Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 59, No 1, 2006

MATHEMATIQUES
Théorie des graphes

FOLKMAN NUMBER $F_e(3,4;8)$ IS EQUAL TO 16

N. Kolev, N. Nenov

(*Submitted by Academician B. Boyanov on October 25, 2005*)

Abstract

The set of the vertices of a graph G is denoted by $V(G)$. The symbol $G \stackrel{e}{\rightarrow} (3,4)$ means that in every 2-colouring of the edges of G there is either a 3-clique in the first colour or a 4-clique in the second colour. Folkman number $F_e(3,4;8)$ is defined by the equality

$$F_e(3,4;8) = \min\{|V(G)| : G \stackrel{e}{\rightarrow} (3,4) \text{ and } K_8 \not\subseteq G\}.$$

In this paper we prove that $F_e(3,4;8) = 16$.

Key words: vertex Folkman numbers, edge Folkman numbers
2000 Mathematics Subject Classification: 05C55

1. Notations. We consider only finite, non-oriented graphs without loops and multiple edges. A set of p vertices of the graph G is called a p-clique if each two of them are adjacent. The greatest positive integer p for which G has a p-clique is called clique number of G and is denoted by $\mathrm{cl}(G)$. We shall use the following notations in this paper:

- $V(G)$ is the vertex set of graph G;
- $E(G)$ is the edge set of graph G;
- $\overline{G}$ is the complementary graph of G;
- $G - V$, $V \subseteq V(G)$ is the subgraph of G induced by the set $V(G) \setminus V$;
- K_n is the complete graph on n vertices;
- C_n is the simple cycle on n vertices;
- $\alpha(G)$ is the independence number of G, i.e. $\alpha(G) = \mathrm{cl}(\overline{G})$;
- $N(v)$, $v \in V(G)$ is the set of all vertices of G adjacent to v.

Let G_1 and G_2 be two graphs without common vertices. We denote by G_1+G_2 the graph G defined as follows: $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x, y] \mid x \in V(G_1), y \in V(G_2)\}$.

2. Main result. Each partition

$$E(G) = E_1 \cup \dots \cup E_r \qquad E_i \cap E_j = \emptyset, \quad i \neq j \tag{2.1}$$

is called an r-colouring of the edges of G. We say that H is a monochromatic subgraph from the i-th colour in the r-colouring (2.1) if $E(H) \subseteq E_i$.

Definition 2.1. Let $a_1, \dots, a_r$ be positive integers, $a_i \geq 2$, $i = 1, \dots, r$. We say that the r-colouring is $(a_1, \dots, a_r)$-free if for each $i \in \{1, \dots, r\}$ there is no a_i-clique in the i-th colour. The symbol $G \overset{e}{\to} (a_1, \dots, a_r)$ means that any r-colouring of $E(G)$ is not $(a_1, \dots, a_r)$-free.

Definition 2.2. Let $a_1, \dots, a_r$ be positive integers, $a_i \geq 2$, $i = 1, \dots, r$. The smallest positive integer n for which $K_n \overset{e}{\to} (a_1, \dots, a_r)$ is denoted by $R(a_1, \dots, a_r)$ and is called Ramsey number.

Note that the number $R(a_1, a_2)$ can also be defined as the smallest positive integer n, such that for every n-vertex graph G either $\mathrm{cl}(G) \geq a_1$ or $\alpha(G) \geq a_2$. The existence of the numbers $R(a_1, \dots, a_r)$ was proved by Ramsey in [19].

An exposition of the results on the numbers $R(a_1, \dots, a_r)$ is given in [18]. In this paper we shall need the following values only:

$$R(3, 4) = R(4, 3) = 9. \tag{2.2}$$

The edge Folkman number $F_e(a_1, \dots, a_r; q)$ is denoted by the equality

$$F_e(a_1, \dots, a_r; q) = \min\{|V(G) : G \overset{e}{\to} (a_1, \dots, a_r) \text{ and } \mathrm{cl}(G) < q\}.$$

