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NORMALLY GENERATED SUBSPACES OF LOGARITHMIC CANONICAL SECTIONS

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The logarithmic-canonical bundle $\Omega^2_{A'}(T')$ of a smooth toroidal compactification $A' = (\mathbb{B}/\Gamma)'$ of a ball quotient $\mathbb{B}/\Gamma$ is known to be sufficiently ample over the Baily-Borel compactification $\widehat{A} = \widehat{\mathbb{B}/\Gamma}$. The present work develops criteria for a subspace $V \subseteq H^0(A', \Omega^2_{A'}(T'))$ to be normally generated over $\widehat{A}$, i.e., to determine a regular immersive projective morphism of $\widehat{A}$ with normal image. These are applied to a specific example $A'_1 = (\mathbb{B}/\Gamma_1)'$ over the Gauss numbers. The first section organizes some preliminaries. The second one provides two sufficient conditions for the normal generation of a subspace $V \subseteq H^0(A', \Omega^2_{A'}(T'))$.

Keywords: Modular forms, holomorphic sections of line bundles, abelian functions, sufficiently ample and normally generated line bundles.

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1. PRELIMINARIES

Throughout, let $\mathbb{B} = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\} = SU_{2,1}/S(U_2 \times U_1)$ be the complex two dimensional ball and $\Gamma \subset SU_{2,1}$ be a lattice, acting freely on $\mathbb{B}$. The compact $\mathbb{B}/\Gamma$ are of general type. The non-compact $\mathbb{B}/\Gamma$ admit smooth toroidal compactification $(\mathbb{B}/\Gamma)'$ by a disjoint union $T' = \cup_{i=1}^h T'_i$ of smooth irreducible elliptic curves T'_i. From now on, we concentrate on $A' = (\mathbb{B}/\Gamma)'$ with abelian minimal model A. In such a case, the lattice Γ, the ball quotient $\mathbb{B}/\Gamma$ and its compactifications are said to be co-abelian.

The contraction $\xi : A' \to A$ of the rational (-1)-curves on A' restricts to a biregular morphism $\xi : T'_i \to \xi(T'_i) = T_i$, as far as an abelian surface A does not support rational curves. In such a way, ξ produces the multi-elliptic divisor $T = \xi(T') = \sum_{i=1}^{h} T_i \subset A$, i.e., a divisor with smooth elliptic irreducible components T_i. According to Kobayashi hyperbolicity of $\mathbb{B}/\Gamma$, any irreducible component of the exceptional divisor of ξ intersects T' in at least two points. Therefore $\xi : A' \to A$ is the blow-up of A at the singular locus $T^{\rm sing} = \sum_{1 \leq i < j \leq h} T_i \cap T_j$ of T. Holzapfel has shown in [5] that the blow-up A' of an abelian surface A at the singular locus $T^{\rm sing} = \sum_{1 \leq i < j \leq h} T_i \cap T_j$ of a multi-elliptic divisor $T = \sum_{i=1}^{h} T_i$ is the toroidal compactification $A' = (\mathbb{B}/\Gamma)'$ of a smooth ball quotient $\mathbb{B}/\Gamma$ if and only if $A = E \times E$ is the Cartesian square of an elliptic curve E and

$$\sum_{i=1}^{h} {\rm card}(T_i \cap T^{\rm sing}) = 4{\rm card}(T^{\rm sing}). \tag{1.1}$$

In order to describe the smooth irreducible elliptic curves T_i on A and their intersections, let us note that the inclusions $T_i \subset A = E \times E$ are morphisms of abelian varieties. Consequently, they lift to affine linear maps of the corresponding universal covers and

$$T_i = \{(u + \pi_1(E), v + \pi_1(E)) \,|\, a_i u + b_i v + c_i \in \pi_1(E)\}$$

for some $a_i, b_i, c_i \in \mathbb{C}$. The fundamental group

$$\pi_1(T_i) = \{t \in \mathbb{C} \,|\, b_i t + \pi_1(E) = -a_i t + \pi_1(E) = \pi_1(E)\} = a_i^{-1}\pi_1(E) \cap b_i^{-1}\pi_1(E).$$

If Γ is an arithmetic lattice then the elliptic curve E has complex multiplication by an imaginary quadratic number field $K = \mathbb{Q}(\sqrt{-d})$, $d \in \mathbb{N}$. As a result, Γ is commensurable with the full Picard modular group $SU_{2,1}(\mathcal{O}_{-d})$ over the integers ring $\mathcal{O}_{-d}$ of $\mathbb{Q}(\sqrt{-d})$. Such Γ are called Picard modular groups. Moreover, all T_i are defined over K. For simplicity, we assume that $\pi_1(E) = \mathcal{O}_{-d}$, in order to have maximal endomorphism ring $End(E) = \mathcal{O}_{-d}$. Since $K = \mathbb{Q}(\sqrt{-d})$ is the fraction field of $\mathcal{O}_{-d}$, one can choose $a_i, b_i \in \mathcal{O}_{-d}$. Thus, $\pi_1(T_i) \supseteq \mathcal{O}_{-d}$, $a_i\pi_1(E) + b_i\pi_1(E) \subseteq \mathcal{O}_{-d}$ and T_i has minimal fundamental group $\pi_1(T_i) = \mathcal{O}_{-d}$ exactly when $a_i\pi_1(E) + b_i\pi_1(E) = \pi_1(E) = \mathcal{O}_{-d}$. In particular, if K is of class number 1, then all the smooth elliptic curves $T_i \subset A = \mathbb{C}^2/(\mathcal{O}_{-d} \times \mathcal{O}_{-d})$, defined over $K = \mathbb{Q}(\sqrt{-d})$, have minimal fundamental groups $\pi_1(T_i) = \mathcal{O}_{-d}$. From now on, we do not restrict the class number of $K = \mathbb{Q}(\sqrt{-d})$, but confine only to smooth irreducible elliptic curves T_i with minimal fundamental groups $\pi_1(T_i) = \pi_1(E) = \mathcal{O}_{-d}$. If $b_i \neq 0$, then

$$T_i^{(1)} = \{(b_i t + \pi_1(E), -a_i t - b_i^{-1} c_i + \pi_1(E)) \,|\, t \in \mathbb{C}\} \subseteq T_i.$$

Moreover, the complete pre-image of $T_i^{(1)}$ in the universal cover $\widetilde{A} = \mathbb{C}^2$ of A is $\pi_1(T_i)$-invariant family of complex lines. Therefore, $T_i^{(1)}$ is an elliptic curve and coincides with T_i.

The notations from the next lemma will be used throughout:

Lemma 1. *Let* $T_s = \{(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) \,|\, a_s u + b_s v + c_s \in \mathcal{O}_d\}$ *and* $D_s = \{(u+\mathcal{O}_{-d}, v+\mathcal{O}_{-d}) \,|\, a_s u + b_s v + c_s + \mu_s \in \mathcal{O}_{-d}\}$ *for* $1 \leq s \leq 3$ *be elliptic curves with minimal fundamental groups* $\pi_1(T_s) = \pi_1(D_s) = \mathcal{O}_{-d}$ *on* $A = (\mathbb{C}/\mathcal{O}_{-d}) \times (\mathbb{C}/\mathcal{O}_{-d})$ *and*

$$\Delta_{ij} := \det \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}, \quad \Delta := \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Then for any even permutation $\{i, j, l\}$ *of* $\{1, 2, 3\}$ *there hold the following:*

(i) *the intersection number is* $T_i.T_j = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{ij})$, *where* $N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})} : \mathbb{Q}(\sqrt{-d}) \to \mathbb{Q}$ *stands for the norm;*

(ii) $T_i \cap T_j \subset D_l$ *if and only if* $\mu_l \in \mathcal{O}_{-d} - \Delta_{ij}^{-1}\Delta$ *and both* $\Delta_{ij}^{-1}\Delta_{jl}$ *and* $\Delta_{ij}^{-1}\Delta_{li}$ *belong to* $End(E) = \mathcal{O}_{-d}$;

(iii) $T_1 \cap T_2 \cap T_3 = \emptyset$ *if and only if* $\Delta \notin \Delta_{12}\mathcal{O}_{-d} + \Delta_{23}\mathcal{O}_{-d} + \Delta_{31}\mathcal{O}_{-d}$.

Proof. (i) If $T_i \cap T_j = \emptyset$, then the liftings of T_i, T_j to the universal cover $\widetilde{A} = \mathbb{C}^2$ of A are discrete families of mutually parallel lines. In such a case, we say briefly that T_i and T_j are parallel. That allows to choose $a_j = a_i$, $b_j = b_i$ and to calculate $N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{ij}) = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(0) = 0 = T_i.T_j$. When $T_i \cap T_j \neq \emptyset$, one can move the origin $\check{o}_A = (\check{o}_E, \check{o}_E) \in A$ in $T_i \cap T_j$ and represent

$$T_i = \{(b_i t + \mathcal{O}_{-d}, -a_i t + \mathcal{O}_{-d}) \,|\, t \in \mathbb{C}\}, \quad T_j = \{(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) \,|\, a_j u + b_j v \in \mathcal{O}_{-d}\}.$$

Then the intersection is

$$T_i \cap T_j = \{(b_i t + \mathcal{O}_{-d}, -a_i t + \mathcal{O}_{-d}) \,|\, \Delta_{ij} t \in \mathcal{O}_{-d} \subset \mathbb{C}\} \simeq$$

$$(\Delta_{ij}^{-1}\mathcal{O}_{-d})/(b_i^{-1}\mathcal{O}_{-d} \cap a_i^{-1}\mathcal{O}_{-d}) = \left(\Delta_{ij}^{-1}\mathcal{O}_{-d}\right)/\mathcal{O}_{-d} \simeq \mathcal{O}_{-d}/\Delta_{ij}\mathcal{O}_{-d}.$$

For an arbitrary lattice $\Lambda \subset \mathbb{C}$, let us denote by $\mathcal{F}(\Lambda)$ a Λ-fundamental domain on $\mathbb{C}$. As far as $\mathcal{F}(\Delta_{ij}\mathcal{O}_{-d})$ is the $\mathcal{O}_{-d}/\Delta_{ij}\mathcal{O}_{-d}$-orbit of $\mathcal{F}(\mathcal{O}_{-d})$, the index equals

$$[\mathcal{O}_{-d} : \Delta_{ij}\mathcal{O}_{-d}] = \frac{\mathrm{vol}\mathcal{F}(\Delta_{ij}\mathcal{O}_{-d})}{\mathrm{vol}\mathcal{F}(\mathcal{O}_{-d})} = \frac{\mathrm{vol}\mathcal{F}(|\Delta_{ij}|\mathcal{O}_{-d})}{\mathrm{vol}\mathcal{F}(\mathcal{O}_{-d})} = |\Delta_{ij}|^2 = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{ij}).$$

