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SATURATED β-SEQUENCES IN GRAPHS

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Abstract

In this article sufficient conditions are obtained under which the mean degree of the members of a given β-sequence (Definition 3) is not least than the the mean degree of all vertices of the graph.

Key words: saturated sequence, generalized r-partite graph, generalized Turan's graph

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1. Introduction. We consider only finite nonoriented graphs without loops and multiple edges. The vertex set and the edge set of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. We shall also use the following notations:

$e(G) = |E(G)|$ – the number of edges of G;

$\Gamma(M)$, $M \subseteq V(G)$ – the set of all vertices of G adjacent to any vertex of M;

$d(v) = |\Gamma(v)|$ – the degree of a vertex v.

The symbol $T_r(n)$ denotes the n-vertex r-partite Turan's graph, i.e. the complete r-partite graph with partition classes $V_1, \dots, V_r$, such that $|p_i - p_j| \leq 1$ for all pairs $\{i,j\}$, where $p_i = |V_i|$, $i = 1, \dots, r$.

Definition 1 ([2]). An n-vertex graph G is called generalized r-partite with partition classes $V_1, \dots, V_r$, if $V(G) = V_1 \cup \dots \cup V_r$, $V_i \cap V_j = \emptyset$, $i \neq j$ and $d(v) \leq n - |V_i|$ for any $v \in V_i$, $i = 1, \dots, r$. If $d(v) = n - |V_i|$ for any $v \in V_i$, $i = 1, \dots, r$, then G is called generalized complete r-partite graph with partition classes $V_1, \dots, V_r$.

Definition 2 ([2]). We call G generalized Turan's r-partite graph, if G is a generalized complete r-partite graph with partition classes $V_1, \dots, V_r$ such that $|p_i - p_j| \leq 1$ for all pairs $\{i,j\}$, where $p_i = |V_i|$, $i = 1, \dots, r$.

In [3] Faudree introduced the following concept:

Definition 3. The sequence of vertices $v_1, \dots, v_r$ in a graph G is called β-sequence, if the following conditions are satisfied: v_1 is a maximal degree vertex in G, and for $i \geq 2$, $v_i \in \Gamma(v_1, \dots, v_{i-1})$ and $d(v_i) = \max\{d(v) \mid v \in \Gamma(v_1, \dots, v_{i-1})\}$.

Definition 4. Let G be an n-vertex graph and $v_1, \dots, v_r \in V(G)$. Then the sequence $v_1, \dots, v_r$ is called saturated, if

$$\frac{1}{r}(d(v_1) + \dots + d(v_r)) \geq \frac{2}{n} e(G). \tag{1.1}$$

Our main results are the following two theorems:

Theorem 1. Let G be an n-vertex graph and let $v_1, \dots, v_r$ be a β-sequence in G, $r \geq 2$ which is not saturated. Then

$$d(v_1) + \dots + d(v_{r-1}) < \frac{(r-1)^2}{r} n.$$

Theorem 2. Let G be an n-vertex graph and let $v_1, \ldots, v_r$ be a β-sequence in G, $r \geq 2$, which is not saturated. Then G is a generalized r-partite graph which is not a generalized r-partite Turan's graph.

We shall use the following:

Proposition 1. Let $v_1, \ldots, v_r$ be a β-sequence in the graph G such that $d(v_1) + \cdots + d(v_k) \leq A$ for some $1 \leq k \leq r$. Then $d(v_1) + \cdots + d(v_r) \leq \frac{A}{k} r$.

Proof. Since $d(v_1) \geq \cdots \geq d(v_r)$, we have $d(v_i) \leq \frac{A}{k}$, $i = k, \ldots, r$. Thus $d(v_1) + \cdots + d(v_r) \leq A + \frac{A}{k}(r-k) = \frac{A}{k} r$.

