the polar coordinates r, θ. The final result is

$$\begin{aligned}
\overline{\sigma}_{rr} =& \frac{1}{2}\overline{p}^{\infty}\Bigg\{1 - \frac{a^2}{r^2} + \left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta \\
&- \frac{2a^2}{\overline{X}r^2}\left[6 + r\overline{R}'(r) - 4\overline{R}(r) + 4\overline{W}V(r)\right]\cos 2\theta\Bigg\}, \\
\overline{\sigma}_{\theta\theta} =& \frac{1}{2}\overline{p}^{\infty}\Bigg\{1 + \frac{a^2}{r^2} - \left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta \\
&- \frac{2a^2}{\overline{X}r^2}\left[r^2\overline{R}''(r) - 4\overline{W}V(r)\right]\cos 2\theta\Bigg\}, \\
\overline{\sigma}_{r\theta} =& -\frac{1}{2}\overline{p}^{\infty}\Bigg\{1 - 3\frac{a^4}{r^4} + \frac{2a^2}{\overline{X}r^2}\Bigg[\overline{U}(r) \\
&+ 2\overline{W}\left(6\frac{a^2}{r^2}G(a) + rG'(r) - 4G(r)\right)\Bigg]\Bigg\}\sin 2\theta,
\end{aligned} \tag{5.20}$$

$$\overline{\sigma}_{\theta r} = -\frac{1}{2}\overline{p}^{\infty}\Bigg\{1 - 3\frac{a^4}{r^4} + \frac{2a^2}{\overline{X}r^2}\Bigg[\overline{U}(r) + 2\overline{W}\left(6\frac{a^2}{r^2}G(a) - r^2G''(r)\right)\Bigg]\Bigg\}\sin 2\theta,$$

$$\overline{\mu}_{rz} = \frac{2\overline{W}a^2}{\overline{X}}G'(r)\overline{p}^{\infty}\sin 2\theta, \quad \overline{\mu}_{\theta z} = \frac{4\overline{W}a^2}{\overline{X}r}G(r)\overline{p}^{\infty}\cos 2\theta;$$

here we have introduced, for the sake of brevity, the functions

$$\overline{U}(r) = 3 - 2\overline{R}(r) + 2r\overline{R}'(r), \tag{5.21}$$

$$V(r) = 3\frac{a^2}{r^2}G(a) + rG'(r) - G(r).$$

The initial– and final–value theorems allow to obtain the stresses σ_{rr}, $\sigma_{\theta\theta}$, etc. at $t = 0$ and $t = \infty$ in explicit form. Indeed, making use of the fact that the tension at infinity is constant, $p^{\infty} = \text{const}$, we have $\overline{p}^{\infty} = p^{\infty}/s$ so that, upon multiplying (5.20) by s and employing (4.6) we get the said stresses at $t = 0$ to be

(5.22)

$$\sigma_{rr} = \frac{1}{2}p^{\infty}\Bigg\{1 - \frac{a^2}{r^2} + \left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta - 4\frac{a^2}{Tr^2}\left(1 + 2c^2V(r)\right)\cos 2\theta\Bigg\},$$

$$\sigma_{\theta\theta} = \frac{1}{2}p^{\infty}\left\{1 + \frac{a^2}{r^2} - \left[1 + 3\frac{a^4}{r^4} - 8\frac{c^2}{T}\frac{a^2}{r^2}V(r)\right]\cos 2\theta\right\},$$

$$\sigma_{r\theta} = -\frac{1}{2}p^{\infty}\Bigg\{1 - 3\frac{a^4}{r^4} + 2\frac{a^2}{Tr^2}\left[1 + 2c^2\left(6\frac{a^2}{r^2}G(a) + rG'(r) - 4G(r)\right)\right]\Bigg\}\sin 2\theta,$$

$$\sigma_{\theta r} = -\frac{1}{2}p^{\infty}\left\{1 - 3\frac{a^4}{r^4} + 2\frac{a^2}{Tr^2}\left[1 + 2c^2\left(6\frac{a^2}{r^2}G(a) - r^2G''(r)\right)\right]\right\}\sin 2\theta,$$

$$\mu_{rz} = \frac{2c^2a^2}{T}G'(r)p^{\infty}\sin 2\theta,$$

$$\mu_{\theta z} = \frac{4c^2a^2}{Tr}G(r)p^{\infty}\cos 2\theta; \quad T = 1 + 4c^2G(a).$$

