MODIFIED VERTEX FOLKMAN NUMBERS

ALEKSANDAR BIKOV AND NEDYALKO NENOV

ABSTRACT. Let $a_1, ..., a_s$ be positive integers. For a graph G the expression

$$G \overset{v}{\rightarrow} (a_1, ..., a_s)$$

means that for every coloring of the vertices of G in s colors (s-coloring) there exists $i \in \{1, ..., s\}$, such that there is a monochromatic a_i-clique of color i. If m and p are positive integers, then

$$G \overset{v}{\rightarrow} m\big|_p$$

means that for arbitrary positive integers $a_1, ..., a_s$ (s is not fixed), such that $\sum\limits_{i=1}^{s}(a_i - 1) + 1 = m$ an $\max\{a_1, ..., a_s\} \leq p$ we have $G \overset{v}{\rightarrow} (a_1, ..., a_s)$. Let

$$\widetilde{\mathcal{H}}(m\big|_p; q) = \{G : G \overset{v}{\rightarrow} m\big|_p \text{ and } \omega(G) < q\}.$$

The modified vertex Folkman numbers are defined by the equality

$$\widetilde{F}(m\big|_p; q) = \min\{|V(G)| : G \in \widetilde{\mathcal{H}}(m\big|_p; q)\}.$$

If $q \geq m$ these numbers are known and they are easy to compute. In the case $q = m - 1$ we know all of the numbers when $p \leq 5$. In this work we consider the next unknown case $p = 6$ and we prove with the help of a computer that

$$\widetilde{F}(m\big|_6; m - 1) = m + 10.$$

1. Introduction

In this paper only finite, non-oriented graphs without loops and multiple edges are considered. The following notations are used:

$\mathrm{V}(G)$ - the vertex set of G;

$\mathrm{E}(G)$ - the edge set of G;

$\overline{G}$ - the complement of G;

$\omega(G)$ - the clique number of G;

$\alpha(G)$ - the independence number of G;

$\chi(G)$ - the chromatic number of G;

$\mathrm{N}(v), \mathrm{N}_G(v), v \in \mathrm{V}(G)$ - the set of all vertices of G adjacent to v;

$d(v), v \in \mathrm{V}(G)$ - the degree of the vertex v, i.e. $d(v) = |\,\mathrm{N}(v)|$;

$G - v, v \in \mathrm{V}(G)$ - subgraph of G obtained from G by deleting the vertex v and all edges incident to v;

$G - e, e \in \mathrm{E}(G)$ - subgraph of G obtained from G by deleting the edge e;

$G + e, e \in \mathrm{E}(\overline{G})$ - supergraph of G obtained by adding the edge e to $\mathrm{E}(G)$.

K_n - complete graph on n vertices;

C_n - simple cycle on n vertices;

$m_0 = m_0(p)$ - see Theorem 2.1;

$G_1 + G_2$ - a graph G for which: $\mathrm{V}(G) = \mathrm{V}(G_1) \cup \mathrm{V}(G_2)$ and $\mathrm{E}(G) = \mathrm{E}(G_1) \cup \mathrm{E}(G_2) \cup E'$, where $E' = \{[x, y] : x \in \mathrm{V}(G_1), y \in \mathrm{V}(G_2)\}$, i.e. G is obtained by connecting every vertex of G_1 to every vertex of G_2.

2000 *Mathematics Subject Classification.* Primary 05C35.

Key words and phrases. Folkman number, clique number, independence number, chromaic number.

All undefined terms can be found in [18].

Let $a_1, ..., a_s$ be positive integers. The expression $G \overset{v}{\rightarrow} (a_1, ..., a_s)$ means that for any coloring of $\mathrm{V}(G)$ in s colors (s-coloring) there exists $i \in \{1, ..., s\}$ such that there is a monochromatic a_i-clique of color i. In particular, $G \overset{v}{\rightarrow} (a_1)$ means that $\omega(G) \geq a_1$.

Define:
$\mathcal{H}(a_1, ..., a_s; q) = \left\{G : G \overset{v}{\rightarrow} (a_1, ..., a_s) \text{ and } \omega(G) < q\right\}$.
$\mathcal{H}(a_1, ..., a_s; q; n) = \{G : G \in \mathcal{H}(a_1, ..., a_s; q) \text{ and } |\mathrm{V}(G)| = n\}$.

