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ON A MODEL OF DAMAGE IN POLYMER MATERIALS

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Abstract. The aim of the paper is to extend the well known reasoning and conclusions of the continuum damage mechanics from metals to polymer materials, when Norton's creep law is replaced by a hereditary type constitutive equation. Treating damage as volume fraction of microvoids appearing and evolving during straining, some simple models of mechanics of composites are employed. A system of coupled differential equation for the longitudinal strain and damage as functions of time is derived as a result. The brittle, ductile and mixed brittle-ductile type of failure are identified in a simplified model example.

1. Introduction

The aim of the paper is to extend the well known reasoning and conclusions of the continuum damage mechanics from metals to polymer materials, when Norton's creep law is replaced by a hereditary type constitutive equation. In the former case the assumption of incompressibility essentially simplifies the analysis. For polymer creep this assumption is inapplicable, moreover, though the constitutive relations are linear, their integral structure makes the damage analysis much more complicated in general.

Here we shall outline an approach which despite its approximate nature allows, hopefully, an adequate study of the damage processes in polymer material in interconnection with straining history. The approach follows the traditions of the continuum damage mechanics, initiated by Kachanov [1,2], invoking in particular a kinetic equation for damage evolution of the same type as in Kachanov's theory. A difference lies in a bit more "physical" interpretation of damage as volume fraction of microvoids that appear and evolve during deformation. This allows us to employ some of the simple models of mechanics of composites, treating the damaged solid as porous. In the classical problem for a deteriorating rod under a fixed tensile force, a system of coupled differential equation for the longitudinal strain and damage as functions of time is derived as a result. The brittle, ductile and mixed brittle-ductile type of failure are identified in a simplified model example.

2. Basic Equations of the Model

Consider a linear viscoelastic material governed by usual hereditary type constitutive equations:

$$\varepsilon(t) = \frac{1}{E}\sigma(t) + \int_0^t \Gamma(t-s)\,\sigma(s)\,ds\,; \tag{2.1}$$

only $1D$ case will be treated hereafter, in the context of a uniaxial tension of circular cylindrical rod, whose current length is l and its radius is r; the initial values (at $t = 0$) of these quantities will be denoted by l_0 and r_0 respectively. In Eq. (2.1) $\varepsilon = dl/l$ is the longitudinal strain, $\sigma = F/A$, $A = \pi r^2$, is the stress, Γ is the creep kernel and E—the Young modulus.

Imagine that due to loading damage appears and evolves in the rod, in the form of spherical microvoids (porosity), homogeneously and isotropically distributed throughout it. The volume concentration of the voids will be denoted by $c = c(t)$. Hence the rod becomes a microporous material whose effective macroscopic properties, say, the creep kernel, the Young modulus, etc., are strongly affected by the magnitude of c. Moreover, though the loading force F remains constant, the tensile stress σ increases since the radius r of the cross-section shrinks. To simplify the calculations hereafter, keeping at the same time the basic features of the problem under study, we shall replace Eq. (2.1) with the simplified relation

$$\varepsilon(t) = K(t,c)\,\sigma(t)\,, \tag{2.2}$$

where, at fixed c, $K(t,c)$ is just the creep curve of the material with porosity c, corresponding to unit stress. The functional dependence of $K(t,c)$ upon c is supplied by the theory of two-phase viscoelastic composite materials in the particular case when one of the constituents represents voids. The appropriate results and approximate analytical relations for the function $K(t,c)$ are discussed in Section 3 below, so that we shall treat this function as known for the moment.

In the case under study

$$\sigma(t) = \frac{F}{\pi r^2} = \frac{\sigma_0}{\rho^2}\,, \qquad \rho = \frac{r}{r_0}\,, \tag{2.3}$$

where $\sigma_0 = F/(\pi r^2)$ is the initial value of the stress and $\rho = \rho(t) \leq 1$ is the dimensionless radius of the cross-section.

Differentiating (2.2) with respect to time, we get

$$\dot{\varepsilon}(t) = \frac{dl/dt}{l} = \frac{\dot{\lambda}}{\lambda} = \frac{d}{dt}\Big[K(t,c)\frac{\sigma_0}{\rho^2}\Big]$$

$$= \frac{\sigma_0}{\rho^3}\left[\rho\frac{\partial K(t,c)}{\partial t} + \rho\dot{c}\frac{\partial K(t,c)}{\partial c} - 2K(t,c)\dot{\rho}\right], \quad \lambda = \frac{l}{l_0}, \tag{2.4}$$

so that $\lambda = \lambda(t) \geq 1$ is the dimensionless length of the rod.

