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NEW RECURRENT INEQUALITY ON A CLASS OF VERTEX FOLKMAN NUMBERS*

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Let G be a graph and $V(G)$ be the vertex set of G. Let $a_1, \dots, a_r$ be positive integers, $m = \sum_{i=1}^{r}(a_i - 1) + 1$ and $p = \max\{a_1, \dots, a_r\}$. The symbol $G \to \{a_1, \dots, a_r\}$ denotes that in every r-coloring of $V(G)$ there exists a monochromatic a_i-clique of color i for some $i = 1, \dots, r$. The vertex Folkman numbers $F(a_1, \dots, a_r; m-1) = \min\{|V(G)| : G \to (a_1 \dots a_r)$ and $K_{m-1} \not\subseteq G\}$ are considered. In this paper we improve the known upper bounds on the numbers $F(2, 2, p; p+1)$ and $F(3, p; p+1)$.

Introduction. We consider only finite, non-oriented graphs without loops and multiple edges. We call a p-clique of the graph G a set of p vertices, each two of which are adjacent. The largest positive integer p, such that the graph G contains a p-clique is denoted by $cl(G)$. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of the graph G respectively. The symbol K_n denotes the complete graph on n vertices.

Let G_1 and G_2 be two graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x, y] \mid x \in V(G_1), y \in V(G_2)\}$.

Definition. *Let $a_1, \dots, a_r$ be positive integers. We say that the r-coloring*

$$V(G) = V_1 \cup \ldots \cup V_r,\ V_i \cap V_j = \emptyset,\ i \neq j,$$

of the vertices of the graph G is $(a_1, \dots, a_r)$ – free, if V_i does not contain an a_i – clique for each $i \in \{1, \dots, r\}$. The symbol $G \to (a_1, \dots, a_r)$ means that there is not an $(a_1, \dots, a_r)$-free coloring of the vertices of G.

We consider for arbitrary natural numbers $a_1, \dots, a_r$ and q

$$H(a_1, \dots, a_r; q) = \{G : G \to (a_1, \dots, a_r) \text{ and } cl(G) < q\}.$$

The vertex Folkman numbers are defined by the equality

$$F(a_1, \dots, a_r; q) = \min\{|V(G)| : G \in H(a_1, \dots, a_r; q)\}.$$

It is clear that $G \to (a_1, \dots, a_r)$ implies $cl(G) \geq \max\{a_1, \dots, a_r\}$. Folkman [1] proved that there exists a graph G such that $G \to (a_1, \dots, a_r)$ and $cl(G) = \max\{a_1, \dots, a_r\}$. Therefore

$$(1) \qquad F(a_1, \dots, a_r; q) \text{ exists if and only if } q > \max\{a_1, \dots, a_r\}.$$

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If $a_1, \dots, a_r$ are positive integers, $r \geq 2$ and $a_i = 1$ then it is easy to see that

(2) $$G \to (a_1, \dots, a_r) \Leftrightarrow G \to (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r).$$

It is also easy to see that for an arbitrary permutation $\varphi \in S_r$ we have

$$G \to (a_1, \dots, a_r) \Leftrightarrow G \to (a_{\varphi(1)}, \dots, a_{\varphi(r)}).$$

That is why

(3) $$F(a_1, \dots, a_r; q) = F(a_{\varphi(1)}, \dots, a_{\varphi(r)}), \quad \text{for each} \quad \varphi \in S_r$$

According to (2) and (3) it is enough to consider just such numbers $F(a_1, \dots, a_r; q)$ for which

(4) $$2 \leq a_1 \leq \dots \leq a_r.$$

For arbitrary positive integers $a_1, \dots, a_r$ define:

(5) $$p = p(a_1, \dots, a_r) = \max\{a_1, \dots, a_r\};$$

(6) $$m = 1 + \sum_{i=1}^{r} (a_i - 1)$$

It is easy to see that $K_m \to (a_1, \dots, a_r)$ and $K_{m-1} \not\to (a_1, \dots, a_r)$. Therefore

$$F(a_1, \dots, a_r; q) = m, \text{ if } q > m.$$

In [4] it was proved that $F(a_1, \dots, a_r; m) = m + p$, where m and p are defined by the equalities (5) and (6). About the numbers $F(a_1, \dots, a_r; m-1)$ we know that $F(a_1, \dots, a_r; m-1) \geq m + p + 2$, $p \geq 2$ and according to [3]

(7) $$F(a_1, \dots, a_r; m-1) \leq m + 3p$$

The exact values of all numbers $F(a_1, \dots, a_r; m-1)$ for which $\max\{a_1, \dots, a_r\} \leq 4$ are known. A detailed exposition of these results was given in [8]. We must add the equality $F(2, 2, 3; 4) = 14$ obtained in [2] to this exposition. We do not know any exact values of $F(a_1, \dots, a_r; m-1)$ in the case when $\max\{a_1, \dots, a_r) \geq 5$.