2. Proof of Theorem 1. Since $(r-2)n < \frac{(r-1)^2}{r} n$, in case $d(v_1) + \cdots + d(v_{r-1}) \leq (r-2)n$, Theorem 1 is obvious. Therefore, we shall assume that

$$d(v_1) + \cdots + d(v_{r-1}) > (r-2)n. \tag{2.1}$$

It follows from Proposition 1 and (2.1) that

$$d(v_1) + \cdots + d(v_{r-2}) > \frac{(r-2)^2 n}{r-1}, r \geq 3. \tag{2.2}$$

Define $V_1 = V(G) \backslash \Gamma(v_1)$, $V_i = \Gamma(v_1, \ldots, v_{i-1}) \backslash \Gamma(v_i)$, $2 \leq i \leq r-1$, and $V_r = \Gamma(v_1, \ldots, v_{r-1})$. Obviously, $v_i \in V_i$, $i = 1, \ldots, r$ and

$$V(G) = V_1 \cup \cdots \cup V_r, V_i \cap V_j = \emptyset, i \neq j. \tag{2.3}$$

Since $V_i \subseteq V(G) \backslash \Gamma(v_i)$, $i = 1, \ldots, r-1$, we have

$$|V_i| \leq n - d(v_i), \quad i = 1, \ldots, r-1. \tag{2.4}$$

It follows from (2.1) and (2.4) that

$$|V_r| = n - \sum_{i=1}^{r-1} |V_i| \geq \sum_{i=1}^{r-1} d(v_i) - (r-2)n > 0.$$

Thus, $V_r \neq \emptyset$. Let V_r' be a subset of V_r such that

$$|V_r'| = \sum_{i=1}^{r-1} d(v_i) - (r-2)n. \tag{2.5}$$

Define $W = V(G) \backslash V_r'$. By (2.5) we have

$$|W| = \sum_{i=1}^{r-1} (n - d(v_i)). \tag{2.6}$$

Since $V_i \subseteq W$, $i = 1, \ldots, r-1$, from (2.3), (2.4) and (2.6) it follows that there exist disjoint sets $V_i' \subseteq W$, $i = 1, \ldots, r-1$, such that $V_i' \supseteq V_i$ and $|V_i'| = n - d(v_i)$, $i = 1, \ldots, r-1$. We have $v_i \in V_i \subset V_i'$, $i = 1, \ldots, r-1$. It follows from (2.6) that $W = \bigcup_{i=1}^{r-1} V_i'$. Hence

$$V(G) = V_1' \cup \cdots \cup V_r', V_i' \cap V_j' = \emptyset, i \neq j. \tag{2.7}$$

Observe that $V_i' \backslash V_i \subseteq V_r = \Gamma(v_1, \ldots, v_{r-1}) \subset \Gamma(v_1, \ldots, v_{i-1})$ and $V_i \subset \Gamma(v_1, \ldots, v_{i-1})$. Thus $V_i' \subset \Gamma(v_1, \ldots, v_{i-1})$, $i = 1, \ldots, r-1$ and $d(v) \leq d(v_i)$, $\forall v \in V_i'$, $i = 1, \ldots, r$. Since $V_r' \subset V_r$, $d(v) \leq d(v_r)$, $\forall v \in V_r'$. So, we have

$$d(v) \leq d(v_i), \forall v \in V_i', i = 1, \ldots, r. \tag{2.8}$$

From (2.7) it follows that

$$2e(G) = \sum_{v\in V(G)} d(v) = \sum_{v\in V_1'} d(v) + \cdots + \sum_{v\in V_r'} d(v).$$

Let $d(v_i) = d_i$, $i = 1, \dots, r$. Since $|V_i'| = n - d(v_i)$, $i = 1, \dots, r-1$, from (2.8) and (2.5) it follows that

$$2e(G) \le \sum_{i=1}^{r-1} d_i(n - d_i) + \left(\sum_{i=1}^{r-1} d_i - (r-2)n\right) d_r. \tag{2.9}$$

Let $\sigma = d_1 + \cdots + d_{r-1}$. Then $d_r < \frac{2re(G)}{n} - \sigma$, because $v_1, \dots, v_r$ is not saturated. Since $\sum_{i=1}^{r-1} d_i^2 \ge \frac{1}{r-1}\sigma^2$, from (2.9) it follows

$$2e(G) \le n\sigma - \frac{1}{r-1}\sigma^2 + (\sigma - (r-2)n)\left(\frac{2re(G)}{n} - \sigma\right).$$

From this inequality one easily gets

$$\frac{2e(G)}{n}((r-1)^2 n - r\sigma) < \frac{\sigma}{r-1}((r-1)^2 n - r\sigma). \tag{2.10}$$

We shall proceed by induction on r with induction base $r = 2$. Substituting $r = 2$ in (2.10) we obtain

$$\frac{2e(G)}{n}(n - 2d_1) < d_1(n - 2d_1).$$

Obviously, $n - 2d_1 \ne 0$. If $n - 2d_1 < 0$, then $d_1 < \frac{2e(G)}{n}$. But this is impossible, as $d_1 = d(v_1)$ is a maximal vertex degree. So, $n - 2d_1 > 0$ and we have done with the basic case $r = 2$.

