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MATHEMATIQUES
Théorie des graphes

ON THE VERTEX FOLKMAN NUMBER $F(3,4)$

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We consider only finite, non-oriented graphs, without loops and multiple edges. $V(G)$ and $E(G)$ denote the set of the vertices and the set of the edges of graph G, respectively. We say that G is an n-vertex graph when $|V(G)| = n$. For $v \in V(G)$ we denote by $\mathrm{Ad}(v)$ the set of all vertices, adjacent to v. We call a p-clique of G a set of p vertices, each two of which are adjacent. The biggest natural number p, such that graph G contains a p-clique is denoted by $\mathrm{cl}(G)$. A set of vertices in a graph G is said to be independent if no two of them are adjacent. The cardinality of any largest independent set of vertices in G is written as $\alpha(G)$.

If $W \subseteq V(G)$, then $G[W]$ denotes the subgraph of graph G, induced by W and by $G - W$ denote the subgraph of G, that is obtained from G by the removal of the vertices belonging to W. The simple cycle of length n is denoted by C_n.

The Ramsey number $R(p,q)$ is the minimum of all natural numbers n, such that for arbitrary n-vertex graph G, either $\mathrm{cl}(G) \geq p$ or $\alpha(G) \geq q$. We need the identities $R(3,3) = 6$ and $R(3,4) = R(4,3) = 9$, [1].

By $\overline{G}$ we denote the complementary graph of G. The complementary graph $\overline{P}$ of P is given in Fig. 2 and the complementary graph $\overline{Q}$ of Q is given in Fig. 3.

Proposition 1 ([7]). There hold $\mathrm{cl}(P) = 4$, $\alpha(P) = 2$ and any 12-vertex graph G with $\mathrm{cl}(G) = 4$, $\alpha(G) = 2$ is a subgraph of graph P.

Proposition 2 ([1]). There hold $\mathrm{cl}(Q) = 4$ and $\alpha(Q) = 2$.

Definition. Let G be a graph and p, q be integers. By $G \to (p,q)$ we denote that in any 2-colouring $V_1 \cup V_2$ of set $V(G)$, either V_1 contains a p-clique or V_2 contains q-clique of graph G.

We put

$$H(p,q) = \{G : G \to (p,q) \text{ and } \mathrm{cl}(G) = \max(p,q)\}$$

$$F(p,q) = \min\{|V(G)| : G \in H(p,q)\}.$$

In [2] Folkman proved that $H(p,q) \neq \emptyset$. Since $F(p,q) = F(q,p)$ we may assume that $p \leq q$. The above $F(p,q)$ are called Folkman numbers. For these numbers the following facts are known:

Theorem A ([3]). For any $p \geq 2$ one has $F(2,p) = 2p + 1$.

Theorem B ([4]). Let $G \in H(2,p)$ and $|V(G)| = 2p + 1$. Then $G = \overline{C}_{2p+1}$.

Theorem C ([8]). Whenever $p \geq 3$ the Folkman numbers $F(p,p) < \lfloor p!e \rfloor - 1$.

N. NENOV constructs in [6] a 14-vertex graph $G \in H(3,3)$, showing that $F(3,3) \leq 14$. In a joint paper with E. NEDIALKOV [9], we proved that $F(3,3) \geq 12$. The work [5] provides a computer proof of the inequality $F(3,3) \geq 14$ and thus $F(3,3) = 14$. $F(3,4)$ is the smallest unknown Folkman number. T. LUCZAK, A. RUCINSKI and S. URBANSKI ([4], Corollary 6) proved that $F(3,4) \leq 33$.

In the present article, we consider the family $H(3,4)$. As a consequence from the proved results, it follows that $F(3,4) = 13$.

Theorem 1. Let G be an n-vertex graph and $G \in H(3,4)$. Then
(a) $\alpha(G) \leq n-9$,
(b) if $\alpha(G) = n-9$, then $n \geq 18$.

Theorem 2. There holds $F(3,4) = 13$.

For the proof of Theorem 2 we will need the following

Lemma. Graph $P \notin H(3,4)$.

