One more proof of Infinitesimal Torelli Theorem for complete intersections in Kähler C-spaces

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Abstract

In the series of papers [5], [6], [7], Konno has proved the Infinitesimal Torelli Theorem for complete intersections $X = X_{d_1,\ldots,d_p}$ in Kähler C-spaces $Y = G/P$ with second Betti number $b_2(Y) = 1$. The argument from [5] makes use of Kii's paper [4], while [6] and [7] exploit Flenner's [2]. After the appearance of [5] in 1986, the author derived the Infinitesimal Torelli Theorem for $X = X_{d_1,\ldots,d_p} \subset Y = G/P$ with $b_2(Y) = 1$ and sufficiently large $d_i \in \mathbb{N}$ from Green's [3] on hypersurfaces of high degree. This proof has been communicated to Konno and never published. In order to make clear my contribution, mentioned in [6] and to emphasize the effectiveness of Konno's arguments from [6], [7], I decided to collect my considerations in the present note.

The simply connected compact Kähler homogeneous spaces $Y = G/H$ for complex Lie groups G are called Kähler C-spaces. A Kähler C-space $Y = G/H$ has second Betti number $b_2(Y) = 1$ or an infinite cyclic Picard group $\mathrm{Pic}(Y) \simeq (\mathbb{Z}, +)$ if and only if G is a simply connected, complex simple Lie group and H is a maximal parabolic subgroup of G. From now on, we consider only Kähler C-spaces Y of $b_2(Y) = 1$ and denote them by $Y = G/P$. More precisely, let r be the rank of the complex simple Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a system of simple roots $\alpha_1, \ldots, \alpha_r$ of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Put Δ^+ for the set of the positive roots of $\mathfrak{g}$, $\Delta^-_{\widehat{l}}$ for the set of the negative roots $\alpha = \sum_{i=1}^{r} n_i\alpha_i$ with $n_i \in \mathbb{Z}^{\leq 0}$, $n_l = 0$ and $\mathfrak{g}_\alpha$ for the root spaces of α. Then the connected Lie subgroup P_l of G with Lie algebra

$$\mathrm{Lie}(P_l) = \mathfrak{h} + \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha + \sum_{\alpha\in\Delta^-_{\widehat{l}}} \mathfrak{g}_\alpha \tag{1}$$

is a maximal parabolic subgroup P_l of G. All Kähler C-spaces Y with $b_2(Y) = 1$ are of the form $Y = G/P_l$ for a simply connected, complex simple Lie group G and a maximal parabolic subgroup P_l with Lie algebra (1) for some $1 \leq l \leq r$. If G is of type $\mathcal{T} = \mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r, \mathbf{D}_r, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8, \mathbf{F}_4$ or $\mathbf{G}_2$ then for any $1 \leq l \leq r = \mathrm{rk}(\mathfrak{g})$ the Kähler C-space $Y = G/P_l$ is said to be of type $(\mathcal{T}, \alpha_l)$.

Let X be a complex projective manifold of $\dim_{\mathbb{C}} X = n$. The period map of X transforms a deformation family for the complex structure on X into deformation family for the Hodge decomposition of $H^n(X, \mathbb{C})$. Let us denote by Θ_X the tangent sheaf of X. The differential of the period map is

$$\tau : H^1(X, \Theta_X) \longrightarrow \sum_{p+q=n} \mathrm{Hom}(H^q(X, \Omega^p_X), H^{q+1}(X, \Omega^{p-1}_X)).$$

If for some $p, q \in \mathbb{Z}^{\geq 0}$ with $p + q = n$ the $\mathbb{C}$-linear map

$$\tau_{(p,q)} : H^1(X, \Theta_X) \longrightarrow \mathrm{Hom}(H^q(X, \Omega_X^p), H^{q+1}(X, \Omega_X^{p-1})) \tag{2}$$

is injective then we say that X satisfies the Infinitesimal Torelli Theorem.