Let $r \ge 3$. By (2.10), $(r-1)^2 n - r\sigma \ne 0$. Assume that $(r-1)^2 n - r\sigma < 0$. Then from (2.10) it follows that $\sigma < \frac{2(r-1)e(G)}{n}$, i.e. $v_1, \dots, v_{r-1}$ is not saturated. By the induction hypothesis, we obtain

$$d(v_1) + \cdots + d(v_{r-2}) < \frac{(r-2)^2 n}{r-1}.$$

This contradicts (2.2) and proves Theorem 1.

3. A corollary from Theorem 1. Let v be a vertex of maximal degree in a graph G. The symbol $l(v)$ denotes the maximal length of β-sequences with first member v.

We shall use the following

Theorem 3 ([1]). Let G be an n-vertex graph, v be a vertex of maximal degree in G and $l(v) = r$. Then $e(G) \le e(T_r(n))$ and the equality appears if and only if $G = T_r(n)$.

Let n and r be positive integers. Define

$$g(n, r) = \frac{n^2(r-1)}{2r}.$$

We obtain the following

Corollary 1. Let G be an n-vertex graph and r be a positive integer, $r \le n$. Let

$$e(G) \ge g(n, r). \tag{3.1}$$

Then

(a) $l(v) \ge r$ for all vertices v of maximal degree. If $l(v) = r$ for some vertex v of maximal degree, then $n \equiv 0 (\bmod r)$ and $G = T_r(n)$.

(b) Each β-sequence with r members in G is saturated.

Proof of (a). Let v be a vertex of maximal degree and $l(v) = s$. By Theorem 3 and (3.1) we have

$$e(T_s(n)) \geq e(G) \geq g(n,r). \tag{3.2}$$

Since $g(n,s) \geq e(T_s(n))$, from (3.2) it follows that $g(n,s) \geq g(n,r)$ which is equivalent to $s \geq r$.

Let $s = r$. Then (3.2) together with the inequality $g(n,r) \geq e(T_r(n))$ implies that $e(T_r(n)) = g(n,r)$ and $e(G) = e(T_r(n))$. From the first equality it follows $n \equiv 0(\operatorname{mod} r)$. The second equality, according to Theorem 3, implies $G = T_r(n)$.

Proof of (b). Let $v_1, \dots, v_r$ be a β-sequence in G. We shall prove that this sequence is saturated by induction on r. If $r = 1$ this is obvious. Let $r \geq 2$. Then from (3.1) it follows that $e(G) > g(n, r-1)$. By the induction hypothesis we obtain that

$$d(v_1) + \cdots + d(v_{r-1}) \geq \frac{2(r-1)e(G)}{n}.$$

This inequality together with (3.1) implies

$$d(v_1) + \cdots + d(v_{r-1}) \geq \frac{(r-1)^2 n}{r}.$$

By Theorem 1, $v_1, \dots, v_r$ is a saturated sequence. This completes the proof of Corollary 1.

Remark 1. A proposition weaker than Corollary 3 (b) is stated in [3] as "Theorem 1". The proof of this theorem given in [3] is not correct, as it has the following considerable defects:

1. The statement $N_1 \cap \cdots \cap N_{k-1} \neq \emptyset$ does not follow from the induction hypothesis (as the author states at p. 330) and needs a proof. The proof is not trivial and follows from Theorem 2 in our paper [1].

2. The proof of the inequality

$$2m \leq \sum_{i=1}^{k-1}(n - d_i)d_i + \left(\sum_{i=1}^{k-1} d_i - (k-2)n\right) d_k$$

is not correct, since $|N_1 \cap \cdots \cap N_{k-1}| \geq \sum_{i=1}^{k-1} d_i - (k-2)n$ (and not conversely).