In turn, the increments of λ and ρ are interconnected through the Poisson ratio $\nu = \nu(c)$:

$$\frac{d\lambda}{\lambda} = -\nu(c)\frac{d\rho}{\rho}, \quad \text{i.e.} \quad \frac{\dot{\lambda}}{\lambda} = -\nu(c)\frac{\dot{\rho}}{\rho}, \tag{2.5}$$

since $d\lambda/\lambda$ is the longitudinal and $d\rho/\rho$ is the transverse strains. We again underline the fact that $\nu = \nu(c)$ depends on the void ratio c; the dependence $\nu(c)$ will also be discussed in Section 3 below.

Finally, an evolution law for the "damage" variable $c = c(t)$ is required. In the best tradition of the continuum damage mechanics we postulate that its rate $\dot{c}$ is determined by the magnitude of the current stress:

$$\dot{c} = B\sigma^n, \tag{2.6}$$

i.e., the same as in Kachanov's damage model [1,2], with B and n denoting material parameters. The difference is only in the fact that while in the latter $\sigma = \sigma_0/(1-\omega)$, where ω is Kachanov's damage, here $\sigma = \sigma_0/\rho^2$ and ρ is a much more complicated function of c that depends implicitly on the creep curve $K(t,c)$, see (2.4) and (2.6).

Note that Eq. (2.6) is tantamount to the equation of the kinetic theory of fracture as developed in detail in [4]. In the latter, one considers the number N of broken bonds in a polymer, assuming that their rupture results from thermal fluctuations induced by the actual applied stress σ which is increased due to the very appearance of the broken bonds. An analysis of the frequency and magnitude of such fluctuations yields that the rate dN/dt is proportional to $\exp(\sigma)$. But broken bonds generate obviously microvoids so that their number is closely connected with the microporosity, i.e., damage in our interpretation, in the polymer. Hence formally we can replace (2.6) by the law

$$\dot{c} = A_1 e^{A_2\sigma}, \tag{2.7}$$

with material parameters A_1, A_2. From the formal point of view, adopted in the continuum damage mechanics, (2.7) is as good as (2.6), since in both these equations the parameters should be specified by an appropriate fit to the experimental data, concerning time-to-rupture for various initial stress values. A more detailed account of the physical situation and fluctuation reasons leading to bonds ruptures allows however to get an interpretation of the parameters A_1 and A_2 in (2.7) (especially, for their temperature dependence [4]) — something which is outside the scope of the more formal continuum damage mechanics in the sense of Kachanov [1,2].

The equations (2.4)–(2.6) form the basic system of differential equations for the unknown functions $\lambda(t)$, $\rho(t)$, $c(t)$ of the proposed model of a deteriorating polymer material. The system should be solved under the obvious initial conditions

$$\lambda(0) = 1, \quad \rho(0) = 1, \quad c(0) = 0. \tag{2.8}$$

The analytical solution of the aforementioned system is impossible even for simplest plausible creep functions $K(t,c)$. However, a qualitative picture of the behaviour of the rod can be easily drawn.

Indeed, this behaviour will be decisively determined by the initial stress value σ_0. If σ_0 is very small, the creep phenomenon will be not pronounced strongly and only the "damage" c will evolve achieving a certain critical magnitude, say $c \approx 0.66$, leading to rod's rupture. This situation corresponds obviously to brittle fracture in this case. The peculiar rheological model—metal or hereditary creep type—is clearly not of special importance as it should be since rupture is governed by damage accumulation that follows Kachanov's type law (2.6). As a consequence, it should be then expected that one of the basic facts of Kachanov's damage mechanics, namely the good approximation of dependence of the time-to-rupture versus stress by a linear fit in the $\log - \log$ coordinates will hold true in the viscoelastic model under study. This will be indeed confirmed in Section 4 for a special and realistic form of the creep function $K(t,c)$ that corresponds to the linear standard body. The peculiarity of the rheology shows up only at higher values of σ_0 when considerable creep deformation evolves and, as a result, the cross-section shrinks down tending to zero; the actual stress increases as a consequence and the rod ruptures due to purely geometrical reasons and fast enough so that there is no time for the damage to attain considerable magnitudes. This situation corresponds obviously to the case of ductile failure. For intermediate values of σ both mechanisms—creep deformation decreasing the cross-section and damage accumulation coexist and interplay and hence a mixed brittle-ductile failure of the rod takes place.