(ii) The intersection $T_i \cap T_j$ consists of the $\pi_1(A)$-equivalence classes of the solutions $(u, v) \in \mathbb{C}^2$ of

$$\left| \begin{array}{l} a_i u + b_i v = \lambda_1 - c_i \\ a_j u + b_j v = \lambda_2 - c_j \end{array} \right.$$

for arbitrary $\lambda_1, \lambda_2 \in \pi_1(E) = \mathcal{O}_{-d}$. A point

$$(\Delta_{ij}^{-1}(b_i c_j - b_j c_i) + \Delta_{ij}^{-1}(b_j \lambda_1 - b_i \lambda_2),\ \Delta_{ij}^{-1}(a_j c_i - a_i c_j) + \Delta_{ij}^{-1}(a_i \lambda_2 - a_j \lambda_1))$$

belongs to the lifting of D_l if and only if

$$\begin{aligned} &-\Delta_{ij}^{-1}\Delta_{jl}\lambda_1 - \Delta_{ij}^{-1}\Delta_{li}\lambda_2 + \Delta_{ij}^{-1}(c_i\Delta_{jl} + c_j\Delta_{li}) + c_l + \mu_l \\ &= -\Delta_{ij}^{-1}\Delta_{jl}\lambda_1 - \Delta_{ij}^{-1}\Delta_{li}\lambda_2 + \Delta_{ij}^{-1}\Delta + \mu_l \in \pi_1(E) = \mathcal{O}_{-d} \end{aligned}$$

for $\forall \lambda_1, \lambda_2 \in \pi_1(E)$. That, in turn, is equivalent to $\Delta_{ij}^{-1}\Delta + \mu_l \in \pi_1(E) = \mathcal{O}_{-d}$ and $\Delta_{ij}^{-1}\Delta_{jl}, \Delta_{ij}^{-1}\Delta_{li} \in End(E) = \mathcal{O}_{-d}$.

(iii) For arbitrary $\lambda_1, \lambda_2, \lambda_3 \in \pi_1(E) = \mathcal{O}_{-d}$, the linear system

$$\left| \begin{array}{l} a_1 u + b_1 v = \lambda_1 - c_1 \\ a_2 u + b_2 v = \lambda_2 - c_2 \\ a_3 u + b_3 v = \lambda_3 - c_3 \end{array} \right.$$

has no solutions exactly when

$$\det \begin{pmatrix} a_1 & b_1 & \lambda_1 - c_1 \\ a_2 & b_2 & \lambda_2 - c_2 \\ a_3 & b_3 & \lambda_3 - c_3 \end{pmatrix} = \Delta_{23}\lambda_1 + \Delta_{31}\lambda_2 + \Delta_{12}\lambda_3 - \Delta \neq 0.$$

Lemma 1 is proved. □

The non-arithmetic lattices $\Gamma \subset SU_{2,1}$ correspond to abelian surfaces $A = E \times E$, whose elliptic factors E have minimal endomorphism rings $End(E) = \mathbb{Z}$. Then the liftings of the elliptic curves $T_i \subset A$ with $\pi_1(T_i) = \pi_1(E)$ to the universal cover $\widetilde{A} = \mathbb{C}^2$ of A are given by $a_i u + b_i v + c_i \in \pi_1(E)$ with $a_i, b_i \in \mathbb{Z}$. As a result, the intersection numbers $T_i.T_j = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{ij})$ are comparatively large and there are very few chances for construction of a multi-elliptic divisor $T = \sum_{i=1}^{h} \subset A$, subject to (1.1). This is a sort of a motivation for restricting our attention to the arithmetic case.

The smooth irreducible elliptic curves $T_i' \subset A'$ contract to the Γ-orbits $\kappa_i = \Gamma(p) \in \partial_\Gamma \mathbb{B}/\Gamma$ of the Γ-rational boundary points $p \in \partial_\Gamma \mathbb{B}$. These κ_i are called cusps. The resulting Baily-Borel compactification $\widehat{A} = \widehat{\mathbb{B}/\Gamma} = (\mathbb{B}/\Gamma) \cup (\partial_\Gamma \mathbb{B}/\Gamma)$ is a normal projective surface.

Definition 2. *Let Γ be a Picard modular group, $\gamma \in \Gamma$ and $Jac(\gamma) = \frac{\partial(\gamma_1, \gamma_2)}{\partial(z_1, z_2)}$ be the Jacobian matrix of $\gamma = (\gamma_1, \gamma_2) : \mathbb{B} \to \mathbb{B} \subset \mathbb{C}^2$. The global holomorphic functions $\delta : \mathbb{B} \to \mathbb{C}$ with transformation law*

$$\gamma^*(\delta)(z) = \delta\gamma(z) = [\det Jac(\gamma)]^{-n}\, \delta(z) \quad \textit{for } \forall \gamma \in \Gamma, \forall z \in \mathbb{B}$$

are called Γ*-modular forms of weight* n*.*

The Γ*-modular forms of weight* n *constitute a* $\mathbb{C}$*-linear space, which is denoted by* $[\Gamma, n]$*.*

Definition 3. *A* Γ*-modular form* $\delta \in [\Gamma, n]$ *is cuspidal if* $\delta(\kappa_i) = 0$ *at all the cusps* $\kappa_i \in \partial_\Gamma \mathbb{B}/\Gamma$*.*

The cuspidal Γ*-modular forms of weight* n *form the subspace* $[\Gamma, n]_{\text{cusp}}$ *of* $[\Gamma, n]$*.*

For any natural number n there is a $\mathbb{C}$-linear embedding

$$j_n : H^0(\mathbb{B}, \mathcal{O}_\mathbb{B}) \longrightarrow H^0\left(\mathbb{B}, \left(\Omega^2_\mathbb{B}\right)^{\otimes n}\right)$$

$$j_n(\delta)(z) = \delta(z)(dz_1 \wedge dz_2)^{\otimes n}$$

of the global holomorphic functions on the ball in the global holomorphic sections of the n-th pluri-canonical bundle $\left(\Omega^2_\mathbb{B}\right)^{\otimes n}$. It restricts to an isomorphism

$$j_n : [\Gamma, n] \longrightarrow H^0\left(\mathbb{B}, \left(\Omega^2_\mathbb{B}\right)^{\otimes n}\right)^\Gamma$$

of the Γ-modular forms of weight n with the Γ-invariant holomorphic sections of $\left(\Omega^2_\mathbb{B}\right)^{\otimes n}$. Note that the subspace $H^0\left(\mathbb{B}, \left(\Omega^2_\mathbb{B}\right)^{\otimes n}\right)^\Gamma$ of $H^0\left(\mathbb{B}/\Gamma, \left(\Omega^2_{\mathbb{B}/\Gamma}\right)^{\otimes n}\right)$ acts on $\widehat{A} = \widehat{\mathbb{B}/\Gamma}$, extending over the cusps $\partial_\Gamma \mathbb{B}/\Gamma$ of codimension 2 in $\widehat{A}$.

The tensor product $\Omega^2_{A'}(T') = \Omega^2_{A'} \otimes_\mathbb{C} \mathcal{O}_{A'}(T')$ is called logarithmic canonical bundle of A', while $\Omega^2_{A'}(T')^{\otimes n}$ are referred to as logarithmic pluri-canonical bundles. Hemperly has observed in [3] that

$$j_n[\Gamma, n] = H^0\left(\mathbb{B}, \left(\Omega^2_\mathbb{B}\right)^{\otimes n}\right)^\Gamma = H^0\left(A', \Omega^2_{A'}(T')^{\otimes n}\right)$$

as long as the holomorphic sections from these spaces have one and the same coordinate transformation law. A classical result of Baily-Borel establishes that $\Omega^2_{A'}(T')$ is sufficiently ample on $\widehat{A}$. The present article provides sufficient conditions for the ampleness of $\Omega^2_{A'}(T')$ on $\widehat{A}$.

Note that the canonical bundle

$$K_{A'} = \xi^* K_A + \mathcal{O}_{A'}(L) = \xi^* \mathcal{O}_A + \mathcal{O}_{A'}(L) = \mathcal{O}_{A'}(L)$$

is associated with the exceptional divisor $L = \xi^{-1}(T^{\text{sing}})$ of $\xi : A' \to A$. If s is a global meromorphic section of $\Omega^2_{A'}$ and t is a global meromorphic section of $\mathcal{O}_{A'}(T')$, then the tensoring

$$(s \otimes_\mathbb{C} t)^{\otimes(-n)} : H^0\left(A', \Omega^2_{A'}(T')^{\otimes n}\right) \longrightarrow \mathcal{L}_{A'}(n(L + T'))$$

is a $\mathbb{C}$-linear isomorphism with

$$\mathcal{L}_{A'}(n(L + T')) = \{f \in \mathfrak{Mer}(A') \,|\, (f) + n(L + T') \geq 0\}.$$

The isomorphism $\xi^* : \mathfrak{Mer}(A) \to \mathfrak{Mer}(A')$ of the meromorphic function fields induces a linear isomorphism

$$(\xi^*)^{-1} : \mathcal{L}_{A'}(n(L+T')) \longrightarrow \mathcal{L}_A(nT, nT^{\rm sing}),$$

where $m_p : {\rm Div}(A) \to \mathbb{Z}$ stands for the multiplicity at a point $p \in A$ and

$$\mathcal{L}_A(nT, nT^{\rm sing}) = \{f \in \mathfrak{Mer}(A) \,|\, (f) + nT \geq 0,\ m_p(f) + n \geq 0 \ \text{ for } \forall p \in T^{\rm sing}\}.$$

The linear isomorphisms

$$\tau_n := (\xi^*)^{-1}(s \otimes_{\mathbb{C}} t)^{\otimes(-n)} : j_n[\Gamma, n] \longrightarrow \mathcal{L}_A(nT, nT^{\rm sing})$$

are called transfers of modular forms of weight n to abelian functions.