It is clear that from $G \overset{e}{\to} (a_1, \dots, a_r)$ it follows $\mathrm{cl}(G) \geq \max\{a_1, \dots, a_r\}$. There exists a graph $G \overset{e}{\to} (a_1, \dots, a_r)$ and $\mathrm{cl}(G) = \max\{a_1, \dots, a_r\}$. In the case $r = 2$ this was proved in [2] and in the general case in [16]. That is why

$$F_e(a_1, \dots, a_r; q) \text{ exists } \iff q > \max\{a_1, \dots, a_r\}.$$

From Definition 2.2 it follows that

$$F_e(a_1, \dots, a_r; q) = R(a_1, \dots, a_r) \text{ if } q > R(a_1, \dots, a_r).$$

In this paper we shall use the equality

$$F_e(3, 4; 9) = 14, \text{ [11] (see also [15]).} \tag{2.3}$$

Besides this value we know only the following edge Folkman numbers of the kind $F_e(a_1, \dots, a_r; R(a_1, \dots, a_r))$

$F_e(3, 3; 6) = 8$ ([3] and [5]);
$F_e(3, 5; 14) = 16$ ([5]);
$F_e(4, 4; 18) = 20$ ([5]);
$F_e(3, 3, 3; 17) = 19$ ([5]).

We know only two edge Folkman numbers of the kind $F_e(a_1, \dots, a_r; R(a_1, \dots, a_r)-1)$, namely $F_e(3,3;5) = 15$ and $F_e(3,3,3;16) = 21$. The inequality $F_e(3,3;5) \leq 15$ was proved in [7] and the inequality $F_e(3,3;5) \geq 15$ was obtained in [17] by the means of a computer. The inequality $F_e(3,3,3;16) \geq 21$ was proved in [5] and the opposite inequality $F_e(3,3,3;16) \leq 21$ in [8].

In this paper we shall compute one more Folkman number of the kind $F_e(a_1, \dots, a_r; R(a_1, \dots, a_r) - 1)$ by proving the following

Main theorem. $F_e(3,4;8) = 16$.

The best previously known upper bound on this number was $F_e(3,4;8) \leq 314$ (see [6]). We also know that $F_e(3,4;8) \geq 15$ which easily follows from (2.3). At the end of this exposition we shall note that $F_e(3,3,3;15) = 23$ ([9]) is the only known number of the kind $F_e(a_1, \dots, a_r; R(a_1, \dots, a_r) - 2)$ and $F_e(3,3,3;14) = 25$ ([10]) is the only known number of the kind $F_e(a_1, \dots, a_r; R(a_1, \dots, a_r) - 3)$.

In order to prove the Main Theorem we shall use

Theorem 2.1. $K_1 + C_5 + C_5 + C_5 \overset{e}{\rightarrow} (3,4)$.

The proof of Theorem 2.1 is very voluminous and we shall prove it additionally in [4].

Let $A \subseteq V(G)$ be an independent set of vertices of the graph G. We denote by G/A the graph $K_1 + (G - A)$. It is easy to see that

$$\mathrm{cl}(G/A) \leq \mathrm{cl}(G) + 1; \tag{2.4}$$

$$G \overset{e}{\rightarrow} (a_1, \dots, a_r) \Rightarrow G/A \overset{e}{\rightarrow} (a_1, \dots, a_r). \tag{2.5}$$

In order to prove the Main Theorem we shall use the following

Proposition 2.1. Let $a_1, \dots, a_r$ be positive integers, $a_i \geq 2$, $i = 1, \dots, r$. Let G be a graph such that $G \overset{e}{\rightarrow} (a_1, \dots, a_r)$ and $\mathrm{cl}(G) \leq q - 2$. Then

$$|V(G)| \geq F_e(a_1, \dots, a_r; q) + \alpha(G) - 1. \tag{2.6}$$

Proof. Let A be an independent set of vertices of G and $|A| = \alpha(G)$. According to (2.4) and (2.5) $G \overset{e}{\rightarrow} (a_1, \dots, a_r)$ and $\mathrm{cl}(G/A) \leq q - 1$. Therefore

$$|V(G/A)| \geq F_e(a_1, \dots, a_r; q).$$

Inequality (2.6) follows from the last inequality as $|V(G/A)| = |V(G)| - \alpha(G) + 1$.