The vertex Folkman number $F_v(a_1, ..., a_s; q)$ is defined by the equality:

$$F_v(a_1, ..., a_s; q) = \min\{|\mathrm{V}(G)| : G \in \mathcal{H}(a_1, ..., a_s; q)\}.$$

Folkman proves in [5] that:

$$F_v(a_1, ..., a_s; q) \text{ exists } \Leftrightarrow q > \max\{a_1, ..., a_s\}. \tag{1.1}$$

Other proofs of (1.1) are given in [4] and [9].
In [10] for arbitrary positive integers $a_1, ..., a_s$ the following are defined

$$m(a_1, ..., a_s) = m = \sum_{i=1}^{s}(a_i - 1) + 1 \quad \text{and} \quad p = \max\{a_1, ..., a_s\}. \tag{1.2}$$

Obviously, $K_m \overset{v}{\rightarrow} (a_1, ..., a_s)$ and $K_{m-1} \overset{v}{\nrightarrow} (a_1, ..., a_s)$. Therefore,

$$F_v(a_1, ..., a_s; q) = m, \quad q \geq m + 1.$$

The following theorem for the numbers $F_v(a_1, ..., a_s; m)$ is true:

Theorem 1.1. *Let $a_1, ..., a_s$ be positive integers and m and p are defined by (1.2). If $m \geq p + 1$, then:*

(a) $F_v(a_1, ..., a_s; m) = m + p$, [10],[9].

(b) $K_{m+p} - C_{2p+1} = K_{m-p-1} + \overline{C}_{2p+1}$
is the only extremal graph in $\mathcal{H}(a_1, ..., a_s; m)$, [9].

The condition $m \geq p+1$ is necessary according to (1.1). Other proofs of Theorem 1.1 are given in [12] and [13].

Very little is known about the numbers $F_v(a_1, ..., a_s; q)$, $q \leq m - 1$. In this work we suggest a method to bound these numbers with the help of the modified vertex Folkman numbers $\widetilde{F}_v(m\big|_p; q)$, which are defined below.

Definition 1.2. *Let G be a graph and m and p be positive integers. The expression*

$$G \overset{v}{\rightarrow} m\big|_p$$

means that for any choice of positive integers $a_1, ..., a_s$ (s is not fixed), such that $m = \sum_{i=1}^{s}(a_i - 1) + 1$ and $\max\{a_1, ..., a_s\} \leq p$, we have

$$G \overset{v}{\rightarrow} (a_1, ..., a_s).$$

Define:
$\widetilde{\mathcal{H}}(m\big|_p; q) = \left\{G : G \overset{v}{\rightarrow} m\big|_p \text{ and } \omega(G) < q\right\}$.
$\widetilde{\mathcal{H}}(m\big|_p; q; n) = \left\{G : G \in \widetilde{\mathcal{H}}(m\big|_p; q) \text{ and } |\mathrm{V}(G)| = n\right\}$.

The modified vertex Folkman numbers are defined by the equality:

$$\widetilde{F}_v(m\big|_p;q) = \min\left\{|\,\mathrm{V}(G)| : G \in \widetilde{\mathcal{H}}(m\big|_p;q)\right\}.$$

The graph G is called a maximal graph in $\widetilde{\mathcal{H}}(m\big|_p;q)$ if $G \in \widetilde{\mathcal{H}}(m\big|_p;q)$, but $G+e \notin \widetilde{\mathcal{H}}(m\big|_p;q), \forall e \in \mathrm{E}(\overline{G})$, i.e. $\omega(G+e) \geq q, \forall e \in \mathrm{E}(\overline{G})$.

For convenience we will also define the following term:

Definition 1.3. *The graph G is called a $(+K_t)$-graph if $G+e$ contains a new t-clique for all $e \in \mathrm{E}(\overline{G})$.*

Obviously, $G \in \widetilde{\mathcal{H}}(m\big|_p;q)$ is a maximal graph in $\widetilde{\mathcal{H}}(m\big|_p;q)$ if and only if G is a $(+K_q)$-graph.

From the definition of the modified Folkman numbers it becomes clear that if $a_1,...,a_s$ are positive integers and m and p are defined by (1.2), then

$$F_v(a_1,...,a_s;q) \leq \widetilde{F}_v(m\big|_p;q). \tag{1.3}$$

Defining and computing the modified Folkman numbers is appropriate because of the following reasons:

1) On the left side of (1.3) there is actually a whole class of numbers, which are bound by only one number $\widetilde{F}_v(m\big|_p;q)$.

2) The upper bound for $\widetilde{F}_v(m\big|_p;q)$ is easier to compute than the numbers $F_v(a_1,...,a_s)$ because of the following

Theorem 1.4. *([1], Theorem 7.2) Let m, m_0, p and q be positive integers, $m \geq m_0$ and $q > \min\{m_0, p\}$. Then*

$$\widetilde{F}_v(m\big|_p;m-m_0+q) \leq \widetilde{F}_v(m_0\big|_p;q)+m-m_0.$$

Therefore, if we know the value of one number $\widetilde{F}_v(m'\big|_p;q)$ we can obtain an upper bound for $\widetilde{F}_v(m\big|_p;q)$ where $m \geq m'$.

3) As we will see below (Theorem 2.1), the computation of the numbers $\widetilde{F}_v(m\big|_p;m-1)$ is reduced to finding the exact values of the first several of these numbers (bounds for the number of exact values needed are given in 2.1 (c)).