3. The inequality $d_k < \frac{2m}{n} - d$ does not imply inequality (6) from [3], since $\sum_{i=1}^{k-1} d_i - (k-2)n \geq 0$ is not proved.

It is interesting to note that although Theorem 1 in [3] is not proved, it is essentially referred to in the proof of Theorem 3.8 and Theorem 3.9 in [4]. However, these two theorems are absurd as it is supposed that the n-vertex graph G has more than $\binom{n}{2}$ edges.

4. Proof of Theorem 2. In the proof of Theorem 2 we shall make use of the following two results:

Theorem 4 ([2]). Let G be an n-vertex generalized r-partite graph with partition classes $V_1, \dots, V_r$. Then

$$e(G) \leq e(T_r(n))$$

and the equality occurs if and only if G is a generalized r-partite Turan's graph with partition classes $V_1, \dots, V_r$.

Lemma 1. Let G be a graph and $v_1, \dots, v_r$ be a β-sequence in G such that

$$d(v_1) + \cdots + d(v_k) \leq \frac{k(r-1)n}{r}, \text{ for some } 1 \leq k \leq r. \tag{4.1}$$

Then G is a generalized r-partite graph. If inequality (4.1) is strict, then G is not a generalized r-partite Turan's graph.

Proof. We shall prove this lemma by induction on r. The base $r = 1$ is clear, because $d(v_1) = 0$, i.e. $e(G) = 0$. Let $r \geq 2$. By (4.1) and Proposition 1 we have

$$d(v_1) + \cdots + d(v_r) \leq (r-1)n. \tag{4.2}$$

Let $d(v_1) + \cdots + d(v_{r-1}) \leq (r-2)n$. Then the β-sequence $v_1, \ldots, v_{r-1}$ satisfies the conditions of Lemma 1 (with $k = r-1$). By the induction hypothesis G is a generalized $(r-1)$-partite graph and, hence, G is a generalized r-partite graph, which is not a generalized r-partite Turan's graph.

Assume that $d(v_1) + \cdots + d(v_{r-1}) > (r-2)n$. Consider the sets $V_i' \subseteq V(G)$, $i = 1, \ldots, r$, which are defined in the proof of Theorem 1. We shall prove that G is a generalized r-partite graph with partition classes $V_1', \ldots, V_r'$. Since $|V_i'| \leq n - d(v_i)$, $i = 1, \ldots, r-1$, by (2.8) we have $d(v) \leq n - |V_i'|$, $\forall v \in V_i'$, $i = 1, \ldots, r-1$. From (4.2) it follows that

$$d(v_r) \leq \sum_{i=1}^{r-1}(n - d(v_i)) = \sum_{i=1}^{r-1}|V_i'| = n - |V_r'|.$$

This inequality together with (2.8) gives $d(v) \leq d(v_r)$, $\forall v \in V_r'$. Thus G is a generalized r-partite graph with partition classes $V_1', \ldots, V_r'$.

Suppose the inequality (4.1) is strict. Then $d(v) < n - |V_r'|$, $\forall v \in V_r'$. Hence, G is not a complete generalized r-partite graph with partition classes $V_1', \ldots, V_r'$. According to Theorem 4, $e(G) < e(T_r(n))$. Thus G is not a generalized r-partite Turan's graph.

Proof of Theorem 2. By Theorem 1, $d(v_1) + \cdots + d(v_{r-1}) < \frac{(r-1)^2}{r}n$. From Lemma 1 (with $k = r-1$) it follows that G is a generalized r-partite graph which is not a generalized r-partite Turan's graph.

Corollary 2. Let G be an n-vertex graph and r be a positive integer such that $2 \leq r \leq n$. If $e(G) \geq e(T_r(n))$, then

(a) $l(v) \geq r$ for any vertex v of maximal degree. If $l(v) = r$ for some vertex v of maximal degree, then $G = T_r(n)$.

(b) Any β-sequence with r members is saturated.

Proof of (a). Let v be a vertex of maximal degree and $l(v) = s$. According to Theorem 3, $e(G) \leq e(T_s(n))$. Since $e(G) \geq e(T_r(n))$, we have $e(T_s(n)) \geq e(T_r(n))$. From this inequality and $2 \leq r \leq n$, $2 \leq s \leq n$ it follows that $s \geq r$. If $r = s$, then we have $e(G) = e(T_r(n))$. By Theorem 3, $G = T_r(n)$.