Similarly, we employ (4.5) to $s\overline{\sigma}_{rr}$, etc. where the stresses $\overline{\sigma}_{rr}$, …, are taken from (5.20). As a result we get the limiting values of these stresses at $t \to \infty$ to be

$$\sigma_{rr} = \frac{1}{2}p^{\infty}\Bigg\{1 - \frac{a^2}{r^2} + \left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta - \frac{2a^2}{Xr^2}\left[6 + rR'(r) - 4R(r) + 4WV(r)\right]\cos 2\theta\Bigg\}, \tag{5.23}$$

…

The dots mean that the needed stresses σ_{rr}, etc. at $t = \infty$, have the same form as $\overline{\sigma}_{rr}$, etc. in (5.20) provided all overbars in the right–hand sides of (5.20) are omitted; cf. for illustration the expression of $\overline{\sigma}_{rr}$ in (5.20) and that of σ_{rr} at $t \to \infty$ in (5.23). The constants X and W in (5.23) correspond to the values of $\overline{X}$ and $\overline{W}$ respectively at $s = 0$

$$\begin{aligned} X &= \overline{X}(0) = 3 - 2R(a) + 4WG(a), \\ W &= \overline{W}(0) = c^2 - \frac{d}{2-m} = q^2 \frac{(\lambda + 2\mu)\xi - \sigma^2}{(\lambda + \mu)\xi - \sigma^2}; \quad q^2 = \frac{\gamma + \varepsilon}{4\mu} \cdot \end{aligned} \tag{5.24}$$

6. AN APPROXIMATE METHOD FOR CALCULATION OF THE STRESSES IN THE WHOLE TIME–INTERVAL $(0,\ \infty)$

The exact evaluation of the stress fields $\sigma_{rr}(x,t)$, etc. for all $t \in (0,\ \infty)$ needs an inversion of (5.20) which seems very difficult and which is not attempted here even numerically. Instead, a simple approximate method for such an inversion is proposed. With this aim in view we notice that there exists a monotonic transition between the stress states at $t = 0$ and $t = \infty$ so that, it may be assumed that the transition follows a simple exponential law

$$\sigma_{rr} = (X_1 - Y_1)\, e^{-a_1 t} + Y_1; \tag{6.1}$$

it suffices to consider in detail the stress σ_{rr} only. Obviously

$$\sigma_{rr}\Big|_{t=0} = X_1, \quad \sigma_{rr}\Big|_{t=\infty} = Y_1 \tag{6.2}$$

so that X_1 and Y_1 are known functions of r, due to (5.22) and (5.23).

We next calculate the Laplace transformation of (6.1)

$$\overline{\sigma}_{rr} = \frac{X_1 s + Y_1 a_1}{s(s + a_1)} \tag{6.3}$$

and compare it to the exact expression (5.20) for $\overline{\sigma}_{rr}$ written as

$$\overline{\sigma}_{rr} = \frac{p^\infty}{2s} \left[A(r,\theta) - \frac{2a^2}{r^2} \overline{B}(r) \cos 2\theta \right],$$

where

$$\begin{aligned} A(r,\theta) &= 1 - \frac{a^2}{r^2} + \left(1 + 3\frac{a^4}{r^4}\right) \cos 2\theta, \\ \overline{B}(r) &= \frac{1}{\overline{X}} \left(6 + r\overline{R}'(r) - 4\overline{R}(r) + 4\overline{W}V(r)\right). \end{aligned} \tag{6.4}$$

Then we should have

$$\frac{X_1 s + Y_1 a_1}{s + a_1} = \frac{1}{2} p^\infty \left[A(r,\theta) - \frac{2a^2}{r^2} \overline{B}(r) \cos 2\theta \right], \tag{6.5}$$

which is satisfied at $s = 0$ and $s = \infty$ due to (6.2). We may want one thing more, namely, both sides of (6.5) to have the same derivative with respect to s at $s = 0$. This requirement yields

$$\frac{1}{a_1} (X_1 - Y_1) = -p^\infty \frac{a^2}{r^2} \cos 2\theta \frac{d}{ds} \overline{B}(s)\Big|_{s=0} \tag{6.6}$$

which allows to specify $a_1 = a_1(r)$ explicitly, making use of (6.2), (6.4), (5.11), (5.13), (5.19), (5.21), etc.

Thus we can calculate σ_{rr}, at least approximately, for all $t \in (0, \infty)$. The evaluation of the respective approximations for the rest of the stresses $\sigma_{\theta\theta}$, etc. is fully similar and therefore it is omitted here.

The foregoing relations (6.1) to (6.4) give a qualitative picture of the stress history in the solid at $t \in (0, \infty)$. We cannot say, however, how accurate is this picture without an exact or numerical inversion of (5.20).