Let A be an independent set of vertices in G. If $V_1 \cup ... \cup V_s$ is $(a_1,...,a_s)$-free s-coloring of $\mathrm{V}(G-A)$ (i.e. V_i does not contain an a_i-clique, $i=1,...,s$), then $A \cup V_1 \cup ... \cup V_s$ is $(2,a_1,...,a_s)$-free $(s+1)$-coloring of $\mathrm{V}(G)$. Therefore

$$G \overset{v}{\rightarrow} (2,a_1,...,a_s) \Rightarrow G-A \overset{v}{\rightarrow} (a_1,...,a_s). \tag{1.4}$$

Further we will need the following

Proposition 1.5. *Let $G \overset{v}{\rightarrow} m\big|_p$ and A is an independent set of vertices in G. Then $G-A \overset{v}{\rightarrow} (m-1)\big|_p$.*

Proof. Let $a_1,..,a_s$ be positive integers, such that

$$m-1 = \sum_{i=1}^{s}(a_i-1)+1 \quad \text{and } 2 \leq a_i \leq p.$$

Then

$$m = (2-1)+\sum_{i=1}^{s}(a_i-1)+1.$$

It follows that $G \overset{v}{\rightarrow} (2,a_1,...,a_s)$ and from (1.4) we obtain $G-A \overset{v}{\rightarrow} (a_1,...,a_s)$. $\square$

It is easy to see that if $q > m$, then $F_v(a_1, ..., a_s; q) = \widetilde{F}_v(m\big|_p; q) = m$. From Theorem 1.1 it follows that $F_v(a_1, ..., a_s; m) = \widetilde{F}_v(m\big|_p; m) = m + p$. In the case $q = m - 1$ the following general bounds are known:

$$m + p + 2 \leq \widetilde{F}_v(m\big|_p; m - 1) \leq m + 3p, \ m \geq p + 2. \tag{1.5}$$

The upper bound follows from the proof of the Main Theorem from [7] and the lower bound follows from (1.3) and $F_v(a_1, ..., a_s; q) \geq m + p + 2$, [12].

We know all the numbers $\widetilde{F}_v(m\big|_p; m - 1)$ where $p \leq 5$ (in the cases $p \leq 4$ see the Remark after Theorem 4.5 and (1.5) from [1], and in the case $p = 5$ see Theorem 7.4 also from [1]). It is also known that

$$m + 9 \leq \widetilde{F}_v(m\big|_6; m - 1) \leq m + 10, \ [1]$$

In this work we complete the computation of the numbers $\widetilde{F}_v(m\big|_6; m - 1)$ by proving

Main Theorem 1. $\widetilde{F}_v(m\big|_6; m - 1) = m + 10, \ m \geq 8.$

2. A theorem for the numbers $\widetilde{F}_v(m\big|_p; m - 1)$

We will need the following fact:

$$G \overset{v}{\rightarrow} (a_1, ..., a_s) \Rightarrow \chi(G) \geq m, \ [13] \text{ (see also [1])}. \tag{2.1}$$

It is easy to prove (see Proposition 4.4 from [1]) that

$$\widetilde{F}_v(m\big|_p; m - 1) \text{ exists } \Leftrightarrow m \geq p + 2. \tag{2.2}$$

In [1](version 1) we formulate without proof the following

Theorem 2.1. *Let $m_0(p) = m_0$ be the smallest positive integer for which*

$$\min_{m \geq p+2} \left\{ \widetilde{F}_v(m\big|_p; m - 1) - m \right\} = \widetilde{F}_v(m_0\big|_p; m_0 - 1) - m_0.$$

Then:

(a) $\widetilde{F}_v(m\big|_p; m - 1) = \widetilde{F}_v(m_0\big|_p; m_0 - 1) + m - m_0, \quad m \geq m_0.$

(b) *if $m_0 > p + 2$ and G is an extremal graph in $\widetilde{\mathcal{H}}(m_0\big|_p; m_0 - 1)$, then*

$G \overset{v}{\rightarrow} (2, m_0 - 2).$

(c) $m_0 < \widetilde{F}_v((p + 2)\big|_p; p + 1) - p.$

In this section we present a proof of Theorem 2.1.

The condition $m \geq p + 2$ is necessary according to (2.2).

Proof. (a) According to the definition of $m_0(p) = m_0$ we have

$\widetilde{F}_v(m\big|_p; m - 1) \geq \widetilde{F}_v(m_0\big|_p; m_0 - 1) + m - m_0, \ m \geq p + 2.$

According to Theorem 1.4 if $m \geq m_0$ the opposite inequality is also true.

(b) Assume the opposite is true and let

$\mathrm{V}(G) = V_1 \cup V_2, V_1 \cap V_2 = \emptyset,$

where V_1 is an independent set and V_2 does not contain an $(m_0 - 2)$-clique. Let $G_1 = G[V_2] = G - V_1$. According to Proposition 1.5, from $G \overset{v}{\rightarrow} m_0\big|_p$ it follows $G_1 \overset{v}{\rightarrow} (m_0 - 1)\big|_p$. Since $\omega(G_1) < m_0 - 2$, $G_1 \in \widetilde{\mathcal{H}}((m_0 - 1)\big|_p; m_0 - 2)$. Therefore

$|\mathrm{V}(G)| - 1 \geq |\mathrm{V}(G_1)| \geq \widetilde{F}_v((m_0 - 1)\big|_p; m_0 - 2).$

Since $|\mathrm{V}(G)| = \widetilde{F}_v(m_0\big|_p; m_0 - 1)$, from these inequalities it follows that

$\widetilde{F}_v(m_0\big|_p; m_0-1) - m_0 \geq \widetilde{F}_v((m_0-1)\big|_p; m_0-2) - (m_0-1)$,
which contradicts the definition of m_0.