For any $\delta \in [\Gamma, 1]$, note that $\delta(\kappa_i) \neq 0$ if and only if $T_i \subset (\tau_1 j_1(\delta))_\infty$. Observe also that $\tau_1 j_1[\Gamma, 1]_{\rm cusp} = \{f \in \mathcal{L}_A(T, T^{\rm sing}) \,|\, (f)_\infty = \emptyset\} = \mathbb{C}$ and fix the cuspidal form $\eta_o = (\tau_1 j_1)^{-1}(1)$ of weight 1.

Towards the construction of abelian functions $f \in \mathcal{L}_A(T, T^{\rm sing})$, let us recall from [7] that any elliptic function $g : E \to \mathbb{P}^1$ can be represented as

$$g(z) = C_o \prod_{i=1}^{k} \frac{\sigma(z-\alpha_i)}{\sigma(z-\beta_i)}, \tag{1.2}$$

where

$$\sigma(z) = z \prod_{\lambda \in \pi_1(E) \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right)^{\frac{z}{\lambda} + \frac{1}{2}\left(\frac{z}{\lambda}\right)^2}$$

is the Weierstrass σ-function, $\alpha_i, \beta_i, C_o \in \mathbb{C}$ and $\sum_{i=1}^{k} \alpha_i \equiv \sum_{i=1}^{k} \beta_i (\bmod\, \pi_1(E))$. The points of $E = \mathbb{C}/\pi_1(E)$ are of the form $\overline{a} = a + \pi_1(E)$ for some $a \in \mathbb{C}$. The elliptic function (1.2) takes all the values from $\mathbb{P}^1$ with one and a same multiplicity k. Moreover, if $g^{-1}(x) = \{\overline{p_i(x)} \in E \,|\, 1 \leq i \leq k\}$ for some $x \in \mathbb{C} \subset \mathbb{P}^1$, then $\sum_{i=1}^{k} \overline{p_i(x)} = \sum_{i=1}^{k} \overline{\beta_i}$. Observe that $\sigma : \mathbb{C} \to \mathbb{C}$ is a non-periodic entire function, but its divisor $(\sigma)_{\mathbb{C}} = \pi_1(E)$ on $\mathbb{C}$ is $\pi_1(E)$-invariant. That enables to define the divisor $(\sigma)_E = \check{o}_E$ of σ on E. In global holomorphic coordinates $(u, v) \in \mathbb{C}^2$, the divisor

$$(\sigma(a_i u + b_i v + c_i))_{\mathbb{C}^2} = \{(u, v) \in \mathbb{C}^2 \,|\, a_i u + b_i v + c_i \in \pi_1(E) = \mathcal{O}_{-d}\}$$

is the complete pre-image of T_i in the universal cover $\widetilde{A} = \mathbb{C}^2$ of A. That allows to define the divisor

$$(\sigma(a_i u + b_i v + c_i)) = (\sigma(a_i u + b_i v + c_i))_A = T_i.$$

Let $f \in \mathcal{L}_A(T)$ be an abelian function with pole divisor $(f)_\infty = \sum_{i=1}^{k} T_i$, after an eventual permutation of the irreducible components of T. Then

$$f_\infty := \prod_{i=1}^{k} \sigma(a_i u + b_i v + c_i) \text{ and } f_0 := f f_\infty \tag{1.3}$$

are (non-periodic) entire functions on $\mathbb{C}^2$. Let $\zeta = \frac{\sigma'}{\sigma}$ be Weierstrass' ζ-function, $\eta : \pi_1(E) \to \mathbb{C}$ be the $\mathbb{Z}$-linear homomorphism, satisfying $\zeta(z+\lambda) = \zeta(z) + \eta(\lambda)$ for all $z \in \mathbb{C}$, $\lambda \in \pi_1(E)$ and

$$\varepsilon(\lambda) = \begin{cases} 1 & \text{for } \lambda \in 2\pi_1(E), \\ -1 & \text{for } \lambda \in \pi_1(E) \setminus 2\pi_1(E). \end{cases}$$

Recall from [6] the $\pi_1(E)$-transformation law

$$\frac{\sigma(z+\lambda)}{\sigma(z)} = \varepsilon(\lambda) e^{\eta(\lambda)\left(z + \frac{\lambda}{2}\right)} \quad \text{for} \quad \forall \lambda \in \pi_1(E), \forall z \in \mathbb{C}.$$

Under the assumption (1.3), the $\pi_1(A)$-periodicity of f is equivalent to

$$\frac{f_0(u+\lambda, v)}{f_0(u,v)} = \frac{f_\infty(u+\lambda, v)}{f_\infty(u,v)} = \prod_{i=1}^{k} \varepsilon(a_i\lambda) e^{\eta(a_i\lambda)\left(a_i u + b_i v + c_i + \frac{a_i \lambda}{2}\right)}$$

and

$$\frac{f_0(u, v+\lambda)}{f_0(u,v)} = \frac{f_\infty(u, v+\lambda)}{f_\infty(u,v)} = \prod_{i=1}^{k} \varepsilon(b_i\lambda) e^{\eta(b_i\lambda)\left(a_i u + b_i v + c_i + \frac{b_i \lambda}{2}\right)}$$

for $\forall \lambda \in \pi_1(E) = \mathcal{O}_{-d}$, $\forall (u,v) \in \mathbb{C}^2$. We choose

$$f_0(u,v) = \prod_{i=1}^{k} \sigma(a_i u + b_i v + c_i + \mu_i)$$

and reduce the $\pi_1(A)$-periodicity of f to

$$1 = \frac{f(u+\lambda, v)}{f(u,v)} = e^{\sum_{i=1}^{k} \eta(a_i\lambda)\mu_i}, \quad 1 = \frac{f(u, v+\lambda)}{f(u,v)} = e^{\sum_{i=1}^{k} \eta(b_i\lambda)\mu_i} \quad \forall \lambda \in \mathcal{O}_{-d}, \forall (u,v) \in \mathbb{C}^2.$$

Let us mention that Holzapfel has studied $f \in \mathcal{L}_A(T)$ of the above form with at most three non-parallel irreducible components of $(f)_\infty$, intersecting pairwise in single points. The next lemma provides a sufficient (but not necessary) condition for $\pi_1(A)$-periodicity of a σ-quotient, whose pole divisor has an arbitrary number of irreducible components with arbitrary intersection numbers.

Lemma 4. *If*

$$\sum_{i=1}^{k} a_i\mu_i = \sum_{i=1}^{k} \overline{a_i}\mu_i = \sum_{i=1}^{k} b_i\mu_i = \sum_{i=1}^{k} \overline{b_i}\mu_i = 0\,, \tag{1.4}$$

then the σ-quotient

$$f(u,v) = \prod_{i=1}^{k} \frac{\sigma(a_i u + b_i v + c_i + \mu_i)}{\sigma(a_i u + b_i v + c_i)} \tag{1.5}$$

is $\mathcal{O}_{-d} \times \mathcal{O}_{-d}$ -periodic.

Proof. Let us recall from [1] that the integers ring of an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ is of the form $\mathcal{O}_{-d} = \mathbb{Z} + 2\omega\mathbb{Z}$ for

$$2\omega = \begin{cases} \sqrt{-d} & \text{for } -d \not\equiv 1(\mathrm{mod}\,4), \\ \frac{-1+\sqrt{-d}}{2} & \text{for } -d \equiv 1(\mathrm{mod}\,4). \end{cases}$$

Any $\nu \in \mathcal{O}_{-d}$ has unique representation $\nu = x + 2\omega y$ with

$$x = \frac{2\omega\overline{\nu} - 2\overline{\omega}\nu}{2\omega - 2\overline{\omega}} \in \mathbb{Z}, \quad y = \frac{\nu - \overline{\nu}}{2\omega - 2\overline{\omega}} \in \mathbb{Z}.$$

Legendre's equality

$$\eta(2\omega) - 2\omega\eta(1) = 2\pi\sqrt{-1},$$

(cf.[6]) implies that

$$\eta(\nu) = \nu\eta(1) + \frac{\nu - \overline{\nu}}{2\omega - 2\overline{\omega}} 2\pi\sqrt{-1} \quad \text{for } \forall \nu \in \mathcal{O}_{-d}.$$

As a result,

$$\sum_{i=1}^{k} \eta(a_i\lambda)\mu_i = \left(\sum_{i=1}^{k} a_i\mu_i\right)\lambda\eta(1) + \left(\sum_{i=1}^{k} a_i\mu_i\right)\frac{2\pi\sqrt{-1}\lambda}{2\omega - 2\overline{\omega}} - \left(\sum_{i=1}^{k} \overline{a_i}\mu_i\right)\frac{\overline{\lambda}2\pi\sqrt{-1}}{2\omega - 2\overline{\omega}}.$$

Lemma 4 is proved □

Mutually parallel smooth elliptic curves $T_1, \ldots, T_k$ admit liftings

$$T_i = \{(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) \,|\, a_1 u + b_1 v + c_i \in \mathcal{O}_{-d}\}.$$

For arbitrary $\mu_j \in \mathbb{C}$ with $\sum_{i=1}^{k} \mu_i = 0$, the σ-quotient

$$f(u,v) = \prod_{i=1}^{k} \frac{\sigma(a_1 u + b_1 v + c_i + \mu_i)}{\sigma(a_1 u + b_1 v + c_i)} \tag{1.6}$$

belongs to $\mathcal{L}_A(T, T^{\rm sing})$ and has smooth pole divisor $(f)_\infty = \sum_{i=1}^{k} T_i$. Following [4], we say that (1.6) is a k-fold parallel σ-quotient. A σ-quotient (1.5) has smooth pole divisor if and only if it is k-fold parallel.

Definition 5. *A special σ-quotient of order k is a function of the form (1.5), which is subject to (1.4), has singular pole divisor $(f)_\infty$ and $\mu_i \notin \mathcal{O}_{-d}$ for all $1 \leq i \leq k$.*

Lemma 6. *If $f \in \mathcal{L}_A(T, T^{\rm sing})$ is a special σ-quotient of order $k \geq 2$, then at any point $p \in (f)_\infty^{\rm sing}$ the multiplicity $m_p(f)_\infty$ satisfies*

$$2 \leq m_p(f)_\infty \leq \left[\frac{k+1}{2}\right],$$

where $\left[\frac{k+1}{2}\right]$ is the greatest natural number, non-exceeding $\frac{k+1}{2}$.

In particular, $\mathcal{L}_A(T, T^{\rm isng})$ does not contain a special σ-quotient of order 2.