3. Vertex Folkman numbers.

Definition 3.1. Let $a_1, \dots, a_r$ be positive integers. We say that the r-colouring

$$V(G) = V_1 \cup \dots \cup V_r, \qquad V_i \cap V_j = \emptyset, \quad i \neq j$$

of the vertices of G is $(a_1, \dots, a_r)$-free if for every $i \in \{1, \dots, r\}$ the set V_i does not contain an a_i-clique. The symbol $G \overset{v}{\rightarrow} (a_1, \dots, a_r)$ means that graph G has no $(a_1, \dots, a_r)$-free colouring.

The vertex Folkman number $F_v(a_1, \dots, a_r; q)$ is defined by the equality

$$F_v(a_1, \dots, a_r; q) = \min\{|V(G) : G \overset{v}{\rightarrow} (a_1, \dots, a_r) \text{ and } \mathrm{cl}(G) < q\}.$$

In [2] Folkman proved that

$$F_v(a_1, \dots, a_r; q) \text{ exists} \iff q > \max\{a_1, \dots, a_r\}.$$

We shall need the following results about the vertex Folkman numbers:

$$F_v(2,2,4;5) = 13 \text{ (see [12])}, \tag{3.1}$$

$$F_v(2,2,p;p+1) \geq 2p+4 \text{ (see [13])}. \tag{3.2}$$

There is an exposition of the results on Folkman numbers in [1]. We shall also add the papers [8–10,14] to this exposition. For the proof of the inequality $F_e(3,4;8) \geq 16$ we shall need the following

Theorem 3.1. Let G be a graph, $\mathrm{cl}(G) \leq p$ and $|V(G)| \geq p+2$, $p \geq 2$. Let G also have the following two properties:

(i) $G \overset{v}{\not\to} (2,2,p)$;

(ii) If $V(G) = V_1 \cup V_2 \cup V_3$ is a $(2,2,p)$-free 3-colouring then $|V_1| + |V_2| \leq 3$.

Then $G = K_1 + G_1$.

Proof. Let $V(G) = V_1 \cup V_2 \cup V_3$ be a $(2,2,p)$-free 3-colouring. According to (ii) we have

$$|V_1| + |V_2| \leq 3. \tag{3.3}$$

Since $|V(G)| \geq p+2 \geq 4$, it follows from (3.3) that

$$V_3 \neq \emptyset. \tag{3.4}$$

It is enough to consider only the situation when $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$. Indeed, let $V_1 = \emptyset$. It follows from (3.4) that there is $w \in V_3$. It is clear that

$$\{w\} \cup V_2 \cup (V_3 - \{w\})$$

is a $(2,2,p)$-free 3-colouring. So, we can assume without loss of generality that

$$1 \leq |V_1| \leq |V_2|. \tag{3.5}$$

It is clear from (3.4) and (3.5) that only the following two cases are possible:

Case 1. $|V_1| = |V_2| = 1$. Let $V_1 = \{a\}$ and $V_2 = \{b\}$. In this case we have $[a,b] \in E(G)$. Assume the opposite. It follows from $|V(G)| \geq p+2$ that $|V_3| \geq p$. As V_3 does not contain a p-clique there exist two non-adjacent vertices $c, d \in V_3$. Then $\{a,b\} \cup \{c,d\} \cup (V_3 - \{c,d\})$ is a $(2,2,p)$-free 3-colouring which contradicts (ii). So, we have

$$[a,b] \in E(G). \tag{3.6}$$

Assume that the statement of Theorem 2.1 is wrong. Then there are $a', b' \in V_3$ such that $[a,a'] \notin E(G)$ and $[b,b'] \notin E(G)$. If $a' \neq b'$ then

$$\{a,a'\} \cup \{b,b'\} \cup (V_3 - \{a',b'\})$$

is a $(2,2,p)$-free 3-colouring which contradicts (ii). It remains to consider only the situation when $a' = b' = c$ and

$$N(a) \supset V_3 - \{c\}, \qquad N(b) \supset V_3 - \{c\}. \tag{3.7}$$

It follows from $\mathrm{cl}(G) \geq p$ and (3.7) that

$$V' = V_3 - \{c\} \text{ does not contain a } (p-1)\text{-clique.} \tag{3.8}$$

As $|V(G)| \geq p+2$ we have $|V'| \geq p+1$. That is why, it follows from (3.8) that V' contains two non-adjacent vertices m and n. Let $V'' = V' - \{m, n\}$. According to (3.8), $V'' \cup \{b\}$ does not contain a p-clique. Therefore,

$$\{a, c\} \cup \{m, n\} \cup (V'' \cup \{b\})$$

is a $(2, 2, p)$-free 3-colouring of $V(G)$ which contradicts (ii).