Theorem 1. (Konno [5]) *Let $Y = G/P$ be a Kähler C-space with $b_2(Y) = 1$ and X be a smooth complete intersection in Y with ample canonical bundle K_X. The Infinitesimal Torelli Theorem holds for the following X:*

(i) X of sufficiently large dimension;

(ii) complete intersections X in irreducible Hermitian symmetric spaces Y of compact type;

(iii) complete intersections X in $Y = G/P$ with $\mathrm{Lie}G = \mathbf{C}_r, \mathbf{E_6}, \mathbf{F}_4$ *or* $\mathbf{G}_2$;

(iv) complete intersections X of $\dim_{\mathbb{C}} X = 2$ *in $Y = G/P$, whose type is different from* $(\mathbf{E}_8, \alpha_4)$.

Let X be a compact Kähler manifold with canonical bundle K_X of the form

$$K_X = E_1^{\otimes n_1} \otimes \ldots \otimes E_k^{\otimes n_k}, \quad n_i \in \mathbb{N}$$

for some holomorphic line bundles $E_i \to X$ with irreducible associated divisors and $\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(E_i)) \geq 2$. In [4] Kii provides a sufficient condition for such X to satisfy the Infinitesimal Torelli Theorem. The proof of Theorem 1 from [5] makes use of the following special case of Kii's result:

Theorem 2. (Special case of Kii's Theorem 1 from [4]:) *Let X be a compact Kähler manifold with* $\dim_{\mathbb{C}} X = n$ *and canonical bundle* $K_X = E_1^{\otimes n_1}$ *for* $n_1 \in \mathbb{N}$ *and a holomorphic line bundle $E_1 \to X$, whose associated linear system has base locus of codimension ≥ 2. If* $\dim_{\mathbb{C}} H^0(X, \Omega_X^{n-1} \otimes E_1) \leq \dim_{\mathbb{C}} H^0(X, E_1) - 2$ *then*

$$\tau_{(n,0)} : H^1(X, \Theta_X) \longrightarrow \mathrm{Hom}(H^0(X, \Omega_X^n), H^1(X, \Omega_X^{n-1}))$$

is injective and X satisfies Infinitesimal Torelli Theorem.

The main result of [6] is the following

Theorem 3. (Konno [6]) *Let $X = X_{d_1,\ldots,d_p}$ be a smooth complete intersection in a Kähler C-space $Y = G/P_l$ with $b_2(Y) = 1$, which is neither a projective space nor a complex quadric. Then Infinitesimal Torelli Theorem holds for the following X:*

(i) complete intersections with non-negative canonical bundle $K_X \geq 0$;

(ii) complete intersections $X = X_{d_1,\ldots,d_p}$ with $d_i \geq 2$ for $\forall 1 \leq i \leq p$, which are different from

(a) a hypersurface of degree 2 in $Y = G/P_l$ of type $(\mathbf{A}_4, \alpha_2)$, $(\mathbf{D}_5, \alpha_4)$, $(\mathbf{E}_6, \alpha_2)$, $(\mathbf{E}_7, \alpha_1)$, $(\mathbf{E}_8, \alpha_8)$, $(\mathbf{F}_4, \alpha_1)$, $(\mathbf{F}_4, \alpha_3)$ *and*

(b) a complete intersection $X = X_{2,2}$ in $Y = G/P_l$ of type $(\mathbf{B}_l, \alpha_2)$, $(\mathbf{D}_l, \alpha_2)$, $(\mathbf{E}_6, \alpha_2)$, $(\mathbf{E}_7, \alpha_1)$, $(\mathbf{E}_8, \alpha_8)$, $(\mathbf{F}_4, \alpha_1)$, $(\mathbf{F}_4, \alpha_3)$.