Proof. The smoothness of the irreducible components $T_1, \dots, T_k$ of $(f)_\infty$ results in $(f)_\infty^{\rm sing} \subset \sum_{1 \leq i < j \leq k} (T_i \cap T_j)$ and implies that $m_p(f)_\infty \geq 2$ for all $p \in (f)_\infty^{\rm sing}$. Suppose that $m_p(f)_\infty = m$ for some $2 \leq m \leq k$. After an eventual permutation of $T_1, \dots, T_k$, one can assume that $p \in T_1 \cap \ldots \cap T_m$ and $p \notin T_{m+1} + \ldots + T_k$. Then

$$m_p(f) + 1 = m_p(f)_0 - m_p(f)_\infty + 1 = m_p(f)_0 - m + 1 \geq 0$$

requires the existence of $D_{m+1}, \dots, D_{2m-1} \subset (f)_0 = \sum_{i=1}^{k} D_i$ with $p \in D_{m+1} \cap \ldots \cap D_{2m-1}$, after a further permutation of $D_{m+1}, \dots, D_k$. Now $2m - 1 \leq k$ implies that $m_p(f)_\infty = m \leq \left[\frac{k+1}{2}\right]$.

In particular, for $k = 2$ the inequality $2 \leq m_p(f)_\infty \leq \left[\frac{3}{2}\right]$ cannot be satisfied. □

Proposition 7. *If*

$$f(u,v) = \prod_{i=1}^{3} \frac{\sigma(a_i u + b_i v + c_i + \mu_i)}{\sigma(a_i u + b_i v + c_i)} \tag{1.7}$$

is a special σ-quotient from $\mathcal{L}_A(T, T^{\rm sing})$, then $T_1 \cap T_2 \cap T_3 = \emptyset$ and the intersection numbers $T_1.T_2 = T_2.T_3 = T_3.T_1 \in \mathbb{N}$ are equal.

Proof. By Lemma 6 there follows $m_p(f)_\infty = 2$ for $\forall p \in (f)_\infty^{\rm sing}$. In particular, $(f)_\infty = T_1 + T_2 + T_3$ has no triple point and $T_1 \cap T_2 \cap T_3 = \emptyset$. Further, for any $p \in T_i \cap T_j$ the condition $m_p(f) + 1 \geq 0$ requires that $p \in D_l$, therefore $\mu_l \in \mathcal{O}_{-d} - \Delta_{ij}^{-1}\Delta$ and $\Delta_{ij}^{-1}\Delta_{jl}, \Delta_{ij}^{-1}\Delta_{li} \in \mathcal{O}_{-d}$, according to Lemma 1 (ii). A cyclic change of the even permutation $\{i,j,l\}$ by $\{j,l,i\}$ and $\{l,i,j\}$ results in $\Delta_{jl}^{-1}\Delta_{li}, \Delta_{jl}^{-1}\Delta_{ij} \in \mathcal{O}_{-d}$ and, respectively, $\Delta_{li}^{-1}\Delta_{ij}, \Delta_{li}^{-1}\Delta_{jl} \in \mathcal{O}_{-d}$. Consequently, $\Delta_{ij}^{-1}\Delta_{jl}, \Delta_{ij}^{-1}\Delta_{li} \in \mathcal{O}_{-d}^*$, whereas $N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{ij}) = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{jl}) = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{li})$. Now, by Lemma 1 (i) it follows that $T_i.T_j = T_j.T_l = T_l.T_i$. □

Definition 8. *The divisor* $T_1 + T_2 + T_3$ *with three smooth elliptic irreducible components is called a triangle if* $T_1 \cap T_2 \cap T_3 = \emptyset$ *and* $T_1.T_2 = T_2.T_3 = T_3.T_1 = 1$.

Examples of special σ-quotients with triangular pole divisors are constructed by Holzapfel in [4]. We show that any triangular divisor can be realized as a pole divisor of a special σ-quotient $f \in \mathcal{L}_A(T, T^{\text{sing}})$ and provide a general formula for such f.

Proposition 9. *Let* $T_i = \{(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) \,|\, a'_i u + b'_i v + c'_i \in \mathcal{O}_{-d}\}$ *with* $1 \leq i \leq 3$ *be the smooth irreducible elliptic components of a triangle* $T_1 + T_2 + T_3$ *and* $a_i = \Delta'_{jl} a'_i$, $b_i = \Delta'_{jl} b'_i$, $c_i = \Delta'_{jl} c'_i$. *Then* $a_1 + a_2 + a_3 = 0$, $b_1 + b_2 + b_3 = 0$, $\Delta_{12}^{-1}\Delta \notin \mathcal{O}_{-d}$ *and for any* $\nu \in \mathcal{O}_{-d}$ *the function*

$$f(u,v) = \prod_{i=1}^{3} \frac{\sigma(a_i u + b_i v + c_i - \Delta_{12}^{-1}\Delta + \nu)}{\sigma(a_i u + b_i v + c_i)} \tag{1.8}$$

is a special σ*-quotient from* $\mathcal{L}_A(T, T^{\text{sing}})$ *with pole divisor* $(f)_\infty = T_1 + T_2 + T_3$.

Proof. Let $v'_i = \begin{pmatrix} a'_i \\ b'_i \end{pmatrix}$ for $1 \leq i \leq 3$. Expanding along the third row, one obtains

$$0 = \begin{vmatrix} a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \\ a'_1 & a'_2 & a'_3 \end{vmatrix} = \Delta'_{23} a'_1 + \Delta'_{31} a'_2 + \Delta'_{12} a'_3 = 0,$$

$$0 = \begin{vmatrix} a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \\ b'_1 & b'_2 & b'_3 \end{vmatrix} = \Delta'_{23} b'_1 + \Delta'_{31} b'_2 + \Delta'_{12} b'_3 = 0,$$

and concludes that

$$v_1 + v_2 + v_3 = \Delta'_{23} v'_1 + \Delta'_{31} v'_2 + \Delta'_{12} v'_3 = 0_{2\times 1}, \quad \Delta_{12} = \Delta_{23} = \Delta_{31}. \tag{1.9}$$

Now, according to Lemma 1 (iii), $T_1 \cap T_2 \cap T_3 = \emptyset$ is equivalent to $\Delta \notin \Delta_{12}\mathcal{O}_{-d}$. Then the condition $m_p(f)_0 \geq m_p(f)_\infty - 1$ for $\forall p \in (f)_\infty^{\text{sing}}$ reduces to $T_i \cap T_j \subset D_l$ for any even permutation $\{i, j, l\}$ of $\{1, 2, 3\}$. Making use of Lemma 1 (ii), one can choose $\mu_1 = \mu_2 = \mu_3 = \nu - \Delta_{12}^{-1}\Delta \notin \mathcal{O}_{-d}$. Then (1.9) implies (1.4) from Lemma 4 and reveals that (1.8) is a special σ-quotient from $\mathcal{L}_A(T, T^{\text{sing}})$. □

Definition 10. *The special* σ*-quotients (1.8) from* $\mathcal{L}_A(T, T^{\text{sing}})$ *with triangular pole divisors* $(f)_\infty = T_1 + T_2 + T_3$ *are called triangular.*

For elliptic curves $T_i = \{(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) \,|\, a_i u + b_i v + c_i \in \mathcal{O}_{-d}\}$, $1 \leq i \leq 2$ with minimal fundamental groups $\pi_1(T_i) = \pi_1(E) = \mathcal{O}_{-d}$ and intersection number $T_1.T_2 = 1$, Lemma 1 (i) implies that

$$M = \begin{pmatrix} a_2 & b_2 \\ a_1 & b_1 \end{pmatrix} \in GL_2(\mathcal{O}_{-d}).$$

As a result, there arises an automorphism

$$\varphi : A \longrightarrow A,$$

$$\varphi(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) = \left[M \left(\begin{array}{c} \overline{u} \\ \overline{v} \end{array} \right) + \left(\begin{array}{c} \overline{c_2} \\ \overline{c_1} \end{array} \right) \right]^t$$

with $\varphi(T_1) = E \times \check{o}_E$, $\varphi(T_2) = \check{o}_E \times E$. Making use of $\sigma(\alpha z) = \alpha\sigma(z)$ for $\forall \alpha \in \mathcal{O}^*_{-d}$, $\forall z \in \mathbb{C}$, one observes that any triangular σ-quotient can be reduced by an automorphism of A to the form

$$f_{012}(u,v) = \frac{\sigma(u + a_0^{-1}c_0)\sigma(v + b_0^{-1}c_0)\sigma(a_0u + b_0v)}{\sigma(u)\sigma(v)\sigma(a_0u + b_0v + c_0)} \tag{1.10}$$

with $a_0, b_0 \in \mathcal{O}^*_{-d}$, $c_0 \notin \mathcal{O}_{-d}$.

We are going to describe the complete divisor of a triangular σ-quotient.

Definition 11. *The divisor* $D = \sum\limits_{i=0}^{2} D_i - \sum\limits_{i=0}^{2} T_i$ *with smooth elliptic irreducible components* D_i, T_j *is called a tetrahedron (cf. Figure 1) if:*

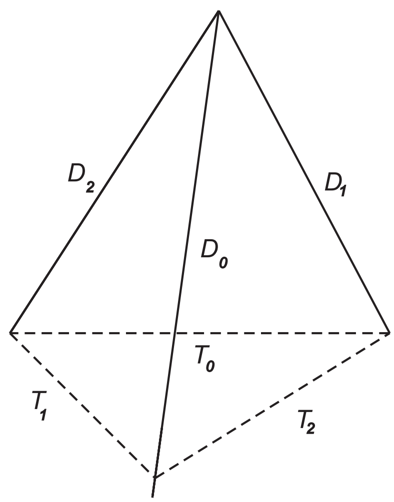

Figure 1: Tetrahedron

(i) $\sum\limits_{i=0}^{2} T_i$ *is a triangle;*

(ii) D_i *are parallel to* T_i *for all* $0 \leq i \leq 2$*;*

(iii) $D_0 \cap D_1 \cap D_2 = D_0 \cap D_1 = D_1 \cap D_2 = D_2 \cap D_0 = \{p_0\}$ *for some point* $p_0 \in A$*;*

(iv) $\left(\sum_{i=0}^{2} D_i\right) \cap \left(\sum_{i=0}^{2} T_i\right) = \left(\sum_{i=0}^{2} T_i\right)^{\text{sing}} \subset \left(\sum_{i=0}^{2} D_i\right)^{\text{smooth}}.$

Definition 12. *An inscribed (ordered) pair of triangles (cf. Figure 2) is a divisor* $D = \sum_{i=0}^{2} D_i - \sum_{i=0}^{2} T_i$*, such that:*