According to (1), $F(a_1, \dots, a_r; m-1)$ exists exactly when $m \geq p + 2$. In this paper we shall improve inequality (7) in the boundary case when $m = p + 2$, $p \geq 5$. From the equality $m = p + 2$ and (4) it easily follows that there are two such numbers only: $F(2, 2, p; p+1)$ and $F(3, p; p+1)$. It is clear that from $G \to (3, p)$ it follows $G \to (2, 2, p)$. Therefore

(8) $$F(2, 2, p; p+1) \leq F(3, p; p+1).$$

The inequality (7) gives us that:

(9) $$F(3, p; p+1) \leq 4p + 2;$$

(10) $$F(2, 2, p : p+1) \leq 4p + 2.$$

Our goal is to improve the inequalities (9) and (10). We shall need the following

Lemma. *Let G_1 and G_2 be two graphs such that*

(11) $$G_1 \to (a_1, \dots, a_{i-1}, a_i', a_{i+1}, \dots, a_r)$$

and

(12) $$G_2 \to (a_1, \dots, a_{i-1}, a_i'', a_{i+1}, \dots, a_r).$$

Then

(13) $$G_1 + G_2 \to (a_1, \dots, a_{i-1}, a_i' + a_i'', a_{i+1}, \dots, a_r).$$

Proof. Assume that (13) is wrong and let

$$V_1 \cup \ldots \cup V_r,\ V_i \cap V_j = \emptyset,\ i \neq j,$$

be a $(a_1, \ldots, a_{i-1}, a'_i + a''_i, a_{i+1}, \ldots, a_r)$-free r-coloring of $V(G_1 + G_2)$. Let $V'_i = V_i \cap V(G_1)$ and $V''_i = V_i \cap V(G_2)$, for $i = 1, \ldots, r$. Then $V'_1 \cup \ldots \cup V'_r$ is an r-coloring of $V(G_1)$, such that V_j does not contain an a_j-clique, $j \neq i$. Thus from (11) it follows that V'_i contains an a'_i-clique. Analogously from the r-colouring $V''_1 \cup \ldots \cup V''_r$ of $V(G_2)$ it follows that V''_i contains an a''_i-clique. Therefore $V_i = V'_i \cup V''_i$ contains a $(a'_i + a''_i)$-clique, which contradicts the assumption that $V_1 \cup \ldots \cup V_r$ is a $(a_1, \ldots, a_{i-1}, a'_i + a''_i, a_{i+1}, \ldots, a_r)$-free r-coloring of $V(G_1 + G_2)$. This contradiction proves the Lemma.

Results. The main result in this paper is the following

Theorem. *Let $a_1 \leq \ldots \leq a_r$, $r \geq 2$ be positive integers and $a_r = b_1 + \ldots + b_s$, where b_i are positive integers, such that $b_i \geq a_{r-1}$, $i = 1, \ldots, s$. Then*

$$F(a_1, \ldots, a_r; a_r + 1) \leq \sum_{i=1}^{s} F(a_1, \ldots, a_{r-1}, b_i; b_i + 1). \tag{14}$$

Proof. We shall prove the Theorem by induction on s. As the inductive step is trivial we shall just prove the inductive base $s = 2$. Let G_1 and G_2 be two graphs such that $cl(G_1) = b_1$ and $cl(G_2) = b_2$, $a_r = b_1 + b_2$, $b_1 \geq a_{r-1}$, $b_2 \geq a_{r-1}$ and

$$G_1 \to (a_1, \ldots, a_{r-1}, b_1),\ |V(G_1)| = F(a_1, \ldots, a_{r-1}, b_1; b_1 + 1),$$
$$G_2 \to (a_1, \ldots, a_{r-1}, b_2),\ |V(G_2)| = F(a_1, \ldots, a_{r-1}, b_2; b_2 + 1).$$

According to the Lemma, $G_1 + G_2 \to (a_1, \ldots, a_{r-1}, a_r)$. As $cl(G_1 + G_2) = cl(G_1) + cl(G_2) = b_1 + b_2 = a_r$, we have

$$F(a_1, \ldots, a_r; a_r + 1) \leq |V(G_1 + G_2)| = |V(G_1)| + |V(G_2)|.$$

From here the inequality (14) trivially follows when $s = 2$ and hence, for arbitrary s, as explained above. The Proof is completed.