Let X be a smooth complete intersection in a Kähler C-space $Y = G/P_l$ with $b_2(Y) = 1$ and N_X be the normal bundle of X in Y. Then there is a short exact sequence of sheaves

$$0 \longrightarrow N_X^* \longrightarrow \Omega_Y^1|_X \longrightarrow \Omega_X^1 \longrightarrow 0 \tag{3}$$

on X. The argument from [6] is based on the application of Flenner's Theorem 1.1 from [2] to (3).

Theorem 4. (Special case of Flenner's Theorem 1.1 from [2]:) *Let X be a complete intersection of $\dim_{\mathbb{C}} X = n$ in a Kähler C-space $Y = G/P_l$ with $b_2(Y) = 1$, N_X be the normal bundle of X in Y, $S^m N_X$ (respectively, $S^m N_X^*$) be the m-th symmetric tensor product of N_X (respectively, N_X^*). If the multiplication map*

$$H^0(X, S^{n-p} N_X \otimes K_X) \otimes H^0(X, S^{p-1} N_X \otimes K_X) \longrightarrow H^0(X, S^{n-1} N_X \otimes K_X^2) \tag{4}$$

is surjective for some $1 \leq p \leq n$ and

$$H^{i+1}(X, SN_X^* \otimes \Omega_Y^{n-i-1} \otimes K_X^{-1}) = 0 \quad \textit{for} \quad \forall 0 \leq i \leq n-2, \tag{5}$$

then

$$\tau_{(p,q)} : H^1(X, \Theta_X) \longrightarrow \mathrm{Hom}(H^q(X, \Omega_X^p), H^{q+1}(X, \Omega_X^{p-1}))$$

with $q = n - p$ is injective and Infinitesimal Torelli Theorem holds for X

The main Theorem 9 of the present note derives the Infinitesimal Torelli Theorem for complete intersections $X = X_{d_1,\ldots,d_p}$ with sufficiently large $d_i \in \mathbb{N}$ in Kähler C-spaces $Y = G/P_l$ with $b_2(Y) = 1$ from Green's work [3] on hypersurfaces of high degree. The proof of Theorem 9 was communicated to Konno in 1986 and cited by him in [6]. The author has decided to collect the proof, in order to specify this citation and to make clear that Konno's results from [5], [6], [7] are much more general than Theorem 9.

More precisely, Green's Lemma 1.14 from [3] asserts that if X is a smooth hypersurface with ample canonical bundle on a smooth compact complex manifold Y of $\dim_{\mathbb{C}} Y = n + 1$, N_X is the normal bundle of X in Y,

$$H^i(X, \Omega_Y^j \otimes S^m N_X^*) = 0 \quad \text{for} \quad \forall i < n, \quad \forall 0 \leq j \leq n, \quad \forall 1 \leq m \leq n, \tag{6}$$

$$H^i(X, \Omega_Y^j \otimes S^m N_X^* \otimes K_X^{-1}) = 0 \quad \text{for} \quad \forall 0 < i < n, \quad \forall 1 \leq j \leq n, \quad \forall 1 \leq m \leq n-2, \tag{7}$$

then there is a commutative diagram

$$\begin{array}{ccc} H^1(X, \Omega_X^{n-1})^* \otimes H^0(X, K_X) & \xleftarrow{\;e\;} & H^0(X, S^{n-1} N_X \otimes K_X) \otimes H^0(X, K_X) \\ \Big\downarrow f & & \Big\downarrow g \\ H^1(X, \Theta_X)^* & \xleftarrow{\;h\;} & H^0(X, S^{n-1} N_X \otimes K_X^2) \end{array} \tag{8}$$

with surjective g, h, so that f is surjective and Infinitesimal Torelli Theorem holds for X due to the injectiveness of the dual map

$$f^* = \tau_{(n,0)} : H^1(X, \Theta_X) \to \mathrm{Hom}(H^0(X, \Omega_X^n), H^1(X, \Omega_X^{n-1})) \simeq H^1(X, \Omega_X^{n-1}) \otimes H^0(X, K_X)^*.$$