(i) $\sum_{i=0}^{2} D_i$ *and* $\sum_{i=0}^{2} T_i$ *are triangles;*

(ii) D_i *are parallel to* T_i *for all* $0 \leq i \leq 2$*;*

(iii) $\left(\sum_{i=0}^{2} D_i\right) \cap \left(\sum_{i=0}^{2} T_i\right) = \left(\sum_{i=0}^{2} T_i\right)^{\text{sing}} \subset \left(\sum_{i=0}^{2} D_i\right)^{\text{smooth}}.$

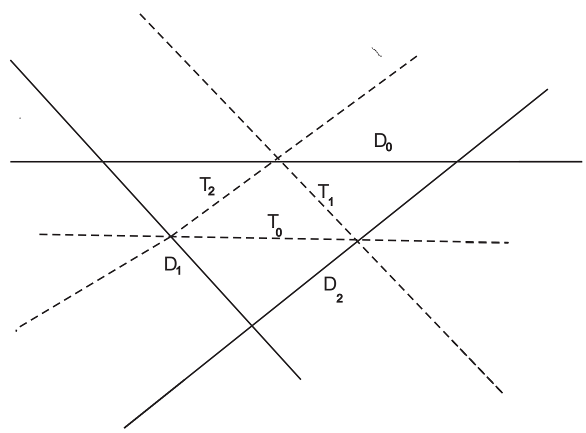

Figure 2: Inscribed (ordered) pair of triangles

An explicit calculation of the singular points of the complete divisor yields the following

Corollary 13. *Let (1.10) with* $a_0, b_0 \in \mathcal{O}_{-d}^*$*,* $c_0 \notin \mathcal{O}_{-d}$ *be a triangular* σ*-quotient with complete divisor* $(f_{012}) = \sum_{i=0}^{2} D_i - \sum_{i=0}^{2} T_i$*. Then:*

(i) $c_0 + \mathcal{O}_{-d} \in E_{2-\text{tor}}$ *is a* 2*-torsion point if and only if* (f_{012}) *is a tetrahedron;*

(ii) $c_0 + \mathcal{O}_{-d} \notin E_{2-tor}$ *exactly when* (f_{012}) *is an inscribed pair of triangles.*

In either case, the multiplicity $m_p(f_{012}) = -1$ *at all* $p \in (f_{012})_\infty \cap T^{\text{sing}}$*.*

In [4] Holzapfel introduces the idea for detecting the linear independence of co-abelian modular forms by the poles of the corresponding transfers to abelian functions. Instead of his strongly descending divisor condition, we use a natural complete decreasing flag on $[\Gamma, 1]$. That enables to supply a criterion for some modular forms to constitute a basis of $[\Gamma, 1]$ and to show that $[\Gamma, 1]$ has always a basis of the considered form.

Observe that the subspaces

$$V_i = j_1[\Gamma, 1]_i := \{\omega \in j_1[\Gamma, 1] \,|\, \omega(\kappa_1) = \ldots = \omega(\kappa_{i-1}) = 0\}$$

of $V_1 = j_1[\Gamma, 1]$ form a non-increasing flag

$$j_1[\Gamma, 1] = V_1 \supseteq V_2 \supseteq \ldots \supseteq V_{m-1} \supseteq V_m \supseteq \ldots \supseteq V_h \supseteq V_{h+1} = j_1[\Gamma, 1]_{\mathrm{cusp}}.$$

For any $\omega, \omega' \in V_i$ one has $\omega'(\kappa_i)\omega - \omega(\kappa_i)\omega' \in V_{i+1}$, so that $0 \leq \dim_{\mathbb{C}} (V_i/V_{i+1}) \leq 1$ for all $1 \leq i \leq h$. We prove that there is a permutation of the cusps $\kappa_1, \ldots, \kappa_h$, so that $V_i/V_{i+1} \simeq \mathbb{C}$ for $1 \leq i \leq m$ and $V_{m+1} = V_{m+2} = \ldots = V_{h+1} = j_1[\Gamma, 1]_{\mathrm{cusp}} \simeq \mathbb{C}$. If so, then $\dim_{\mathbb{C}}[\Gamma, 1] = m + 1$.

Proposition 14. *If the pole divisors of $f_i \in \mathcal{L}_A(T, T^{\mathrm{sing}})$ are subject to*

$$T_i \subset (f_i)_\infty \subseteq T_i + T_{i+1} + \ldots + T_h \quad \text{for all} \quad 1 \leq i \leq m,$$

then $\omega_i = \tau_1^{-1}(f_i) \in j_1[\Gamma, 1]$ with $1 \leq i \leq m$ form a basis of a complement of $V_{m+1} = j_1[\Gamma, 1]_{m+1}$.

In particular, if $V_{m+1} = V_{h+1} = j_1[\Gamma, 1]_{\mathrm{cusp}}$, then $j_1(\eta_o), \omega_1, \ldots, \omega_m$ is a $\mathbb{C}$-basis of $j_1[\Gamma, 1]$.

Proof. It suffices to show that for arbitrary $b_1, \ldots, b_m \in \mathbb{C}$ the linear system

$$\sum_{i=1}^{m} \omega_i(\kappa_j) t_i = b_j, \quad 1 \leq j \leq m \tag{1.11}$$

has a unique solution $(t_1, \ldots, t_m)$. On one hand, that implies the linear independence of $\omega_1, \ldots, \omega_m$ over $\mathbb{C}$. On the other hand, for any $\omega \in j_1[\Gamma, 1]$ there is uniquely determined $\sum_{i=1}^{m} c_i \omega_i$ with $\omega_0 = \omega - \sum_{i=1}^{m} c_i \omega_i \in j_1[\Gamma, 1]_{m+1} = V_{m+1}$. In other words, $j_1[\Gamma, 1] = \mathrm{Span}_{\mathbb{C}}(\omega_1, \ldots, \omega_m) \oplus V_{m+1}$, so that $\omega_1, \ldots, \omega_m$ is a basis of the complement $\mathrm{Span}_{\mathbb{C}}(\omega_1, \ldots, \omega_m)$ of V_{m+1}.

Towards the existence of a unique solution of (1.11), note that the requirement $T_i \subset (\tau_1(\omega_i))_\infty \subseteq T_i + T_{i+1} + \ldots + T_h$ is equivalent to $\omega_i(\kappa_i) \neq 0$ and $\omega_i(\kappa_1) = \omega_i(\kappa_2) = \ldots = \omega_i(\kappa_{i-1}) = 0$. Thus, (1.11) is of the form

$$\begin{pmatrix} \omega_1(\kappa_1) & \ldots & 0 & \ldots & 0 \\ \omega_1(\kappa_i) & \ldots & \omega_i(\kappa_i) & \ldots & 0 \\ \omega_1(\kappa_m) & \ldots & \omega_i(\kappa_m) & \ldots & \omega_m(\kappa_m) \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_i \\ \vdots \\ t_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{pmatrix},$$

with non-degenerate, lower-triangular coefficient matrix and has unique solution for all $b_1, \dots, b_m \in \mathbb{C}$.

In the case of $V_{m+1} = V_{h+1} = j_1[\Gamma, 1]_{\text{cusp}}$, note that $j_1[\Gamma, 1]_{\text{cusp}} = \mathbb{C} j_1(\eta_o)$ with $\tau_1 j_1(\eta_o) = 1 \in \mathcal{L}_A(T, T^{\text{sing}})$, so that $j_1(\eta_o), \omega_1, \dots, \omega_m$ is a $\mathbb{C}$-basis of $j_1[\Gamma, 1]$. □

The next proposition establishes that $j_1[\Gamma, 1]$ has always a $\mathbb{C}$-basis of the considered form.

Proposition 15. *Let $\Gamma \subset SU_{2,1}$ be a freely acting, co-abelian Picard modular group and $\dim_{\mathbb{C}}[\Gamma, 1] = m+1$. Then there is a permutation $\{\kappa_1, \dots, \kappa_m, \kappa_{m+1}, \dots, \kappa_h\}$ of the Γ-cusps, such that*

$$V_1/V_2 \simeq V_2/V_3 \simeq \dots \simeq V_m/V_{m+1} \simeq \mathbb{C}, \quad V_{m+1} = V_{m+2} = \dots = V_{h+1} = j_1[\Gamma, 1]_{\text{cusp}}.$$

Any $\omega_i \in V_i \setminus V_{i+1}$ transfers to $\tau_1(\omega_i) \in \mathcal{L}_A(T, T^{\text{sing}})$ with

$$T_i \subset (\tau_1(\omega_i))_\infty \subseteq T_i + T_{i+1} + \dots + T_h \quad \text{for} \quad 1 \leq i \leq m$$

and $j_1(\eta_o), \omega_1, \dots, \omega_m$ is a $\mathbb{C}$-basis of $V_1 = j_1[\Gamma, 1]$.

In particular, if $T_{h-1}.T_h = 1$ then $V_{h-1} = j_1[\Gamma, 1]_{\text{cusp}}$ and $\dim[\Gamma, 1] \leq h - 1$.

Proof. If $V_1 = V_{h+1}$, then there is nothing to be proved. From now on, we assume that $\dim V_1/V_{h+1} = m \in \mathbb{N}$. By induction on $1 \leq i \leq m$, we establish the existence of $\omega_j \in V_j \setminus V_{j+1}$ for all $1 \leq j \leq i$. First of all, for any $\omega_1 \in V_1 \setminus V_{h+1}$ there exists a cusp κ_1 with $\omega_1(\kappa_1) \neq 0$. Then for an arbitrary permutation of the remaining cusps, one has $\omega_1 \in V_1 \setminus V_2$. If we have chosen $\omega_j \in V_j \setminus V_{j+1}$ for $1 \leq j \leq i-1$ and $V_i \supsetneq V_{h+1}$, then for an arbitrary $\omega_i \in V_i \setminus V_{h+1}$ there exists a permutation of $\{\kappa_i, \kappa_{i+1}, \dots, \kappa_h\}$, such that $\omega_i(\kappa_i) \neq 0$. Clearly, $\omega_i \in V_i \setminus V_{i+1}$ and we have obtained a basis $j_1(\eta_o), \omega_1, \dots, \omega_m$ of $V_1 = j_1[\Gamma, 1]$. The conditions $\omega_i \in V_i \setminus V_{i+1}$ amount to $T_i \subset (\tau_1(\omega_i))_\infty$ and $T_j \not\subseteq (\tau_1(\omega_i))_\infty$ for all $1 \leq j \leq i-1$.