3. Some Facts from Mechanics of Porous Solids

To take into account the influence of the porosity c, treated here as damage parameter, on the effective properties of the rod, we shall invoke some of the simplest relations of mechanics of composite materials. Such relations follow, e.g., from the model of "concentric" spheres as discussed in [5]. This model, as a matter of fact, corresponds to the effective field approach in mechanics of composites [5,6], which in the scalar conductivity case coincides with the well-known Maxwell (or Clausius-Mossoti) relation. The predictions of the model are extremely simple if we assume additionally that the Poisson ratio, ν_0, of the undamaged rod is 0.2. In this case it turns out that $\nu(c) \equiv \nu_0 = 0.2$, independently of c. Hence, from (2.5),

$$\frac{d\lambda}{\lambda} = -0.2\,\frac{d\rho}{\rho}$$

which means that

$$\rho\lambda^5 = 1. \tag{3.1}$$

A somewhat similar relation between ρ and λ exists in the incompressible case, namely, $\rho^2\lambda = 1$. Recall that the latter is essentially used in the Hoff's model of ductile rupture of metals in creep [1–3].

Note that (2.3) and (3.1), when employed in (2.6), yield

$$\dot{c} = B_1 \varepsilon_0^n \lambda^\nu, \quad B_1 = B E_0^n, \quad \nu = 10n, \tag{3.2}$$

thus excluding the function $\rho(t)$. In (3.2) $\varepsilon_0 = \sigma_0/E_0$ is the initial strain of the rod (at $t = 0$).

In the same concentric shell model at $\nu_0 = 0.2$, the instantaneous Young modulus of the rod is a very simple function of c:

$$E = E(c) = E_0 \frac{1-c}{1+c} \tag{3.3}$$

and similarly for the long-time modulus E_∞. That is why we can assume the creep curves of the undamaged and damaged rod to be proportional

$$K(t, c) = \frac{1+c}{1-c}\, K(t, 0)\,, \tag{3.4}$$

with a factor, reciprocal to that in (3.3). (The reason is that $K(0, c) = 1/E$, $K(\infty, c) = 1/E_\infty$, where $E = E(c)$ and $E_\infty = E_\infty(c)$ are respectively the instantaneous and long-time elastic moduli of the damaged rod.) The formula (3.4) holds true at $t = 0$ and $t \to \infty$. The detailed analysis of (3.4), performed in [8,9], demonstrates that it provides a very good approximation to the creep curves of a two-phase linear viscoelastic composite and hence it is fully appropriate for the present study which already contains a number of simplifying assumptions.

With (3.3) and (3.4) taken into account, (2.4) becomes

$$\dot{\varepsilon}(t) = \frac{\dot{\lambda}}{\lambda} = \varepsilon_0 \frac{d}{dt}\Big[\lambda^{10}\, \frac{1+c}{1-c}\, f(t)\Big], \tag{3.5}$$

where $f(t) = K(t, 0)\, E_0$ is the dimensionless creep curve, $f(0) = 1$, and $\varepsilon_0 = \sigma_0/E_0$ is the initial strain, defined in (3.2).

A simple differentiation of the right-side of (3.5) yields, taking into account (3.2),

$$\dot{\lambda} = \frac{\varepsilon_0 \lambda^{11} [2 B_1 \varepsilon_0^n \lambda^\nu f(t) + (1 - c^2) \dot{f}(t)]}{(1 - c)[1 - c - 10 \varepsilon_0 \lambda^{10} (1 + c) f(t)]}\,. \tag{3.6}$$

Together with (3.2) we thus get a system of differential equations for $\lambda(t)$ and $c(t)$ that governs the tensile behaviour of the deteriorating viscoelastic rod. The system takes into account the creep characteristics of the latter through the function $f(t)$ and the damage accumulation features through the material parameters B_1 and n. A bit more detailed analysis of this system will be performed in the next Section for a simple and plausible form of the function $f(t)$.

Note that (3.6) makes sense only at $\varepsilon_0 < 0.1$ since at $\varepsilon_0 = 0.1$ the denominator in the right-side of (3.6) vanishes at $t = 0$ which means that at $\sigma_0 = E_0/10$ the rod ruptures instantaneously.