The following Proposition 5 generalizes Green's Lemma 1.14 from [3] to smooth complete intersections:

Proposition 5. *Let $Y = G/P_l$ be a Kähler C-space with $b_2(Y) = 1$ and canonical bundle $K_Y = \mathcal{O}_Y(-k(Y))$, $k(Y) \in \mathbb{N}$. If $X = X_{d_1,\dots,d_p}$ is a smooth complete intersection in Y with $\dim_{\mathbb{C}} X = n$, normal bundle*

$$N_X = \mathcal{O}_X(d_1) \oplus \ldots \oplus \mathcal{O}_X(d_p),$$

non-negative canonical bundle

$$K_X = \mathcal{O}_X\left(-k(Y) + \sum_{s=1}^{p} d_s\right), \quad -k(Y) + \sum_{s=1}^{p} d_s \geq 0,$$

(6) and (7), then there is a commutative diagram (8) with surjective g, h, f and Infinitesimal Torelli Theorem holds for X.

Proof. Note that

$$f = \tau^*_{(n,0)} : H^1(X, \Omega_X^{n-1})^* \otimes H^0(X, K_X) \longrightarrow H^1(X, \Theta_X)$$

is the dual of $\tau_{(n,0)}$ from (2), so that the surjectiveness of f is equivalent to the injectiveness of $\tau_{(n,0)}$ and implies Infinitesimal Torelli Theorem for X. Towards the surjectiveness of f, it suffices to establish the existence of a commutative diagram (8) with surjective g and h. By Lemma 1.5 [3], (1-6) implies the presence of a $\mathbb{C}$-linear map

$$\mu : H^1(X, \Omega_X^{n-1})/A \longrightarrow \left(H^0(X, S^{n-1}N_X \otimes K_X)/B\right)^*$$

for appropriate $\mathbb{C}$-subspaces $A \subset H^1(X, \Omega_X^{n-1})$ and $B \subset H^0(X, S^{n-1}N_X \otimes K_X)$. Let

$$\mu^* : H^0(X, S^{n-1}N_X \otimes K_X)/B \longrightarrow \left(H^1(X, \Omega_X^{n-1})/A\right)^*.$$

be the dual map. Consider the natural embedding

$$\varepsilon : \left(H^1(X, \Omega_X^{n-1})/A\right)^* \hookrightarrow H^1(X, \Omega_X^{n-1})^*$$

of the $\mathbb{C}$-linear functionals on $H^1(X, \Omega_X^{n-1})$, vanishing on A in all the $\mathbb{C}$-linear functionals $H^1(X, \Omega_X^{n-1})^*$ on $H^1(X, \Omega_X^{n-1})$. Denote by π_B the natural projection

$$\pi_B : H^0(X, S^{n-1}N_X \otimes K_X) \longrightarrow H^0(X, S^{n-1}N_X \otimes K_X)/B,$$

with kernel $\ker(\pi_B) = B$. Then the composition

$$\varepsilon\mu^*\pi_B : H^0(X, S^{n-1}N_X \otimes K_X) \longrightarrow H^1(X, \Omega_X^{n-1})^*$$

is a $\mathbb{C}$-linear map. Tensoring with $\mathrm{Id}_{H^0(X,K_X)}$, one obtains a $\mathbb{C}$-linear map $e = (\varepsilon\mu^*\pi_B) \otimes \mathrm{Id}_{H^0(X,K_X)}$,

$$e : H^0(X, S^{n-1}N_X \otimes K_X) \otimes H^0(X, K_X) \longrightarrow H^1(X, \Omega_X^{n-1})^* \otimes H^0(X, K_X).$$

According to Lemma 1.10 from [3], (7) suffices for the presence of a $\mathbb{C}$-linear isomorphism

$$H^1(X, \Theta_X) \simeq \left[\frac{H^0(X, S^{n-1}N_X \otimes K_X^2)}{\mathrm{im} H^0(X, S^{n-2}N_X \otimes \Theta_Y \otimes K_X^2)}\right]^*.$$