If $T_{h-1}.T_h = 1$, then up to an automorphism of A, one can assume that $T_{h-1} = E \times \check{o}_E$ and $T_h = \check{o}_E \times E$. We claim that $\mathcal{L}_A((E \times \check{o}_E) + (\check{o}_E \times E)) = \mathbb{C}$, so that $\dim_{\mathbb{C}}[\Gamma, 1] = m + 1 \leq h - 1$. Indeed, for an arbitrary $Q \in E \setminus \check{o}_E$ the restriction $f|_{E \times Q}$ is an elliptic function of order 1. Therefore $f|_{E \times Q} \equiv C(Q) \in \mathbb{C}$ is a constant. Similarly, $f|_{P \times E} \equiv C'(P) \in \mathbb{C}$ for any $P \in E \setminus \check{o}_E$. As a result, $C'(P) = f(P, Q) = C(Q)$ for all $Q \in E$ and $f|_A$ is constant. □

Proposition 16. (Holzapfel [5]) *Let us fix the half-periods $\omega_1 = \frac{1}{2}$, $\omega_2 = \frac{i}{2}$, $\omega_3 = \omega_1 + \omega_2$ of the lattice $\pi_1(E) = \mathcal{O}_{-1} = \mathbb{Z} + i\mathbb{Z}$, the 2-torsion points $Q_0 := 0 (\operatorname{mod} \mathbb{Z} + i\mathbb{Z}) \in E$, $Q_j := \omega_j (\operatorname{mod} \mathbb{Z} + i\mathbb{Z}) \in E$ for $1 \leq j \leq 3$ and $Q_{ij} := (Q_i, Q_j) \in A$. Consider the elliptic curves*

$$T_k = \{(u + \pi_1(E), v + \pi_1(E)) \mid u - i^k v \in \pi_1(E)\} \quad \text{for} \quad 1 \leq k \leq 4,$$

$$T_{4+k} = \{(u + \pi_1(E), v + \pi_1(E)) \mid u - \omega_k \in \pi_1(E)\} \quad \text{for} \quad 1 \leq k \leq 2,$$

$$T_{6+k} = \{(u + \pi_1(E), v + \pi_1(E)) \,|\, v - \omega_k \in \pi_1(E)\} \quad \text{for} \quad 1 \leq k \leq 2.$$

Then the blow-up of A at the singular points

$$S_1 = Q_{00}, \quad S_2 = Q_{33}, \quad S_3 = Q_{11}, \quad S_4 = Q_{12}, \quad S_5 = Q_{21}, \quad S_6 = Q_{22}$$

of $T_{\sqrt{-1}}^{(6,8)} = \sum_{k=1}^{8} T_k$ *is the toroidal compactification* $(\mathbb{B}/\Gamma_1)'$ *of a ball quotient* $\mathbb{B}/\Gamma_1$ *by a freely acting Picard modular group* Γ_1 *over the Gaussian integers* $\mathbb{Z}[i]$.

The self-intersection matrix $M(6,8) \in \mathbb{Z}_{6\times 8}$ of $T_{\sqrt{-1}}^{(6,8)}$ is defined to have entries $M(6,8)_{ij} = 1$ for $S_i \in T_j$ and $M(6,8)_{ij} = 0$ for $S_i \not\in T_j$. Straightforwardly,

$$M(6,8) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

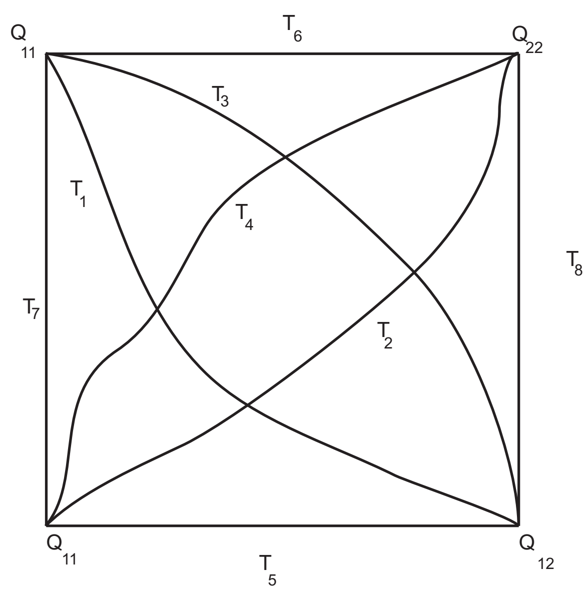

Figure 3: The incidence relations of $T_{\sqrt{-1}}^{(6,8)}$ and $\sum_{i=1}^{2}\sum_{j=1}^{2} Q_{ij} \subset \left(T_{\sqrt{-1}}^{(6,8)}\right)^{\text{sing}}$.

According to $Q_{00}, Q_{33} \in T_k$ or $\forall 1 \leq k \leq 4$, there are no triangles $T_i + T_j + T_k \subset T_{\sqrt{-1}}^{(6,8)}$ with $1 \leq i < j \leq 4$, $1 \leq i < j < k \leq 8$. Bearing in mind that

$\left(T^{(6,8)}_{\sqrt{-1}}\right)^{\mathrm{sing}} \cap \left(\sum\limits_{k=5}^{8} T_k\right) = \sum\limits_{i=1}^{2}\sum\limits_{j=1}^{2} Q_{ij}$, one makes use of Figure 3 and recognizes the triangles $T_{2k-1} + T_{4+m} + T_{6+m}$, $T_{2k} + T_{4+m} + T_{9-m}$ with $1 \leq k, m \leq 2$. An immediate application of Proposition 9 with $\nu = 2\omega_m$ and, respectively, $\nu = \omega_3 + \omega_m + (-1)^{k+1}\omega_{3-m}$, yields the following

Corollary 17. *The space* $\mathcal{L}_A\left(T^{(6,8)}_{\sqrt{-1}}, \left(T^{(6,8)}_{\sqrt{-1}}\right)^{\mathrm{sing}}\right)$ *contains the binary parallel*

$$f_{56}(u,v) = \frac{\sigma(u-\omega_1-\mu_1)\sigma(u-\omega_2+\mu_1)}{\sigma(u-\omega_1)\sigma(u-\omega_2)},$$

$$f_{78}(u,v) = \frac{\sigma(v-\omega_1-\mu_2)\sigma(v-\omega_2+\mu_2)}{\sigma(v-\omega_1)\sigma(v-\omega_2)}$$

and the triangular σ*-quotients*

$$f_{2k-1,4+m,6+m}(u,v) = \frac{\sigma(u+(-1)^k iv+\omega_3)\sigma(-u+\omega_m+\omega_3)\sigma((-1)^{k+1}iv+(-1)^k i\omega_m+\omega_3)}{\sigma(u+(-1)^k iv)\sigma(-u+\omega_m)\sigma((-1)^{k+1}iv+(-1)^k i\omega_m)}$$

$$f_{2k,4+m,9-m}(u,v) = \frac{\sigma(u+(-1)^{k+1}v+\omega_3)\sigma(-u+\omega_m+\omega_3)\sigma((-1)^k v+(-1)^{k+1}\omega_{3-m}+\omega_3)}{\sigma(u+(-1)^{k+1}v)\sigma(-u+\omega_m)\sigma((-1)^k v+(-1)^{k+1}\omega_{3-m})}$$

with arbitrary $1 \leq k, m \leq 2$.

Proposition 14 provides the following

Corollary 18. *If* f_{pq} *and* f_{ijk} *are the binary parallel and triangular* σ*-quotients from the space* $\mathcal{L}_A\left(T^{(6,8)}_{\sqrt{-1}}, \left(T^{(6,8)}_{\sqrt{-1}}\right)^{\mathrm{sing}}\right)$ *and* $\omega_{pq} = \tau_1^{-1}(f_{pq})$, $\omega_{ijk} = \tau_1^{-1}(f_{ijk})$, *then*

$$\omega_{157}, \quad \omega_{258}, \quad \omega_{368}, \quad \omega_{467}, \quad \omega_{56}, \quad \omega_{78}, \quad j_1(\eta_o)$$

is a $\mathbb{C}$*-basis of* $j_1[\Gamma_1, 1]$.

In particular, $\dim_{\mathbb{C}}[\Gamma, 1] = 7$.

2. SUFFICIENT CONDITIONS FOR THE NORMAL GENERATION OF A SPACE OF LOGARITHMIC CANONICAL SECTIONS

Definition 19. *A holomorphic line bundle* $\mathcal{E}$ *on an algebraic variety* X *is sufficiently ample if the holomorphic sections of a sufficiently large tensor power* $\mathcal{E}^{\otimes m}$ *provide a projective embedding of* X.

Definition 20. *A holomorphic line bundle $\mathcal{E}$ over an algebraic variety X is globally generated if the global holomorphic sections of $\mathcal{E}$ determine a regular projective morphism.*

A subspace $V \subseteq H^0(X, \mathcal{E})$ is globally generated if some (and therefore any) basis of V provides a regular projective morphism $X \to \mathbb{P}(V)$.

Definition 21. *A holomorphic line bundle $\mathcal{E}$ over an algebraic manifold X is normally generated if $\mathcal{E}$ is globally generated and $H^0(X, \mathcal{E})$ defines a projective immersion of X with normal image.*

A subspace $V \subseteq H^0(X, \mathcal{E})$ is normally generated if it is globally generated and the morphism $X \to \mathbb{P}(V)$ is a projective immersion with normal image.

The normal generation of a sufficiently ample line bundle is a classical topic under study. Various works provide normally generated and non-normally generated line bundles over curves and abelian varieties. According to [2], if $\mathcal{E}$ is a sufficiently ample line bundle on an abelian variety of dimension n, then $\mathcal{E}^{\otimes(n-1)}$ is normally generated. In particular, any sufficiently ample line bundle on an abelian surface is normally generated.

Our aim is to provide sufficient conditions for the normal generation of a subspace $V \subseteq H^0(A', \Omega^2_{A'}(T'))$ over the Baily-Borel compactification $\widehat{A}$. That cannot be derived from the normal generation of a subspace $W \subseteq H^0(A, \mathcal{E})$ of holomorphic sections of a line bundle $\mathcal{E} \to A$. Namely, $\xi^* W$ cannot be a normally generated space of global holomorphic sections of $\xi^*\mathcal{E}$, as far as the morphism, associated with $\xi^* W$ is not immersive on the exceptional divisor $L = \xi^{-1}(T^{\text{sing}})$ of $\xi : A' \to A$.