Case 2. $|V_1| = 1$, $|V_2| = 2$. Let $|V_1| = \{a\}$ and $V_2 = \{b, c\}$. We shall first prove that

$$N(a) \supset V_3. \tag{3.9}$$

Assume that (3.9) is wrong and let $[a, d] \notin E(G)$, $d \in V_3$. Then

$$\{a, d\} \cup \{b, c\} \cup (V_3 - \{d\})$$

is a $(2, 2, p)$-free 3-colouring of $V(G)$ which is a contradiction.

If $\{a, b, c\}$ is an independent set and $d \in V_3$ then

$$\{a, b, c\} \cup \{d\} \cup (V_3 - \{d\})$$

is a $(2, 2, p)$-free 3-colouring of $V(G)$ which is a contradiction. Therefore, we can assume that $[a, b] \in E(G)$. We have from (3.9) that $N(a) \supset V_3 \cup \{b\}$. Since $\mathrm{cl}(G) \leq p$, $V_3 \cup \{b\}$ does not contain a p-clique. Thus $\{a\} \cup \{c\} \cup (V_3 \cup \{b\})$ is a $(2, 2, p)$-free 3-colouring of $V(G)$ and we are in the situation of case 1. Theorem 3.1 is proved.

4. Proof of the Main Theorem.

I. Proof of the inequality $F_e(3, 4; 8) \leq 16$. We consider the graph $H = K_1 + C_5 + C_5 + C_5$. By Theorem 2.1, $H \stackrel{e}{\rightarrow} (3, 4)$. Since $\mathrm{cl}(H) = 7$ we have that $F_e(3, 4; 8) \leq |V(H)| = 16$.

II. Proof of the inequality $F_e(3, 4; 8) \geq 16$. Assume that this inequality is wrong. Then there is a graph G such that $G \stackrel{e}{\rightarrow} (3, 4)$, $\mathrm{cl}(G) \leq 7$ and $|V(G)| \leq 15$. It follows from $|V(G)| \leq 15$, Proposition 2.1 ($q = 9$) and (2.3) that

$$|V(G)| = 15, \tag{4.1}$$

$$\alpha(G) = 2. \tag{4.2}$$

We shall prove that G suffices the conditions of Theorem 3.1 for $p = 7$. We have from (3.2) that $F_e(2, 2, 7; 8) \geq 18$. Since $\mathrm{cl}(G) \leq 7$, from this inequality and (4.1) it follows $G \stackrel{v}{\not\rightarrow} (2, 2, 7)$. Let $V_1 \cup V_2 \cup V_3$ be $(2, 2, 7)$-free 3-colouring of $V(G)$. We define the graphs $G_1 = G/V_1$ and $G_2 = G_1/V_2$. We see from (2.5) that $G_2 \stackrel{e}{\rightarrow} (3, 4)$. As V_3 does not contain a 7-clique, $\mathrm{cl}(G_2) \leq 8$. Thus, it follows from (2.3) that

$$|V(G_2)| \geq 14. \tag{4.3}$$

We see from (4.2) that $|V_1| \leq 2$ and $|V_2| \leq 2$. Therefore, if $|V_1| \leq 1$ or $|V_2| \leq 1$ we have that $|V_1| + |V_2| \leq 3$. It remains just to consider the situation when $|V_1| = |V_2| = 2$. Since