7. THE STRESS CONCENTRATION FACTOR AROUND THE HOLE

Let K be the classical stress concentration factor at $t \to \infty$ around the hole, i. e.

$$K = \frac{1}{p^{\infty}} \max \sigma_{\theta\theta}. \tag{7.1}$$

From (5.23) it follows that

$$\max \sigma_{\theta\theta} = \sigma_{\theta\theta}\Big|_{r=a,\, \theta=\pm\pi/2} \tag{7.2}$$

and thus

$$K = 3 + \frac{a^2 R''(a) - 8WG(a)}{3 - 2R(a) + 4WG(a)}\,. \tag{7.3}$$

In virtue of (5.24), (5.13) and (5.16) the relations (7.3) can be recast to the following nondimensional form

$$K = 3 - \frac{\left(4 + H^2\right) F(L) - F(L)F(H) + 8Q^2(1-\nu)(4N+1)F(H)}{2F(L) + 2NF(L)F(H) + 4Q^2(1-\nu)(4N+1)F(H)}, \tag{7.4}$$

where

$$\begin{aligned} &H = \frac{a}{h}, \quad L = \frac{a}{l}, \quad Q = \frac{q}{l}, \\ &F(z) = 4 + z^2 + 2z\frac{K_0(z)}{K_1(z)}, \quad \nu = \frac{\lambda}{2(\lambda+\mu)}, \end{aligned} \tag{7.5}$$

so that ν is the classical Poisson ratio. The well–known formulae for the modified Bessel functions

$$\begin{aligned} K_2(z) &= \frac{2}{z}K_1(z) + K_0(z), \\ K_2'(z) &= -\frac{2}{z}K_0(z) - \left(1 + \frac{4}{z^2}\right)K_1(z), \\ K_2''(z) &= \left(1 + \frac{6}{z^2}\right)K_0(z) + \frac{3}{z}\left(1 + \frac{4}{z^2}\right)K_1(z) \end{aligned} \tag{7.6}$$

have been used when deriving (7.4).

Note that the coefficient N in (7.5) first appeared in (5.13). It corresponds to a coefficient, introduced and denoted also by N by S. Cowin [9, p. 455], when the usual transition from plane strain, considered herein, to plane stress, considered in [9], is made. That is why $N > 0$ as it was shown in [9], so that $m > 2$ — a fact already mentioned in §5.

The nondimensional parameters, introduced in (7.5), can be interpreted as follows: $H = a/h$ is a certain measure of, so to say, "degree of dilatationity" of the solid in the sense that if $h \to \infty$, i. e. $H \to 0$, the dilatation effects can be neglected. Similarly, $L = a/l$ together with the ratio q/l characterize how "micropolar" is the solid again in the sense that at $a/l \to \infty$, $q/l \to 0$ the micropolar effects disappear. Note that the parameter $Q = q/l$ is introduced here after V. Palmov [11].

It is easily seen that

$$\lim_{H\to 0} K = \frac{3+P}{1+P}, \tag{7.7}$$

where

$$P = 8(1-\nu)Q^2\left[4 + L^2 + 2L\frac{K_0(L)}{K_1(L)}\right]^{-1}, \tag{7.8}$$

which means that at $H \to 0$ we get as a particular case of (7.4) the result of V. Palmov [11, p. 406] for the stress concentration factor around a circular hole in a micropolar solid. This is fully natural because at $H \to 0$ the dilatation effects, as already mentioned, disappear and the solid becomes micropolar. From (7.7) it follows that the maximum of K is obtained when P is minimum; but $\min P = 0$ and it is reached at $l \to \infty$, i. e. at $\alpha = 0$. In the latter case we get the value $K = 3$ which is well–known from the classical theory of elasticity. In turn, the minimum of K is attained when P attains its maximum value; the latter corresponds, in virtue of (7.8), to $L \to 0$, $\nu = 0$, $l = q$, so that $\max P = 2$ and thus $\frac{5}{3}$ is the minimum value of the stress concentration factor around a circular hole in a micropolar solid. Therefore $\frac{5}{3} < K < 3$ in such a solid. Note that all these results belong to V. Palmov [11, p. 407].

Consider the other limiting case $H \to \infty$, i. e. $h \to 0$ which corresponds to very strong dilatation effects in the solid. The general formula for K, cf. (7.4), then reads

$$\lim_{H\to\infty} K = 1 + \frac{8N}{4N + P(4N+1)}, \tag{7.9}$$

where P is defined in (7.8). Keeping in mind the foregoing result of V. Palmov

$(P \in (0, 2),\ K \in \left(\frac{5}{3},\ 3\right))$ and the fact that $N > 0$, we conclude that in the micropolar–dilatation model under consideration

$$(7.10) \qquad K > 1,$$

so that K cannot be less than one. Moreover, the dilatation effects, when taken into account together with the micropolar ones, can decrease the factor K making it less than $\frac{5}{3}$. The minimum value $K = 1$ is obtained at $N \to 0$, i. e. at $m \to 2$. This is illustrated in Fig. 1.