Proof of (b). Assume that there exists a β-sequence $v_1, \ldots, v_r$ which is not saturated. Then from Theorem 2 and Theorem 4 it follows that $e(G) < e(T_r(n))$ which is a contradiction. This completes the proof of Corollary 2.

5. An improvement of Corollary 1. Let n and r be positive integers, $2 \leq r \leq n$. Define

$$f(n,r) = g(n,r) - \frac{n}{2(r-1)}, \text{ if } n \equiv 0 (\operatorname{mod} r);$$

$$f(n,r) = g(n,r) - \frac{\nu n}{2r(r-1)}, \text{ if } n \equiv \nu (\operatorname{mod} r), 1 \leq \nu \leq r-1,$$

where $g(n,r)$ is defined in section 3.

It is straightforward to show that

$$f(n,r) > g(n,r-1). \tag{5.1}$$

Theorem 5. Let G be an n-vertex graph and r be a positive integer, $2 \leq r \leq n$. If $e(G) > f(n,r)$, then

(a) $l(v) \geq r$ for any vertex v of maximal degree;

(b) any β-sequence with r members is saturated.

Proof of (a). From $e(G) > f(n,r)$ and (5.1) it follows that $e(G) > g(n, r-1)$. Hence, $G \neq T_{r-1}(n)$. By Corollary 1 (a), $l(v) > r - 1$ for any vertex v of maximal degree.

Proof of (b). Assume that $v_1, \dots, v_r$ is a β-sequence which is not saturated. Then by Theorem 1 we have

$$d(v_1) + \cdots + d(v_{r-1}) < \frac{(r-1)^2}{r} n. \tag{5.2}$$

According to (5.1) and Corollary 1, $v_1, \dots, v_{r-1}$ is saturated, i.e.

$$d(v_1) + \cdots + d(v_{r-1}) \geq \frac{2(r-1)e(G)}{n}. \tag{5.3}$$

Case 1. $n \equiv 0 (\operatorname{mod} r)$. Since $\dfrac{(r-1)^2}{r} n$ is a positive integer, from (5.2) it follows that

$$d(v_1) + \cdots + d(v_{r-1}) \leq \frac{(r-1)^2}{r} n - 1. \tag{5.4}$$

Inequality (5.4) together with (5.3) gives $e(G) \leq f(n,r)$ which is a contradiction.

Case 2. $n \equiv \nu (\operatorname{mod} r)$, $1 \leq \nu \leq r-1$. In this case we have

$$\left\lfloor \frac{(r-1)^2}{r} n \right\rfloor = \frac{(n-\nu)(r-1)^2}{r} + \nu(r-2).$$

Thus, from (5.2) it follows that

$$d(v_1) + \cdots + d(v_{r-1}) \leq \frac{(n-\nu)(r-1)^2}{r} + \nu(r-2).$$

This inequality, together with (5.3) implies $e(G) \leq f(n,r)$ which is a contradiction. This completes the proof of Theorem 5.

Remark 2. If $n \equiv 0 (\operatorname{mod} r)$, then $f(n,r) < e(T_r(n))$. Therefore, in this case Corollary 2 (b) follows from Theorem 5. If $n \equiv \nu (\operatorname{mod} r)$, $1 \leq \nu \leq r-1$, then the equality

$$e(T_r(n)) = g(n,r) - \frac{\nu(r-\nu)}{2r}$$

implies that if $\dfrac{\nu(r-\nu)}{2r} < \dfrac{\nu n}{2r(r-1)}$, i.e. $n > (r-\nu)(r-1)$, then we have $f(n,r) < e(T_r(n))$. Hence, if $n > (r-\nu)(r-1)$, Corollary 2 (b) follows from Theorem 5 (b).

REFERENCES

[1] Khadzhiivanov N., N. Nenov. Serdica Math. J., **30**, 2004, No 1, 95–102. [2] Id. Compt. rend. Acad. bulg. Sci., **57**, 2004, No 2, 19–24. [3] Faudree R. J. Graph Theory, **16**, 1992, 327–334. [4] Erdos P., R. Faudree, C. Rousseau. Discrete Math., **127**, 1994, 139–152.

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