(c) If $m_0 = p+2$, then from (1.5) we have $\widetilde{F}_v((p+2)\big|_p; p+1) \geq 2p+4 = p+2+m_0$ and therefore in this case the inequality (c) is true.

Let $m_0 > p+2$ and G be an extremal graph in $\widetilde{\mathcal{H}}(m_0\big|_p; m_0-1)$. If $a_1, ..., a_s$ are positive integers, such that $m = \sum_{i=1}^{s}(a_i - 1) + 1$ and $\max\{a_1, ..., a_s\} \leq p$, then $G \overset{v}{\rightarrow} (a_1, ..., a_s)$ and according to (2.1), $\chi(G) \geq m_0$. From (b) and Theorem 1.1 we see that $|\,\mathrm{V}(G)| \geq 2m_0 - 3$ and $|\,\mathrm{V}(G)| = 2m_0 - 3$ only if $G = \overline{C}_{2m_0-3}$. However, the last equality is not possible because $\chi(G) \geq m_0$ and $\chi(\overline{C}_{2m_0-3}) = m_0 - 1$. Therefore

$|\,\mathrm{V}(G)| = \widetilde{F}_v(m_0\big|_p; m_0-1) \geq 2m_0 - 2$

Since $m_0 > p+2$ from the definition of m_0 we have

$\widetilde{F}_v(m_0\big|_p; m_0-1) - m_0 < \widetilde{F}_v((p+2)\big|_p; p+1) - p - 2$.

From these inequalities the inequality (c) follows easily. □

3. Algorithms

In this section we present algorithms for finding all maximal graphs in $\widetilde{\mathcal{H}}(m\big|_p; q; n)$ with the help of a computer. The remaining graphs in this set can be obtained by removing edges from the maximal graphs. The idea for these algorithms comes from [14] (see Algorithm 1). Similar algorithms are used in [1], [2], [19], [8], [15]. Also with the help of the computer, results for Folkman numbers are obtained in [6], [17], [16] and [3].
The following proposition for maximal graphs in $\widetilde{\mathcal{H}}(m\big|_p; q; n)$ will be useful

Proposition 3.1. *Let G be a maximal graph in $\widetilde{\mathcal{H}}(m\big|_p; q; n)$. Let $v_1, v_2, ..., v_k$ be independent vertices of G and $H = G - \{v_1, v_2, ..., v_k\}$. Then:*

(a) *$H \in \widetilde{\mathcal{H}}((m-1)\big|_p; q; n-k)$*

(b) *H is a $(+K_{q-1})$-graph*

(c) *$\mathrm{N}_G(v_i)$ is a maximal K_{q-1}-free subset of $\mathrm{V}(H)$, $i = 1, ..., k$*

Proof. The proposition (a) follows from Proposition 1.5, (b) and (c) follow from the maximality of G. □

We will define an algorithm, which is based on Proposition 3.1, and generates all maximal graphs in $\widetilde{\mathcal{H}}(m\big|_p; q; n)$ with independence number at least k.

Algorithm 3.2. *Finding all maximal graphs in $\widetilde{\mathcal{H}}(m\big|_p; q; n)$ with independence number at least k by adding k independent vertices to the $(+K_{q-1})$-graphs in $\widetilde{\mathcal{H}}((m-1)\big|_p; q; n-k)$.*

1. Denote by $\mathcal{A}$ the set of all $(+K_{q-1})$-graphs in $\widetilde{\mathcal{H}}((m-1)\big|_p; q; n-k)$. The obtained maximal graphs in $\widetilde{\mathcal{H}}(m\big|_p; q; n)$ will be output in $\mathcal{B}$, let $\mathcal{B} = \emptyset$.

2. For each graph $H \in \mathcal{A}$:

2.1. Find the family $\mathcal{M}(H) = \{M_1, ..., M_t\}$ of all maximal K_{q-1}-free subsets of $\mathrm{V}(H)$.

2.2. Consider all the k-tuples $(M_{i_1}, M_{i_2}, ..., M_{i_k})$ of elements of $\mathcal{M}(H)$, for which $1 \leq i_1 \leq ... \leq i_k \leq t$ (in these k-tuples some subsets M_i can coincide). For every such k-tuple construct the graph $G = G(M_{i_1}, M_{i_2}, ..., M_{i_k})$ by adding to