Proof. Let $V(P) = V_1 \cup V_2$, where $V_1 = \{v_5, v_6, v_9, v_{10}, v_{12}\}$, $V_2 = \{v_1, v_2, v_3, v_4, v_7, v_8, v_{11}\}$. Then $P[V_1] = C_5$ and $P[V_2] = \overline{C}_7$. Since $\mathrm{cl}(C_5) = 2$ and $\mathrm{cl}(\overline{C}_7) = 3$, we have $P \notin H(3,4)$.

Proof of Theorem 1. Let A be an independent set of graph G, $|A| = \alpha(G)$ and $G_1 = G - A$. From $G \in H(3,4)$ it follows that $G_1 \in H(2,4)$. According to Theorem A, we have $|V(G_1)| \geq 9$. Since $|V(G_1)| = n - \alpha(G)$, this inequality implies $\alpha(G) \leq n-9$.

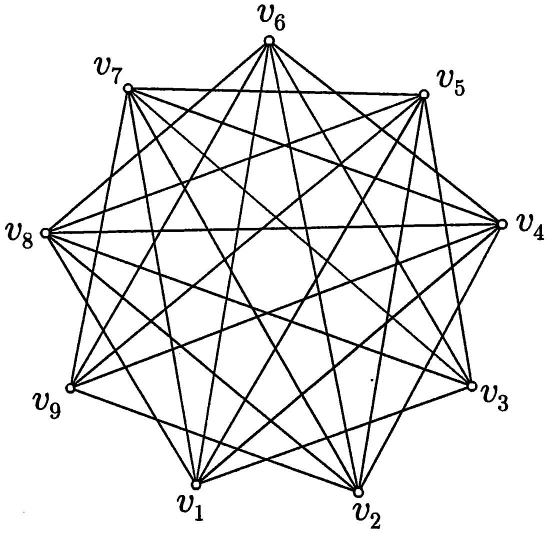

Fig. 1. Graph $\overline{C_9}$

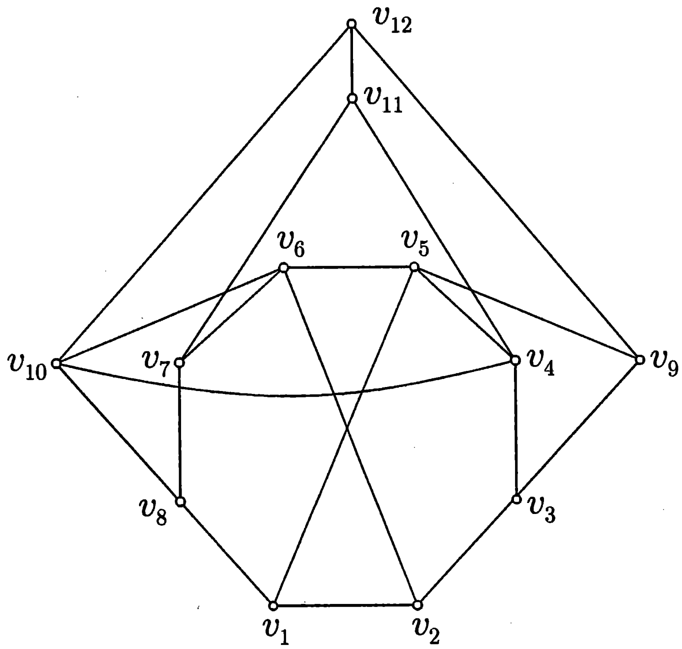

Fig. 2. Graph $\overline{P}$

Suppose that $\alpha(G) = n-9$. Than $|V(G_1)| = 9$. Since $G_1 \in H(2,4)$, according to Theorem B, we have $G_1 = \overline{C}_9$. We consider the set $M_1 = \{v_1, v_3, v_4, v_7, v_8\}$ of vertices of the graph $G_1 = \overline{C}_9$ (Fig. 1). The map σ defined by $\sigma(v_i) = v_{i+1}$, $i = 1, \ldots, 8$ and $\sigma(v_9) = v_1$ is obviously an automorphism of the graph $G_1 = \overline{C}_9$. Set $M_i = \sigma^{i-1}(M_1)$, $i = 1, \ldots, 9$. We will prove that for each M_i, $i = 1, \ldots, 9$ there is a vertex $u_i \in A$ such that $\mathrm{Ad}(u_i) \supseteq M_i$. Since $\sigma^{i-1}(M_1) = M_i$, $i = 1, \ldots, 9$, it suffices to show that for some of the vertices $u \in A$ we have $\mathrm{Ad}(u) \supseteq M_1$. Assume the opposite. Set $V_1' = \{v_4, v_5, v_6, v_7\}$ and $V_2' = \{v_1, v_2, v_3, v_8, v_9\}$. We denote by W_1 the set of those of vertices $u \in A$ for which $v_1, v_3, v_8 \in \mathrm{Ad}(u)$ and $W_2 = A \setminus W_1$. We put $V_1 = V_1' \cup W_1$, $V_2 = V_2' \cup W_2$. It is clear that $V_1 \cup V_2$ is a 2-colouring of $V(G)$. It is clear also that V_2 contains no 4-cliques. Let $u \in W_1$. Then $v_1, v_3, v_8 \in \mathrm{Ad}(u)$. From $\mathrm{cl}(G) = 4$ it follows that u is not adjacent to the vertices v_5 and v_6. Since we assume that u is not adjacent to some of the vertices of M_1, it follows that u is not adjacent to one of the vertices v_4, v_7. Consequently V_1 contains no 3-cliques, which is a contradiction.