$|V(G_2)| = |V(G)| - |V_1| - |V_2| + 2 = 17 - |V_1| - |V_2|$ from (4.3) we obtain $|V_1| + |V_2| \leq 3$. So, G suffices the conditions of Theorem 3.1 for $p = 7$. Thus, $G = K_1 + H_1$. It follows from $\mathrm{cl}(G) \leq 7$ that $\mathrm{cl}(H_1) \leq 6$. Now we shall prove that H_1 suffices the conditions of Theorem 3.1 for $p = 6$. From (3.2) we have $F_v(2,2,6;7) \geq 16$. Since $|V(H_1)| = 14$ and $\mathrm{cl}(H_1) \leq 6$ we have $H_1 \overset{v}{\not\to} (2,2,6)$. Let $V(K_1) = \{a\}$ and $V_1 \cup V_2 \cup V_3$ be $(2,2,6)$-free 3-colouring of $V(H_1)$. It is clear that $V_1 \cup V_2 \cup (V_3 \cup \{a\})$ is $(2,2,7)$-free 3-colouring of $V(G)$. As we proved above $|V_1| + |V_2| \leq 3$. According to Theorem 3.1 $H_1 = K_1 + H_2$ and $G = K_2 + H_2$. From $\mathrm{cl}(H_1) \leq 6$ it follows $\mathrm{cl}(H_2) \leq 5$. From (3.2) we obtain that $H_2 \overset{v}{\not\to} (2,2,5)$. Repeating about H_2 the above considerations about H_1 we see that H_2 suffices the condition (ii) of Theorem 3.1 for $p = 5$, too. Hence, $H_2 = K_1 + H_3$ and $G = K_3 + H_3$. Now consider the graph H_3. Since $|V(H_3)| = 12$, from (3.1) we have $H_3 \overset{v}{\not\to} (2,2,4)$. As above we see that H_3 suffices the condition (ii) of Theorem 3.1 for $p = 4$, too. That is why $H_3 = K_1 + H_4$ and $G = K_4 + H_4$. As $\mathrm{cl}(G) \leq 7$ we have $\mathrm{cl}(H_4) \leq 3$. It follows from (4.2) that $\alpha(H_4) = 2$. This contradicts (2.2) as $|V(H_4)| = 11$.

The Main Theorem is proved.

REFERENCES

[1] Coles J. Algorithms for bounding Folkman numbers. Master Thesis (see `http://www.jpcoles.com/uni/rit/thesis/jpc-thesis.folkman.pdf`). [2] Folkman J. SIAM J. Appl. Math., **18**, 1970, 19–24. [3] Graham R. L. J. Combin. Theory, **4**, 1968, 300. [4] Kolev N., N. Nenov. Ann. Univ. Sofia, Fac. Math. Inform. (submitted). [5] Lin S. J. Combin. Theory, **B12**, 1972, 82–92. [6] Luczak T., A. Rucinski, S. Urbanski. Discrete Math., **236**, 2001, 245–262. [7] Nenov N. Compt. rend. Acad. bulg. Sci., **34**, 1981, 1487–1489 (in Russian). [8] Id. Ibid., **33**, 1980, No 9, 1171–1174. [9] Id. Ibid., **34**, 1981, 1209–1212. [10] Id. Ann. Univ. Sofia, Fac. Math. Mec., **75**, 1981, 27–38 (in Russian). [11] Id. Ann. Univ. Sofia, Fac. Math. Inform., **85**, 1991, No 1–2, 71–81. [12] Id. Serdica Bulgaricae math. publicationes, **27**, 2001, 131–136. [13] Id. Ann. Univ. Sofia, Fac. Math. Inform., **94**, 2000, 15–25. [14] Id. Serdica Math. J., **28**, 2002, 219–232. [15] Id. Ann. Univ. Sofia, Fac. Math. Inform., **95**, 2001, 59–69. [16] Nešetřil J., V. Rödl. J. Comb. Th., **B20**, 1976, 243–249. [17] Piwakowski K., S. Radziszowski, S. Urbanski. J. Graph Theory, **32**, 1999, 41–49. [18] Radziszowski S. Electronic J. Combinatorics, DS1, 2002 (`http://www.combinatorics.org/Surveys/`). [19] Ramsey P. Proc. London Math. Soc., **30**, 1930, 264–268.

Faculty of Mathematics and Informatics
St. Kliment Ohridski University of Sofia
5, James Bourchier Blvd
1164 Sofia, Bulgaria
e-mail: `nenov@fmi.uni-sofia.bg`