$\mathrm{V}(H)$ *new independent vertices* $v_1, v_2, ..., v_k$*, so that* $\mathrm{N}_G(v_j) = M_{i_j}, j = 1, ..., k$ *(see Proposition 3.1 (c)). If* $\omega(G + e) = q, \forall e \in \mathrm{E}(\overline{G})$*, then add* G *to* $\mathcal{B}$*.*

3. Exclude the isomorph copies of graphs from $\mathcal{B}$*.*

4. Exclude from $\mathcal{B}$ *all graphs which are not in* $\widetilde{\mathcal{H}}(m\big|_p; q; n)$*.*

Theorem 3.3. *Upon completion of Algorithm 3.2 the obtained set* $\mathcal{B}$ *is equal to the set of all maximal graphs in* $\widetilde{\mathcal{H}}(m\big|_p; q; n)$ *with independence number at least* k*.*

Proof. From step 4 we see that $\mathcal{B} \subseteq \widetilde{\mathcal{H}}(m\big|_p; q; n)$ and from step 2.2 it becomes clear, that $\mathcal{B}$ contains only maximal graphs in $\widetilde{\mathcal{H}}(m\big|_p; q; n)$ with independence number at least k. Let G be an arbitrary maximal graph in $\widetilde{\mathcal{H}}(m\big|_p; q; n)$ with independence number in k. We will prove that $G \in \mathcal{B}$. Let $v_1, ..., v_k$ be independent vertices of G and $H = G - \{v_1, ..., v_k\}$. According to Proposition 3.1(a) and (b), $H \in \widetilde{\mathcal{H}}((m-1)\big|_p; q; n-k)$ and H is a $(+K_{q-1})$-graph. Therefore in step 1 we have $H \in \mathcal{A}$. According to Proposition 3.1(c), $\mathrm{N}_G(v_i) \in \mathcal{M}(H)$ for all $i \in \{1, ..., k\}$, hence in step 2 G is added to $\mathcal{B}$. □

Let us note that if $G \in \widetilde{\mathcal{H}}(m\big|_p; q; n)$ and $n \geq q$, then $G \neq K_n$ and therefore $\alpha(G) \geq 2$. In this case, with the help of Algorithm 3.2 we can obtain all maximal graphs in $\widetilde{\mathcal{H}}(m\big|_p; q; n)$ by adding to independent vertices to the $(+K_{q-1})$-graphs in $\widetilde{\mathcal{H}}((m-1)\big|_p; q; n-2)$.

It is clear that if G is a graph for which $\alpha(G) = 2$ and H is a subgraph of G obtained by removing independent vertices, then $\alpha(H) \leq 2$. We modify Algorithm 3.2 in the following way to obtain the maximal graphs in $\widetilde{\mathcal{H}}(m\big|_p; q; n)$ with independence number 2:

Algorithm 3.4. *A modification of Algorithm 3.2 for finding all maximal graphs in* $\widetilde{\mathcal{H}}(m\big|_p; q; n)$ *with independence number 2 by adding* 2 *independent vertices to the* $(+K_{q-1})$*-graphs in* $\widetilde{\mathcal{H}}((m-1)\big|_p; q; n-2)$ *with independence number not greater than 2.*

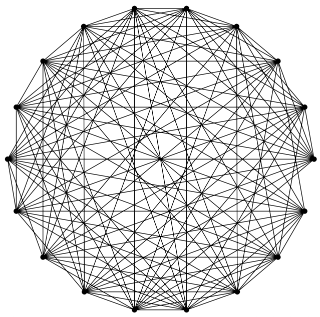
In step 1 of Algorithm 3.2 we add the condition that the set $\mathcal{A}$ *contains only the* $(+K_{q-1})$*-graphs* $\widetilde{\mathcal{H}}((m-1)\big|_p; q; n-k)$ *with independence number not greater than 2, and at the end of step 2.2 after the condition* $\omega(G + e) = q, \forall e \in \mathrm{E}(\overline{G})$ *we also add the condition* $\alpha(G) = 2$*.*

Thus, finding all maximal graphs in $\widetilde{\mathcal{H}}(m\big|_p; q; n)$ with independence number 2 is reduced to finding all $(+K_{q-1})$-graphs with independence number not greater than 2 in $\widetilde{\mathcal{H}}(m-1\big|_p; q; n-2)$ and finding the remaining maximal graphs in $\widetilde{\mathcal{H}}(m\big|_p; q; n)$ with independence number greater than or equal to 3 is reduced to finding all $(+K_{q-1})$-graphs in $\widetilde{\mathcal{H}}(m-1\big|_p; q; n-3)$. In this way we can obtain all maximal graphs in $\widetilde{\mathcal{H}}(m\big|_p; q; n)$ in steps, starting from graphs with a small number of vertices.

The *nauty* programs [11] have an important role in this work. We use them for fast generation of non-isomorphic graphs and for graph isomorph rejection.