So, there are $u_1, \ldots, u_9 \in A$ such that $\mathrm{Ad}(u_i) \supseteq M_i$, $i = 1, \ldots, 9$. We will prove that $u_i \neq u_j$, $i \neq j$ and, hence, $n \geq 18$. Assume the opposite and let, for example $u_1 = u_i$, $i \neq 1$. Then $\mathrm{Ad}(u_1) \supseteq M_1 \cup M_i$, $i \neq 1$. The set M_1 contains the 3-clique $\{v_1, v_3, v_8\}$. Since for $i \neq 1$ either $v_5 \in M_i$ or $v_6 \in M_i$, it follows that either $\mathrm{Ad}(u_1)$ contains the 4-clique $\{v_1, v_3, v_5, v_8\}$ or $\mathrm{Ad}(u_1)$ contains the 4-clique $\{v_1, v_3, v_6, v_8\}$. Hence, $\mathrm{cl}(G) \geq 5$, which is a contradiction.

Remark. Let $N_1 = \{v_2, v_3, v_4, v_5, v_8, v_9\} \subseteq V(\overline{C}_9)$ and $N_i = \sigma^{i-1}(N_1)$, $i = 1, \ldots, 9$. Denote by Γ the extension of graph $\overline{C}_9$ (Fig. 1), obtained by adding nine new vertices to $V(\overline{C}_9)$, namely $u_1, \ldots, u_9$, each pair of which is not adjacent and such that $\mathrm{Ad}(u_i) = N_i$, $i = 1, \ldots, 9$. It is true that 18-vertex graph $\Gamma \in H(3,4)$ and obviously $\alpha(\Gamma) = 9$.

The proof of Theorem 2. Let $G \in H(3,4)$. From $\mathrm{cl}(G) = 4$ it follows that $\alpha(G) \geq 2$. According to Theorem 1, $|V(G)| \geq 12$. We will prove that $|V(G)| \neq 12$. Assume the opposite, i.e., $|V(G)| = 12$. According to Theorem 1, $\alpha(G) < n - 9$ and, hence, $\alpha(G) = 2$. It follows from Proposition 1 that G is a subgraph of graph P (Fig. 2). This contradicts the Lemma and proves the inequality $F(3,4) \geq 13$.

We prove the inequality $F(3,4) \leq 13$ by showing that $Q \in H(3,4)$ (Fig. 3). Let $V_1 \cup V_2$ be the 2-colouring of the vertices of graph Q. Put $G_1 = Q[V_1]$, $G_2 = Q[V_2]$. From $\alpha(Q) = 2$ it follows that $\alpha(G_1) \leq 2$ and $\alpha(G_2) \leq 2$.