4. Example and Discussion

For the sake of simplicity, assume that the solid part of the rod follows the well-known standard-linear model, so that its creep curve is

$$K_0(t) = K(t,0) = f(t)/E_0, \quad f(t) = 1 + p(1 - e^{-mt}), \tag{4.1}$$

where p and m are positive material parameters; moreover $m = 1/t_*$ and t_* is the retardation time of the model [10]. Introducing the dimensionless time $\tau = t/t_*$, the basic system (3.2), (3.6) becomes

$$\frac{d\lambda}{d\tau} = \frac{\varepsilon_0 \lambda^{11}[2\widetilde{B}_1 \varepsilon_0^n \lambda^\nu f(t) + (1-c^2)pe^{-\tau}]}{(1-c)[1-c-10\varepsilon_0\lambda^{10}(1+c)f(\tau)]},$$

$$\frac{dc}{d\tau} = \widetilde{B}_1 \varepsilon_0^n \lambda^\nu, \tag{4.2}$$

where $f(\tau) = 1 + p(1 - e^{-\tau})$ and $\widetilde{B}_1 = B_1 t_* = BE_0^n t_*$ is a dimensionless material parameter.

It is important to point out that to have a reasonable picture of the rod's behaviour we should take $n < 1$. Then at $\varepsilon_0 \ll 1$, the right side of $(4.2)_1$ will be very small also which means that $\lambda = l/l_0$ will remain close to 1. At the same time $\varepsilon_0^n > \varepsilon_0$, and the damage will increase linearly in time: $c \approx \widetilde{B}_1 \varepsilon_0^n t$, as it follows from $(4.2)_2$. The failure occurs when c attains a critical value, say, $c \approx 2/3$, while λ remains close to 1, that is the longitudinal strain is very small. The time-to-rupture, t_R, will be specified then by the relation

$$B\sigma_0^n t_R = 2/3,$$

which shows that $\log \sigma_0$ is a linear function of $\log t_R$. If the kinetic equation of damage (2.7) is adopted, then $\log t_R$ and σ_0 will be interconnected by a linear function, which is often observed in experiments [4].

The aforesaid is demonstrated in Figs. 1 to 3, which correspond to somewhat arbitrarily chosen (for the illustrative purposes) material constants $\widetilde{B}_1 = 0.1$, $n = 0.1$ and $p = 2$ (the latter means that the creep is well pronounced, i.e. $E_0/E_\infty = 3$), varying the initial strain ε_0. In Fig. 1 the functions $\lambda(\tau)$ and $c(\tau)$ are shown for $\varepsilon_0 = 0.0001$. The time-to-rupture is $t_R \approx 11t_*$, corresponding to the moment when the damage c attains the critical value of 2/3. Up to this moment $\lambda = l/l_0$ remains very close to 1 which means that the rod indeed fails due to damage accumulation without showing considerable macroscopic deformation. At $\varepsilon_0 = 0.01$ the situation is entirely different, as is well seen from Fig. 2, where the same functions $\lambda(\tau)$ and $c(\tau)$ are plotted. In this case the function $\lambda(\tau)$ has a vertical asymptote, not shown there, at $\tau \approx 2.6$; at this moment the volume concentration of damage remains low—around 0.2 and hence a typical ductile failure takes place. The intermediate case is shown in Fig. 3, where the initial strain is $\varepsilon_0 = 0.001$ in between those of Figs. 1 and 2. In this

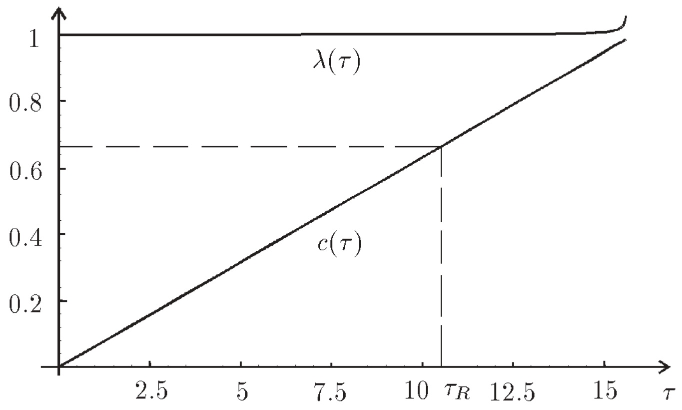

Fig. 1. Longitudinal extension $\lambda = l/l_0$ and damage c as functions of dimensionless time $\tau = t/t_*$; initial strain $\varepsilon_0 = 0.0001$ corresponding to "brittle" rupture.