Corollary 22. *Let X be an irreducible normal projective variety X and $f : X \to Y$ be a finite, regular, generically injective morphism onto Y. Then $f : X \to Y$ is a regular immersion with normal image Y.*

Proof. If $f : X \to Y$ is a regular morphism of degree $d \in \mathbb{N}$, then the generic fiber of f consists of d points, while the exceptional ones are constituted by $\leq d$ points. In particular, for $d = 1$, any regular, generically injective morphism is bijective onto its image. As a result, $f : X \to Y$ is a regular immersion with normal image. ☐

Our specific considerations will be based on the following immediate consequence of Corollary 22

Corollary 23. *Let X be an irreducible normal projective variety, $\mathcal{E} \to X$ be e holomorphic line bundle over X and $V \subseteq H^0(X, \mathcal{E})$ be a space of global holomorphic sections of $\mathcal{E}$. If $f : X \to \mathbb{P}(V)$ is a finite, regular, generically injective morphism then V is normally generated.*

Lemma 24. *A subspace $V \subseteq H^0(A', \Omega^2_{A'}(T'))$, containing the cuspidal form $j_1(\eta_o)$, is globally generated over $\widehat{A}$ if and only if it satisfies simultaneously the following two geometric conditions:*

(i) *for any irreducible component* T_i *of* T *there is* $\omega_i \in V$ *with* $(\tau_1(\omega_i))_\infty \supset T_i$*;*

(ii) *for any* $p \in T^{\rm sing}$ *there exists* $\omega_p \in V$ *with* $m_p(\tau_1(\omega_p)) = -1$.

Proof. The space V is globally generated over $\widehat{A}$ exactly when for any point $q \in \widehat{A}$ there is $v_q \in V$ with $v_q(q) \neq 0$. If $q \in \left(\widehat{\mathbb{B}/\Gamma}\right) \setminus \left(L \cup \sum_{i=1}^{h} \kappa_i\right)$, then $j_1(\eta_o)(q) \neq 0$. A modular form $\omega_i \in V$ does not vanish on the cusp κ_i if and only if $T_i \subset (\tau_1(\omega_i))_\infty$. A modular form $\omega_p \in V$ takes non-zero values on the rational (-1)-curve $\xi^{-1}(p)$ exactly when the multiplicity $m_p(\tau_1(\omega_p)) = -1$. □

From now on, we say briefly that a modular form $\omega \in H^0(A', \Omega^2_{A'}(T'))$ is binary parallel or triangular if its transfer $\tau_1(\omega) \in \mathcal{L}_A(T, T^{\rm sing})$ is binary parallel or, respectively, triangular.

Proposition 25. *Let us suppose that the subspace* $V \subseteq H^0(A', \Omega^2_{A'}(T'))$ *contains the cuspidal form* $j_1(\eta_o)$*, two binary parallel forms* ω_{13}*,* ω_{24}*, a triangular* ω_{012} *with* $T_0 \cap T_3 \cap T_4 = \emptyset$ *and satisfies the following three conditions:*

(i) *for any* $i \notin \{0, 1, \ldots, 4\}$ *there exists* $\omega_i \in V$ *with* $(\tau_1(\omega_i))_\infty \supset T_i$*;*

(ii) *for any* $p \in T^{\rm sing} \setminus \left(\sum_{j=0}^{4} T_j\right)$ *there exists* $\omega_p \in V$ *with* $m_p(\tau_1(\omega_p)) = -1$*;*

(iii) *for any* $1 \leq i < j \leq h$ *there is* $\omega_{ij} \in V$*, such that* $(\tau_1(\omega_{ij}))_\infty$ *contains exactly one of* T_i *or* T_j*.*

Then V *is normally generated.*

Proof. In the presence of Corollary 23, it suffices to establish that the projective morphism $f : \widehat{A} \to \mathbb{P}(V)$, associated with V is regular, finite and generically injective. Assumption (i) from the present proposition and $(\tau_1(\omega_{ij}))_\infty = T_i + T_j$, $(\tau_1(\omega_{012}))_\infty = T_0 + T_1 + T_2$ imply assumption (i) from Lemma 24. Further, noone $p \in T^{\rm sing} \cap (T_1 + T_3)$ belongs to $(\tau_1(\omega_{13}))_0 = D_1 + D_3$, as far as T_1, T_3, D_1 and D_3 are mutually parallel and distinct. Therefore, $m_p(\tau_1(\omega_{13})) = -1$. Similarly, $m_p(\tau_1(\omega_{24})) = -1$ for $p \in T^{\rm sing} \cap (T_2 + T_4)$. By Corollary 13, $m_p(\tau_1(\omega_{012})) = -1$ for all $p \in T^{\rm sing} \cap \left(\sum_{i=0}^{2} T_i\right)$. Combining with assumption (ii) from the present proposition, one obtains (ii) from Lemma 24. That allows to conclude that $f : \widehat{A} \to \mathbb{P}(V)$ is regular.

The assumption (iii) guarantees that $f : \widehat{A} \to f(\widehat{A}) \subset \mathbb{P}(V)$ is finite. First of all, on $\widehat{A} \setminus [L + (\partial_\Gamma \mathbb{B}/\Gamma)] = (\mathbb{B}/\Gamma) \setminus L = A \setminus T$, the morphism

$$\left(\frac{\omega_{13}}{j_1(\eta_o)} = f_{13} \circ \xi = f_{13}, \frac{\omega_{24}}{j_1(\eta_o)} = f_{24} \circ \xi = f_{24}\right) : (\mathbb{B}/\Gamma) \setminus L = A \setminus T \longrightarrow \mathbb{C}^2$$

is of degree 4. More precisely, if

$$f_{13}(u,v) = \frac{\sigma(u-\mu_1)\sigma(u-c_3+\mu_1)}{\sigma(u)\sigma(u-c_3)},\ f_{24}(u,v) = \frac{\sigma(v-\mu_2)\sigma(v-c_4+\mu_2)}{\sigma(v)\sigma(v-c_4)}, \quad (2.1)$$

then for any $x, y \in \mathbb{P}^1$ the fiber is

$$(f_{13}, f_{24})^{-1}(x,y) = \{(P_i(x), Q_j(y)) \,|\, 1 \leq i, j \leq 2\}$$

with

$$P_1(x) + P_2(x) = \overline{c_3}, \quad Q_1(y) + Q_2(y) = \overline{c_4}.$$

The condition (iii) provides the injectiveness of $f : \partial_\Gamma \mathbb{B}/\Gamma \to f(\partial_\Gamma \mathbb{B}/\Gamma)$, which suffices for $f : L \to f(L)$ to be discrete and, therefore, finite. Otherwise, f contracts some irreducible component $\xi^{-1}(p)$, $p \in T^{\mathrm{sing}}$ of L. If $p \in T_i \cap T_j$ then $\kappa_i, \kappa_j \in \xi^{-1}(p)$, whereas $f(\kappa_i) = f(\kappa_j)$. Thus, $f : L \cup (\partial_\Gamma \mathbb{B}/\Gamma) \to f(L \cup (\partial_\Gamma \mathbb{B}/\Gamma))$ and, therefore, $f : \widehat{A} \to f(\widehat{A})$ is a finite regular morphism.

The generic injectiveness of the projective morphism $f : \widehat{A} \to f(\widehat{A})$ follows from the generic injectiveness of the affine morphism

$$F = \left(\frac{\omega_{13}}{j_1(\eta_o)} = f_{13}, \frac{\omega_{24}}{j_1(\eta_o)} = f_{24}, \frac{\omega_{012}}{j_1(\eta_o)} = f_{012} \right) : (\mathbb{B}/\Gamma) \setminus L = A \setminus T \longrightarrow \mathbb{C}^3.$$

This, in turn, is equivalent to the generic injectiveness of the rational surjective morphism

$$F = (f_{13}, f_{24}, f_{012}) : A \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Let us consider also the rational surjection $F_1 = (f_{13}, f_{24}) : A \to \mathbb{P}^1 \times \mathbb{P}^1$ and its factorization

$$\begin{array}{ccc} A & \xrightarrow{F} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ {\scriptstyle F_1}\downarrow & \swarrow {\scriptstyle \mathrm{pr}_{12}} & \\ \mathbb{P}^1 \times \mathbb{P}^1 & & \end{array}$$

through F and the projection $\mathrm{pr}_{12} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ onto the first two factors. The irreducible components T_1 and T_2 of the triangle $T_0 + T_1 + T_2$ have intersection number $T_1.T_2 = 1$. That allows to assume that $T_1 = \check{o}_E \times E$, $T_2 = E \times \check{o}_E$ and (1.10).

Suppose that $F : A \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is not generically injective. By $F_1 = \mathrm{pr}_{12} \circ F$ and $\deg F_1 = 4$, the generic fiber of F on $F_1^{-1}(x,y)$ consists of 2 or 4 points. In either case, for any $(x,y) \in \mathbb{P}^1 \times \mathbb{P}^1$ there holds at least one of the following pairs of relations:

Case (i):
$$f_{012}(P_1(x), Q_2(y)) = f_{012}(P_2(x), Q_1(y)),$$
$$f_{012}(P_1(x), Q_1(y)) = f_{012}(P_2(x), Q_2(y));$$

Case (ii): $f_{012}(P_1(x), Q_2(y)) = f_{012}(P_2(x), Q_2(y)),$
$f_{012}(P_1(x), Q_1(y)) = f_{012}(P_2(x), Q_1(y));$

Case (iii): $f_{012}(P_1(x), Q_2(y)) = f_{012}(P_1(x), Q_1(y)),$
$f_{012}(P_2(x), Q_2(y)) = f_{012}(P_2(x), Q_1(y)).$

We claim that the relations from at least one case are satisfied identically on $\mathbb{P}^1 \times \mathbb{P}^1$. Otherwise, the locus of either case is a proper analytic subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$ and their union is also a proper analytic subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$. The contradiction implies that for any $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$ there holds identically at least one of the Cases (i), (ii) or (iii). Note that (ii) and (iii) are equivalent under the transposition of the factors of $\mathbb{P}^1 \times \mathbb{P}^1$ and, respectively, of $A = E \times E$.