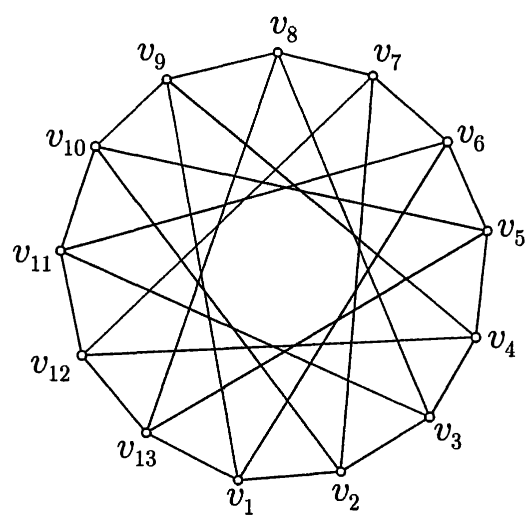

Fig. 3. Graph $\overline{Q}$

Case 1. $|V_1| \leq 4$, i.e., $|V_2| \geq 9$. From $R(4,3) = 9$ and $\alpha(G_2) \leq 2$, it follows that $\mathrm{cl}(G_2) \geq 4$, i.e., V_2 contains a 4-clique.

Case 2. $|V_1| \geq 6$. From $R(3,3) = 6$ and $\alpha(G_1) \leq 2$, it follows that $\mathrm{cl}(G_1) \geq 3$, i.e., V_1 contains a 3-clique of Q.

Case 3. $|V_1| = 5$. Assume that V_1 contains no 3-clique, i.e., $\mathrm{cl}(G_1) = 2$. From $\mathrm{cl}(G_1) = 2$ and $\alpha(G_1) \leq 2$ it follows that $G_1 = \overline{G}_1 = C_5$. Let E_1 denote the set of edges of the 13-cycle $v_1, v_2, \ldots, v_{13}$ of graph $\overline{Q}$ (Fig. 3) and $E_2 = E(\overline{Q}) \setminus E_1$. From $|E(\overline{G}_1)| = 5$, there follows either $|E(\overline{G}_1) \cap E_1| \geq 3$ or $|E(\overline{G}_1) \cap E_2| \geq 3$. The map

$$\varphi = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} \\ v_1 & v_6 & v_{11} & v_3 & v_8 & v_{13} & v_5 & v_{10} & v_2 & v_7 & v_{12} & v_4 & v_9 \end{pmatrix}$$

is obviously an automorphism of graph $\overline{Q}$. Since $\varphi(E_1) = E_2$ it is enough to consider the case $|E(\overline{G}_1) \cap E_1| \geq 3$. Suppose that $e_1, e_2, e_3 \in E(\overline{G}_1) \cap E_1$. From $|V_1| = 5$ it follows that two of the edges e_1, e_2, e_3 are incident. Let e_1 and e_2 be incident edges of the 13-cycle $v_1, v_2, \ldots, v_{13}, v_1$ of the graph $\overline{Q}$. We may assume without a loss of generality that $e_1 = [v_1, v_2]$, $e_2 = [v_2, v_3]$. From $\overline{G}_1 = C_5$ it follows that there are the following possibilities:

Subcase 3.a. $e_3 = [v_3, v_4]$. From $\overline{G}_1 = C_5$ it follows that $V_1 = \{v_1, v_2, v_3, v_4, v_9\}$. Hence, the set V_2 contains the 4-clique $\{v_6, v_8, v_{10}, v_{12}\}$.

Subcase 3.b. $e_3 = [v_8, v_9]$. In this subcase the set V_2 contains the 4-clique $\{v_4, v_7, v_{10}, v_{13}\}$.

So, graph $Q \in H(3,4)$. From $|V(Q)| = 13$ it follows that $F(3,4) \leq 13$ and hence, $F(3,4) = 13$.

Set $F(4,4;6) = \min\{|V(G)| : G \to (4,4) \text{ and } \mathrm{cl}(6) < 6\}$. In [4], it is proved that $F(4,4;6) \leq 35$. It is clear that $Q \to (3,4)$ implies $K_1 + Q \to (4,4)$. Hence, $F(4,4;6) \leq 14$.

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