4. Computation of the number $\widetilde{F}_v(8\big|_6; 7)$

From Theorem 2.1 it becomes clear that in order to compute the numbers $\widetilde{F}_v(m\big|_6; m-1)$ we need the exact value of the number $m_0(6)$. According to Theorem 2.1 (c), to obtain an upper bound for this number we need to know $\widetilde{F}_v(8\big|_6; 7)$. In this section we compute this number by proving the following



0	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0
1	0	0	1	1	1	0	1	1	1	1	1	1	1	1	0	1	0
1	0	0	1	1	0	1	1	1	1	1	1	1	1	0	1	0	1
1	1	1	0	1	1	1	1	1	1	0	1	0	1	0	1	1	0
1	1	1	1	0	1	1	1	1	1	1	0	1	0	1	0	0	1
1	1	0	1	1	0	1	0	1	1	1	1	0	1	1	1	1	0
1	0	1	1	1	1	0	1	0	1	1	1	1	0	1	1	0	1
1	1	1	1	1	0	1	0	1	0	0	1	1	1	0	1	1	1
1	1	1	1	1	1	0	1	0	0	1	0	1	1	1	0	1	1
1	1	1	1	1	1	1	0	0	0	1	1	0	0	1	1	1	1
1	1	1	0	1	1	1	0	1	1	0	0	1	0	1	1	1	1
1	1	1	1	0	1	1	1	0	1	0	0	0	1	1	1	1	1
1	1	1	0	1	0	1	1	1	0	1	0	0	1	1	1	1	1
1	1	1	1	0	1	0	1	1	0	0	1	1	0	1	1	1	1
0	1	0	0	1	1	1	0	1	1	1	1	1	1	0	1	1	1
0	0	1	1	0	1	1	1	0	1	1	1	1	1	1	0	1	1
0	1	0	1	0	1	0	1	1	1	1	1	1	1	1	1	0	1
0	0	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	0

Γ_1

FIGURE 1. Graph $\Gamma_1 \in \widetilde{\mathcal{H}}(8\big|_6; 7; 18)$

Theorem 4.1. $\widetilde{F}_v(8\big|_6; 7) = 18$.

Proof. The inequality $\widetilde{F}_v(8\big|_6; 7) \leq 18$ is proved in [1] with the help of the graph Γ_1 which is given on Figure 1(see the proof of Theorem 1.10 in version 1 or the proof of Theorem 1.9 in version 2). To obtain the lower bound we will prove with the help of a computer that $\widetilde{\mathcal{H}}(8\big|_6; 7; 17) = \emptyset$.

First, we search for maximal graphs in $\widetilde{\mathcal{H}}(8\big|_6; 7; 17)$ with independence number greater than 2. It is clear that K_6 and $K_6 - e$ are the only $(+K_6)$-graphs in $\widetilde{\mathcal{H}}(3\big|_6; 7; 6)$. With the help of Algorithm 3.2 we add 2 independent vertices to these graphs to find all maximal graphs in $\widetilde{\mathcal{H}}(4\big|_6; 7; 8)$. By removing edges from them we find all $(+K_6)$-graphs in $\widetilde{\mathcal{H}}(4\big|_6; 7; 8)$. In the same way, we successively obtain all maximal and all $(+K_6)$-graphs in the sets:
$\widetilde{\mathcal{H}}(5\big|_6; 7; 10)$, $\widetilde{\mathcal{H}}(6\big|_6; 7; 12)$, $\widetilde{\mathcal{H}}(7\big|_6; 7; 14)$.
In the end, with the help of Algorithm 3.2 we add 3 independent vertices to the obtained $(+K_6)$-graphs in $\widetilde{\mathcal{H}}(7\big|_6; 7; 14)$ to find all maximal graphs in $\widetilde{\mathcal{H}}(8\big|_6; 7; 17)$ with independence number greater than 2.

After that, we search for maximal graphs in $\widetilde{\mathcal{H}}(8\big|_6; 7; 17)$ with independence number 2. It is clear that K_5 is the only $(+K_6)$-graph in $\widetilde{\mathcal{H}}(2\big|_6; 7; 5)$. With the help of Algorithm 3.4 we add 2 independent vertices this graph to find all maximal graphs in $\widetilde{\mathcal{H}}(3\big|_6; 7; 7)$ with independence number 2. By removing edges from them we find all $(+K_6)$-graphs in $\widetilde{\mathcal{H}}(3\big|_6; 7; 7)$ with independence number 2. In the same way, we successively obtain all maximal and all $(+K_6)$-graphs with independence number 2 in the sets:
$\widetilde{\mathcal{H}}(4\big|_6; 7; 9)$, $\widetilde{\mathcal{H}}(5\big|_6; 7; 11)$, $\widetilde{\mathcal{H}}(6\big|_6; 7; 13)$, $\widetilde{\mathcal{H}}(7\big|_6; 7; 15)$ and $\widetilde{\mathcal{H}}(8\big|_6; 7; 17)$.