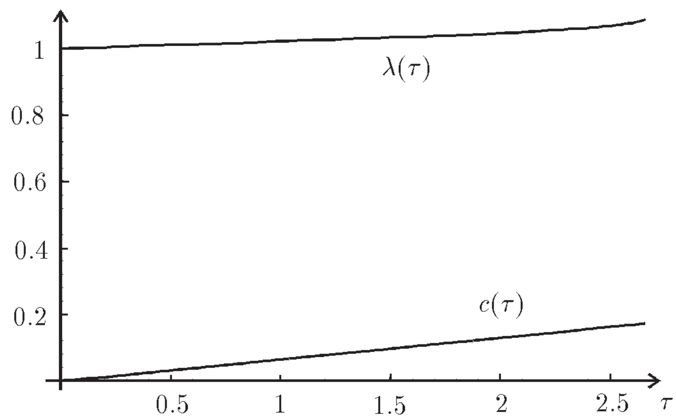

Fig. 2. The same as in Fig. 1 for initial strain $\varepsilon_0 = 0.01$ corresponding to "ductile" rupture.

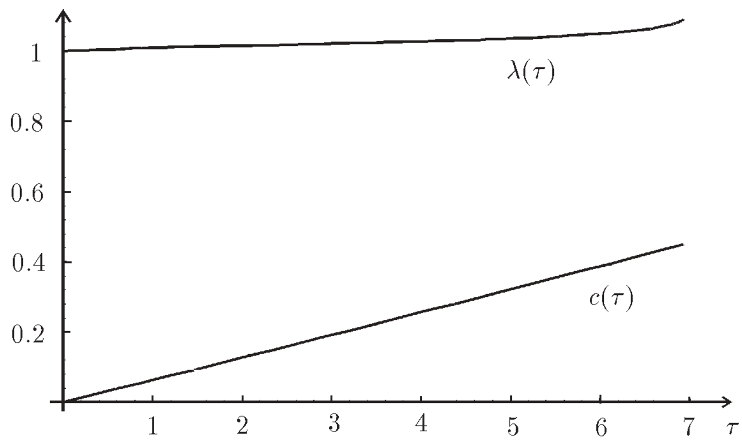

Fig. 3. The same as in Fig. 1 for initial strain $\varepsilon_0 = 0.005$ corresponding to mixed "brittle-ductile" rupture.

case again the function $\lambda(\tau)$ has a vertical asymptotics (not shown) at $\tau \approx 7$, but the volume concentration of damage accumulated up to this moment is already very considerable—about 0.4. This is obviously a mixed brittle-ductile type of failure of the rod.

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References

1. L. M. Kachanov, *Izv. AN SSSR, Otd. Tehn. Nauk* (1958) (No 8) 26. (in Russian.)
2. L. M. Kachanov, *Introduction to Continuum Damage Mechanics*, Klüwer Acad. Publ., 1986.
3. J. Lemaitre and J.-L. Chaboche, *Mécanique des Materiaux Solids*, Dunod, Paris, 2-ème edition, 1988.
4. V. R. Regel, A. V. Slutcker and V. V. Tomashevskii, *Kinetical Nature of Strength of Solids*, Nauka, Moscow, 1974. (in Russian.)
5. R. Christensen, *Mechanics of Composite Materials*, John Wiley, New York, 1979.
6. K. Z. Markov, In *Continuum Models and Discrete Systems*, eds. O. Brulin and R. Hsieh, North-Holland, 1981, p. 441.
7. S. K. Kanaun and V. M. Levin, In *Advances in Mathematical Modelling of Composite Materials*, ed. K. Z. Markov, World Sci., 1994, p. 1.
8. K. Z. Markov, *Theor. and Appl. Mech., Bulg. Acad. Sciences*, Year 7 (No 4) (1976) 51.
9. K. Z. Markov, *Theor. Appl. Mechanics, Bulg. Acad. Sci.*, Year 14 (No 4) (1983) 45.
10. R. Christensen, *Theory of Viscoelasticity*, John Wiley, New York, 1979.