Fig. 1. The dependence of the stress concentration factor K on the dimensionless "degree of dilatationity" $H = a/h$ at various degrees of micropolarity $L = a/l$; $L = 1, 2, 3, 4, 5$; $Q = q/l = 1$, $N = 0.0025$; Poisson ratio $\nu = 0$

Let us suppose now that $L = a/l \to \infty$, i. e. $l \to 0$, and $q/l \to 0$. This corresponds to neglecting of micropolar effects. In this limiting case the general formula (7.4) becomes

$$(7.11) \qquad \lim_{\substack{a/l \to \infty \\ q/l \to 0}} K = 3 + \left[2N + \left(1 + N(4 + H^2)\right) \frac{K_1(H)}{H K_0(H)}\right]^{-1},$$

which coincides with the respective formula of S. Cowin [9, p. 456] for the stress concentration factor around a circular hole in a dilatation–elastic solid, provided the usual transition from plane strain to plane stress is performed. Note that the same limiting value (7.11) of K is obtained in the other limiting case $a/l \to \infty$, $a/q \to \infty$. A detailed and well–illustrated numerical analysis of the formula (7.11) is performed in [9].

The dependence of the stress concentration factor K upon some of the parameters (7.5), according to the formula (7.4), is demonstrated in Fig. 2 and 3.

8. CONCLUDING REMARKS

In this paper we have employed a certain microstructural model, which generalizes both micropolar and dilatation theories of elasticity, in order to investigate the stress concentration effects around a circular hole. The obtained results (7.4) and (7.10) show that the stress concentration factor K in this case could closely approach 1, cf. Fig. 1. This means first of all that microstructural effects could, in principle, strongly influence the stress fields around stress concentrators. If we recall also the interpretation of microstructural theories as certain models of damaged solids (see §1), we can conclude that inhomogeneous damage fields that appear in vicinity of such concentrators could strongly affect the stress field and in

Fig. 2. The same as in Fig. 1, i. e. $K = K(H)$, but at $L = 6$, $Q = 0.1$, $v = 0.25$ and N between 0.1875 and 2.5

Fig. 3. The dependence of the stress concentration factor K on the dimensionless degree of micropolarity $L = a/l$ at various N between 0.1875 and 2.5; $Q = q/l = 0.1$, $\nu = 0.25$ and $H = a/h = 6$

some cases the stress concentration could be drastically reduced due to the presence of defects like microcracks and microvoids.

Second, the fact that K can approach 1 means that the hole does not concentrate stresses at all in certain cases. This seems strange intuitively, but a plausible explanation can be proposed immediately in order to elucidate the situation. Note that we have used the classical definition (7.1) of the stress concentration factor K in a nonclassical theory of elasticity. If we introduce a similar factor, say K_{rz}, as the ratio of the maximum micropolar stress μ_{rz} in the solid to the value of μ_{rz} at infinity, we shall have $K_{rz} = \infty$ since $\mu_{rz} \to \infty$ at $r \to \infty$, cf. (5.1). Moreover, the coefficient K_{rz} looks, in the framework of the micropolar theory at least, as reasonable to be introduced as the classical stress concentration factor K is. Thus, in the authors' view, the classical stress concentration factor K, given in (7.1), loses its privilege position if microstructural extensions of the classical elasticity are employed. It is hardly probable that K can be replaced by a certain generalized coefficient one and the same in all situations. The generalization of K should depend on the concrete situation and context of the microstructural theory which is applied. For example, it may be useful, instead of K, to introduce in our context, the stress concentration factor in the form

$$\widetilde{K} = \frac{1}{p^\infty} \max \left[(\boldsymbol{T}_\sigma : \boldsymbol{T}_\sigma)^{1/2} + (\boldsymbol{T}_\mu : \boldsymbol{T}_\mu)^{1/2} + (\boldsymbol{h} \cdot \boldsymbol{h})^{1/2} \right], \tag{8.1}$$

i. e. to compare the uniaxial tension at infinity with the maximum value of the sum of the second invariants of the tensors $\boldsymbol{T}_\sigma$ and $\boldsymbol{T}_\mu$ and the length of the hydrostatic stress field $\boldsymbol{h}$.

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