The number of graphs found in each step is described in Table 1 in []. In both cases we do not obtain any maximal graphs in $\widetilde{\mathcal{H}}(8|_6; 7; 17)$, therefore $\widetilde{\mathcal{H}}(8|_6; 7; 17) = \emptyset$. □

Corollary 4.2. $8 \leq m_0(6) \leq 11$

Proof. The inequality $m_0(6) \geq 8$ follows from the definition of m_0 and the upper bound follows from Theorem 2.1 (c), $p = 6$. □

5. Proof of the Main Theorem

Since $\widetilde{F}_v(8|_6; 7) = 18$, according to Theorem 2.1 (a) it is enough to prove $m_0(6) = 8$. According to Corollary 4.2 this equality will be proved if we prove $\widetilde{F}_v(9|_6; 8) > 18$, $\widetilde{F}_v(10|_6; 9) > 19$ and $\widetilde{F}_v(11|_6; 10) > 20$. The proof of these inequalities is similar to the proof of $\widetilde{F}_v(8|_6; 7) > 17$ from Theorem 4.1. It is clear that it is enough to prove $\widetilde{\mathcal{H}}(m|_6; m-1; m+9) = \emptyset$ for $m = 9, 10, 11$.

First, we search for maximal graphs in $\widetilde{\mathcal{H}}(m|_6; m-1; m+9)$ with independence number greater than 2. It is clear that K_{m-2} and $K_{m-2}-e$ are the only $(+K_{m-2})$-graphs in $\widetilde{\mathcal{H}}((m-5)|_6; m-1; m-2)$. With the help of Algorithm 3.2 we successively obtain all maximal and all $(+K_{m-2})$-graphs in the sets:
$\widetilde{\mathcal{H}}((m-4)|_6; m-1; m)$
$\widetilde{\mathcal{H}}((m-3)|_6; m-1; m+2)$
$\widetilde{\mathcal{H}}((m-2)|_6; m-1; m+4)$
$\widetilde{\mathcal{H}}((m-1)|_6; m-1; m+6)$
In the end, with the help of Algorithm 3.2 we add 3 independent vertices to the obtained $(+K_{m-2})$-graphs in $\widetilde{\mathcal{H}}((m-1)|_6; m-1; m+6)$ to find all maximal graphs in $\widetilde{\mathcal{H}}(m|_6; m-1; m+9)$ with independence number greater than 2.

After that, we search for maximal graphs in $\widetilde{\mathcal{H}}(m|_6; m-1; m+9)$ with independence number 2. It is clear that K_{m-3} is the only $(+K_{m-2})$-graph in $\widetilde{\mathcal{H}}((m-6)|_6; m-1; m-3)$. With the help of Algorithm 3.4 we successively obtain all maximal and all $(+K_{m-2})$-graphs with independence number 2 in the sets:
$\widetilde{\mathcal{H}}((m-5)|_6; m-1; m-1)$
$\widetilde{\mathcal{H}}((m-4)|_6; m-1; m+1)$
$\widetilde{\mathcal{H}}((m-3)|_6; m-1; m+3)$
$\widetilde{\mathcal{H}}((m-2)|_6; m-1; m+5)$
$\widetilde{\mathcal{H}}((m-1)|_6; m-1; m+7)$
$\widetilde{\mathcal{H}}(m|_6; m-1; m+9)$.

The number of graphs found in each step is given in Table 2, Table 3 and Table 4 in []. In both cases we do not obtain any maximal graphs in the sets $\widetilde{\mathcal{H}}(m|_6; m-1; m+9)$, $m = 9, 10, 11$, hence it follows $\widetilde{F}_v(9|_6; 8) > 18$, $\widetilde{F}_v(10|_6; 9) > 19$, $\widetilde{F}_v(11|_6; 10) > 20$ and $m_0(6) = 8$. Thus we finish the proof of the Main Theorem.

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Aleksandar Bikov
asbikov@fmi.uni-sofia.bg, Corresponding author
Nedyalko Nenov
nenov@fmi.uni-sofia.bg

Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski
5, James Bourchier Blvd.
1164 Sofia, Bulgaria

Appendix A. Results of the computations

set	independence number	maximal graphs	$(+K_6)$-graphs
$\widetilde{\mathcal{H}}(3\|_6;7;6)$	-		2
$\widetilde{\mathcal{H}}(4\|_6;7;8)$	-	2	13
$\widetilde{\mathcal{H}}(5\|_6;7;10)$	-	8	324
$\widetilde{\mathcal{H}}(6\|_6;7;12)$	-	56	104 271
$\widetilde{\mathcal{H}}(7\|_6;7;14)$	-	18	1825
$\widetilde{\mathcal{H}}(8\|_6;7;17)$	≥ 3	0	
$\widetilde{\mathcal{H}}(2\|_6;7;5)$	≤ 2		1
$\widetilde{\mathcal{H}}(3\|_6;7;7)$	$= 2$	1	3
$\widetilde{\mathcal{H}}(4\|_6;7;9)$	$= 2$	2	22
$\widetilde{\mathcal{H}}(5\|_6;7;11)$	$= 2$	5	468
$\widetilde{\mathcal{H}}(6\|_6;7;13)$	$= 2$	24	97 028
$\widetilde{\mathcal{H}}(7\|_6;7;15)$	$= 2$	468	2 395 573
$\widetilde{\mathcal{H}}(8\|_6;7;17)$	$= 2$	0	
$\widetilde{\mathcal{H}}(8\|_6;7;17)$	-	0	