We shall derive some corollaries from the Theorem. Let $p \geq 4$ and $p = 4k + l$, $0 \leq l \leq 3$. Then from (14) it easily follows that

$$F(3, p; p + 1) \leq (k - 1)F(3, 4, ; 5) + F(3, 4 + l; 5 + l) \tag{15}$$

$$F(2, 2, p; p + 1) \leq (k - 1)F(2, 2, 4; 5) + F(2, 2, 4 + l; 5 + l). \tag{16}$$

From (15), (9) ($p = 5, 6, 7$) and the equality $F(3, 4; 5) = 13$ (see [6]), we obtain

Corollary 1. *Let $p \geq 4$. Then:*

$$F(3, p; p + 1) \leq \frac{13p}{4} \quad \textit{for} \quad p \equiv 0 \mod 4;$$
$$F(3, p; p + 1) \leq \frac{13p + 23}{4} \quad \textit{for} \quad p \equiv 1 \mod 4;$$
$$F(3, p; p + 1) \leq \frac{13p + 26}{4} \quad \textit{for} \quad p \equiv 2 \mod 4;$$
$$F(3, p; p + 1) \leq \frac{13p + 29}{4} \quad \textit{for} \quad p \equiv 3 \mod 4.$$

From (16), the equality $F(2,2,4;5) = 13$ (see [7]), the inequality (10) ($p = 5$) and both inequalities $F(2,2,6;7) \leq 22$ and $F(2,2,7;8) \leq 27$ (see [9]) we obtain

Corollary 2. *Let $p \geq 4$. Then*

$$F(2,2,p;p+1) \leq \frac{13p}{4} \quad \textit{for} \quad p \equiv 0 \mod 4;$$
$$F(2,2,p;p+1) \leq \frac{13p+23}{4} \quad \textit{for} \quad p \equiv 1 \mod 4;$$
$$F(2,2,p;p+1) \leq \frac{13p+10}{4} \quad \textit{for} \quad p \equiv 2 \mod 4;$$
$$F(2,2,p;p+1) \leq \frac{13p+17}{4} \quad \textit{for} \quad p \equiv 3 \mod 4.$$

We conjecture that the following inequalities hold:

(17) $$F(3,p;p+1) \leq \frac{13p}{4} \quad \text{for } p \geq 4;$$

(18) $$F(2,2,p;p+1) \leq \frac{13p}{4} \quad \text{for } p \geq 4.$$

From the Theorem it follows that

(19) $$F(3,p;p+1) \leq F(3,p-4;p-3) + F(3,4;5), p \geq 8;$$

(20) $$F(2,2,p;p+1) \leq F(2,2,p-4;p-3) + F(2,2,4;5), p \geq 8.$$

From $F(3,4;5) = 13$ (see [6]) and (19) we obtain

Corollary 3. *If the inequality (17) holds for $p = 5$, 6 and 7, then (17) is true for every $p \geq 4$.*

From $F(2,2,4;5) = 13$ (see [7]) and from (20) it follows

Corollary 4. *If the inequality (18) holds for $p = 5$, 6 and 7 then (18) is true for every $p \geq 4$.*

At the end in regard with (8) we shall pose the following

Problem. *Is there a positive integer p, for which $F(2,2,p;p+1) \neq F(3,p;p+1)$?*

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НОВА РЕКУРЕНТНА ВРЪЗКА ЗА КЛАС ОТ ВЪРХОВИ ФОЛКМАНОВИ ЧИСЛА

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Нека G е граф и $V(G)$ е множеството от върховете на G. Нека $a_1, \dots, a_r$ са естествени числа и $m = \sum_{i=1}^{r}(a_i - 1) + 1$ и $p = \max\{a_1, \dots, a_r\}$. Символът $G \to \{a_1, \dots, a_r\}$ означава, че във всяко r-оцветяване на $V(G)$ има едноцветна a_i-клика от цвят i за някое $i = 1, \dots, r$. Разглеждат се върховите Фолкманови числа $F(a_1, \dots, a_r; m-1) = \min\{|V(G)| : G \to (a_1 \dots a_r)$ и $K_{m-1} \not\subseteq G\}$. В тази работа подобряваме известните оценки от горе за числата $F(2,2,p;p+1)$ и $F(3,p;p+1)$.