Without loss of generality, one can suppose that $P_1(\infty) = \check{o}_E$ and $P_2(\infty) = \overline{c_3}$. In Case (i), up to a relabeling of $Q_1(y)$, $Q_2(y)$, one has $Q_1(\infty) = \check{o}_E$, $Q_2(\infty) = \overline{c_4}$. Then

$$\infty = f_{012}(\check{o}_E, \check{o}_E) = f_{012}(P_1(\infty), Q_1(\infty)) = f_{012}(P_2(\infty), Q_2(\infty)) = f_{012}(\overline{c_3}, \overline{c_4}).$$

However, $\overline{c_3} \neq \check{o}_E$, $\overline{c_4} \neq \check{o}_E$ and $T_3 \cap T_4 = \{(\overline{c_3}, \overline{c_4})\} \subsetneqq T_0$ reveal that $f_{012}(\overline{c_3}, \overline{c_4}) \neq \infty$, so that Case (i) does not hold identically on A. Similarly, in Case (ii), there follows

$$\infty = f_{012}(\check{o}_E, \overline{c_4}) = f_{012}(P_1(\infty), Q_2(\infty)) = f_{012}(P_2(\infty), Q_2(\infty)) = f_{012}(\overline{c_3}, \overline{c_4}).$$

The contradiction implies that $F : A \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is generically injective. $\square$

Here is another sufficient condition for a subspace $V \subseteq H^0(A', \Omega^2_{A'}(T'))$ to be normally generated.

Proposition 26. *Let V be a subspace of $H^0(A', \Omega^2_{A'}(T'))$, containing the cuspidal form $j_1(\eta_o)$, a binary parallel ω_{13}, triangular ω_{012}, ω_{234} with $T_0 \cap T_1 \cap T_4 = \emptyset$ and satisfying the following three conditions:*

(i) *for any $i \notin \{0, 1, \dots 4\}$ there exists $\omega_i \in V$ with $(\tau_1(\omega_i))_\infty \supset T_i$;*

(ii) *for any $p \in T^{\text{sing}} \setminus \left(\sum_{j=0}^{4} T_j \right)$ there exists $\omega_p \in V$ with $m_p(\tau_1(\omega_p)) = -1$;*

(iii) *for any $1 \leq i < j \leq h$ there is $\omega_{ij} \in V$, such that $(\tau_1(\omega_{ij}))_\infty$ contains exactly one of T_i or T_j.*

Then V is normally generated.

Proof. As in Proposition 25, first we establish the regularity of the projective morphism $f : \widehat{A} \to f(\widehat{A})$.

Further, $f : \widehat{A} \to f(\widehat{A})$ is finite, as far as the fibers of its restriction on $(\mathbb{B}/\Gamma) \setminus L = A \setminus T$ are contained in the fibers of

$$\left(\frac{\omega_{13}}{j_1(\eta_o)} = f_{13}, \frac{\omega_{012}}{j_1(\eta_o)} = f_{012} \right) : A \setminus T \longrightarrow \mathbb{C}^2.$$

Let $f_{012}(u, v)$ be of the form (1.10) and f_{13} be as in (2.1). Then for any $x, y \in \mathbb{P}^1$ the fiber

$$(f_{13}, f_{012})^{-1}(x, y) = \{(P_i(x), Q_{ij}(x, y)) \,|\, 1 \leq i, j \leq 2\}$$

with

$$P_1(x) + P_2(x) = \overline{c_3}, \quad Q_{i1}(x, y) + Q_{i2}(x, y) = -a_0 b_0^{-1} P_i(x) - b_0^{-1} \overline{c_0}$$

consists of at most four points. The reason is that for any fixed $P_i(x) \in E$ the elliptic function $f_{012}(P_i(x), \ )$ is of order 2. Thus, $(f_{13}, f_{012}) : A \setminus T \to \mathbb{C}^2$ is finite. The assumption (iii) implies that $f : L \cup (\partial_\Gamma \mathbb{B}/\Gamma) \to f\,(L \cup (\partial_\Gamma \mathbb{B}/\Gamma))$ is finite, so that $f : \widehat{A} \to f(\widehat{A})$ is a finite regular morphism.

We derive the generic injectiveness of $f : \widehat{A} \to f(\widehat{A})$ from the generic injectiveness of the affine morphism

$$F = \left(\frac{\omega_{13}}{j_1(\eta_o)} = f_{13}, \frac{\omega_{012}}{j_1(\eta_o)} = f_{012}, \frac{\omega_{234}}{j_1(\eta_o)} = f_{234} \right) : (\mathbb{B}/\Gamma) \setminus L = A \setminus T \longrightarrow \mathbb{C}^3.$$

To this end, let us factor the rational surjection $F_1 = (f_{13}, f_{012}) : A \to \mathbb{P}^1 \times \mathbb{P}^1$ through the rational surjection $F = (f_{13}, f_{012}, f_{234}) : A \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and the projection $\mathrm{pr}_{12} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, along the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{F} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ {\scriptstyle F_1}\downarrow & \swarrow {\scriptstyle \mathrm{pr}_{12}} & \\ \mathbb{P}^1 \times \mathbb{P}^1 & & \end{array} .$$

If F is not generically injective, then at least one of the following three cases holds identically on $\mathbb{P}^1 \times \mathbb{P}^1$:

Case (i):
$$f_{234}(P_1(x), Q_{12}(x, y)) = f_{234}(P_2(x), Q_{21}(x, y)),$$
$$f_{234}(P_1(x), Q_{11}(x, y)) = f_{234}(P_2(x), Q_{22}(x, y));$$

Case (ii):
$$f_{234}(P_1(x), Q_{12}(x, y)) = f_{234}(P_2(x), Q_{22}(x, y)),$$
$$f_{234}(P_1(x), Q_{11}(x, y)) = f_{234}(P_2(x), Q_{21}(x, y));$$

Case (iii):
$$f_{234}(P_1(x), Q_{12}(x, y)) = f_{234}(P_1(x), Q_{11}(x, y)),$$
$$f_{234}(P_2(x), Q_{22}(x, y)) = f_{234}(P_2(x), Q_{21}(x, y)).$$

In either case, denote by $P_1(\infty) = \check{o}_E$ and $P_2(\infty) = \overline{c_3}$ the poles of the elliptic function f_{13} and note that $T_1 = P_1(\infty) \times E$, $T_3 = P_2(\infty) \times E$. Further, let

$Q_{i1}(\infty,\infty) = \check{o}_E$, so that $T_2 = E \times Q_{11}(\infty,\infty) = E \times Q_{21}(\infty,\infty)$. Finally, let $Q_{i2}(\infty,\infty) = -a_0 b_0^{-1} P_i(\infty) - b_0^{-1}\overline{c_0}$, in order to have

$$\{q_{10}\} = T_1 \cap T_0 = \{(P_1(\infty), Q_{12}(\infty,\infty))\},$$

$$\{q_{30}\} = T_3 \cap T_0 = \{(P_2(\infty), Q_{22}(\infty,\infty))\}.$$

Denote also

$$\{q_{12}\} = T_1 \cap T_2 = \{(P_1(\infty), Q_{11}(\infty,\infty))\},$$

$$\{q_{32}\} = T_3 \cap T_2 = \{(P_2(\infty), Q_{21}(\infty,\infty))\}.$$

Bearing in mind that $(f_{234})_\infty = T_2 + T_3 + T_4$, note that $f_{234}(q_{ij}) = \infty$ whenever $\{i,j\} \cap \{2,3,4\} \neq \emptyset$. In the Case (i) one has $f_{234}(q_{10}) = f_{234}(q_{32}) = \infty$. If $q_{10} \in T_2$, then $q_{10} \in T_0 \cap T_1 \cap T_2$, contrary to the assumption that $T_0 + T_1 + T_2$ is a triangle. On the other hand, $T_3 \cap T_1 = \emptyset$ guarantees that $q_{10} \notin T_3$. Therefore $q_{10} \in T_4$ and $q_{10} \in T_0 \cap T_1 \cap T_4 = \emptyset$. The contradiction rejects the Case (i). If the first relation of Case (ii) is identical on $\mathbb{P}^1 \times \mathbb{P}^1$, then $f_{234}(q_{10}) = f_{234}(q_{30}) = \infty$. As in the Case (i), that leads to an absurd. Finally, $f_{234}(q_{10}) = f_{234}(q_{12}) = \infty$ contradicts the hypotheses and establishes that $F = (f_{13}, f_{012}, f_{234}) : A \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is generically injective. □

An immediate application of Proposition 26 to the example from Proposition 16 yields the following

Corollary 27. *In the terms of Proposition 16, the subspace*

$$V_1 = \mathrm{Span}_{\mathbb{C}}(j_1(\eta_o), \omega_{56}, \omega_{157}, \omega_{267}, \omega_{368}, \omega_{458}) \subset H^0(A_1', \Omega^2_{A_1'}(T'))$$

is normally generated, i.e., determines a regular projective immersion

$$f : \widehat{\mathbb{B}/\Gamma_1} \to \mathbb{P}(V_1) = \mathbb{P}^5$$

with normal image.

If one applies Proposition 25 to the cuspidal form $j_1(\eta_o)$, the binary parallel ω_{56}, ω_{78} and triangular ω_{157}, then one needs to adjoin the triangular $\omega_{2,4+k,9-k}$, $\omega_{3,4+l,6+l}$, $\omega_{4,4+m,9-m}$ for some $k, l, m \in \{1, 2\}$. The span of these modular forms is 7-dimensional and depletes the entire $[\Gamma_1, 1]$. It is clear that the normal generation of V_1 implies the normal generation of $H^0(A_1', \Omega^2_{A_1'}(T')) = j_1[\Gamma_1, 1]$.

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3. REFERENCES

1. Borevich, Z. I., I. R. Shafarevich: Number theory. Nauka, 1985.
2. Ewald, G., U. Wessels: On the ampleness of invertible sheaves in complete projective toric varieties. *Results Math.*, **19**, 1991, 275–278.
3. Hemperly, J.: The parabolic contribution to the number of independent automorphic forms on a certain bounded domain. *Amer. J. Math.*, **94**, 1972, 1078–1100.
4. Holzapfel, R.-P.: Jacobi theta embedding of a hyperbolic 4-space with cusps. In : Third International Conference in Geometry, Integrability and Quantization, Varna, June 14-23, 2001, pp. 11–63.
5. Holzapfel, R.-P.: Complex hyperbolic surfaces of abelian type. *Serdica Math. J.*, **30**, 2004, 207–238.
6. Lang, S.: Elliptic functions. Addison-Wesley, 1973.
7. Privalov, I. I.: Introduction to the theory of functions of a complex variable, GITTL, Moscow, 1940 (In Russian).

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