Table 1. Steps in the search of all maximal graphs in $\widetilde{\mathcal{H}}(8|_6;7;17)$

set	independence number	maximal graphs	$(+K_7)$-graphs
$\widetilde{\mathcal{H}}(4\|_6;8;7)$	-		2
$\widetilde{\mathcal{H}}(5\|_6;8;9)$	-	2	13
$\widetilde{\mathcal{H}}(6\|_6;8;11)$	-	8	326
$\widetilde{\mathcal{H}}(7\|_6;8;13)$	-	56	105 125
$\widetilde{\mathcal{H}}(8\|_6;8;15)$	-	20	1844
$\widetilde{\mathcal{H}}(9\|_6;8;18)$	≥ 3	0	
$\widetilde{\mathcal{H}}(3\|_6;8;6)$	≤ 2		1
$\widetilde{\mathcal{H}}(4\|_6;8;8)$	$= 2$	1	3
$\widetilde{\mathcal{H}}(5\|_6;8;10)$	$= 2$	2	22
$\widetilde{\mathcal{H}}(6\|_6;8;12)$	$= 2$	5	489
$\widetilde{\mathcal{H}}(7\|_6;8;14)$	$= 2$	25	119 124
$\widetilde{\mathcal{H}}(8\|_6;8;16)$	$= 2$	506	2 747 120
$\widetilde{\mathcal{H}}(9\|_6;8;18)$	$= 2$	0	
$\widetilde{\mathcal{H}}(9\|_6;8;18)$	-	0	

Table 2. Steps in the search of all maximal graphs in $\widetilde{\mathcal{H}}(9|_6;8;18)$

set	independence number	maximal graphs	$(+K_8)$-graphs
$\widetilde{\mathcal{H}}(5\|_6; 9; 8)$	-		2
$\widetilde{\mathcal{H}}(6\|_6; 9; 10)$	-	2	13
$\widetilde{\mathcal{H}}(7\|_6; 9; 12)$	-	8	327
$\widetilde{\mathcal{H}}(8\|_6; 9; 14)$	-	56	105 281
$\widetilde{\mathcal{H}}(9\|_6; 9; 16)$	-	20	1845
$\widetilde{\mathcal{H}}(10\|_6; 9; 19)$	≥ 3	0	
$\widetilde{\mathcal{H}}(4\|_6; 9; 7)$	≤ 2		1
$\widetilde{\mathcal{H}}(5\|_6; 9; 9)$	$= 2$	1	3
$\widetilde{\mathcal{H}}(6\|_6; 9; 11)$	$= 2$	2	22
$\widetilde{\mathcal{H}}(7\|_6; 9; 13)$	$= 2$	5	496
$\widetilde{\mathcal{H}}(8\|_6; 9; 15)$	$= 2$	25	121 498
$\widetilde{\mathcal{H}}(9\|_6; 9; 17)$	$= 2$	509	2 749 155
$\widetilde{\mathcal{H}}(10\|_6; 9; 19)$	$= 2$	0	
$\widetilde{\mathcal{H}}(10\|_6; 9; 19)$	-	0	

TABLE 3. Steps in the search of all maximal graphs in $\widetilde{\mathcal{H}}(10|_6; 9; 19)$

set	independence number	maximal graphs	$(+K_9)$-graphs
$\widetilde{\mathcal{H}}(6\|_6; 10; 9)$	-		2
$\widetilde{\mathcal{H}}(7\|_6; 10; 11)$	-	2	13
$\widetilde{\mathcal{H}}(8\|_6; 10; 13)$	-	8	327
$\widetilde{\mathcal{H}}(9\|_6; 10; 15)$	-	56	105 314
$\widetilde{\mathcal{H}}(10\|_6; 10; 17)$	-	20	1845
$\widetilde{\mathcal{H}}(11\|_6; 10; 20)$	≥ 3	0	
$\widetilde{\mathcal{H}}(5\|_6; 10; 8)$	≤ 2		1
$\widetilde{\mathcal{H}}(6\|_6; 10; 10)$	$= 2$	1	3
$\widetilde{\mathcal{H}}(7\|_6; 10; 12)$	$= 2$	2	22
$\widetilde{\mathcal{H}}(8\|_6; 10; 14)$	$= 2$	5	498
$\widetilde{\mathcal{H}}(9\|_6; 10; 16)$	$= 2$	25	121 863
$\widetilde{\mathcal{H}}(10\|_6; 10; 18)$	$= 2$	509	2 749 171
$\widetilde{\mathcal{H}}(11\|_6; 10; 20)$	$= 2$	0	
$\widetilde{\mathcal{H}}(11\|_6; 10; 20)$	-	0	

TABLE 4. Steps in the search of all maximal graphs in $\widetilde{\mathcal{H}}(11|_6; 10; 20)$