As a result, one obtains a $\mathbb{C}$-linear isomorphism

$$\left[\frac{H^0(X, S^{n-1}N_X \otimes K_X^2)}{\mathrm{im} H^0(X, S^{n-2}N_X \otimes \Theta_Y \otimes K_X^2)}\right] \simeq H^1(X, \Theta_X)^*,$$

whereas a $\mathbb{C}$-linear surjection

$$h : H^0(X, S^{n-1}N_X \otimes K_X^2) \longrightarrow H^1(X, \Theta_X)^*.$$

Towards the existence of a surjection

$$g : H^0(X, S^{n-1}N_X \otimes K_X) \otimes H^0(X, K_X) \longrightarrow H^0(X, S^{n-1}N_X \otimes K_X^2), \tag{9}$$

let us note that the normal bundle $N_X = \mathcal{O}_X(d_1) \oplus \ldots \oplus \mathcal{O}_X(d_p)$ and its symmetric poser $S^{n-1}N_X$ decompose in direct sums of holomorphic line bundles. More precisely, denote by ω_l the fundamental weight, corresponding to the simple root α_l of $\mathfrak{g} = Lie(G)$. The positive generator of the Picard group $Pic(Y) \simeq (\mathbb{Z}, +)$ of $Y = G/P_l$ is associated with the irreducible representation of P_l with dominant weight ω_l. There is a sheaf decomposition

$$S^{n-1}N_X = \oplus_{\substack{k=(k_1,\ldots,k_p)\in(\mathbb{Z}^{\geq 0})^p \\ k_1+\ldots+k_p=n-1}} \quad \mathcal{O}_X\left(\sum_{s=1}^{p} k_s d_s\right),$$

whereas

$$S^{n-1}N_X \otimes K_X = \oplus_{\substack{k=(k_1,\ldots,k_p)\in(\mathbb{Z}^{\geq 0})^p \\ k_1+\ldots+k_p=n-1}} \quad \mathcal{O}_X\left(-k(Y) + \sum_{s=1}^{p}(k_s+1)d_s\right).$$

As a result, there arises a decomposition

$$H^0(X, S^{n-1}N_X \otimes K_X) =$$

$$= \oplus_{\substack{k=(k_1,\ldots,k_p)\in(\mathbb{Z}^{\geq 0})^p \\ k_1+\ldots+k_p=n-1}} \quad H^0\left(X, \mathcal{O}_X\left(-k(Y) + \sum_{s=1}^{p}(k_s+1)d_s\right)\right).$$

of the corresponding holomorphic sections, as well as

$$H^0(X, S^{n-1}N_X \otimes K_X^2) =$$

$$= \oplus_{\substack{k=(k_1,\ldots,k_p)\in(\mathbb{Z}^{\geq 0})^p \\ k_1+\ldots+k_p=n-1}} H^0\left(X, \mathcal{O}_X\left(-2k(Y) + \sum_{s=1}^{p}(k_s+2)d_s\right)\right).$$

The surjectiveness of

$$g_l : H^0\left(X, \mathcal{O}_X\left(-k(Y) + \sum_{s=1}^{p}(k_s+1)d_s\right)\right) \otimes H^0\left(X, \mathcal{O}_X\left(-k(Y) + \sum_{s=1}^{p} d_s\right)\right) \longrightarrow$$

$$\longrightarrow H^0\left(X, \mathcal{O}_X\left(-2k(Y) + \sum_{s=1}^{p}(k_s+2)d_s\right)\right)$$

for all $k = (k_1, \ldots, k_p) \in (\mathbb{Z}^{\geq 0})^p$ with $\sum_{s=1}^{p} k_s = n-1$ implies the surjectiveness of (9). The commutativity of the diagram (8) follows from the fact that the multiplication by $H^0(X, K_X)$ commutes with the differentials of the spectral sequence, associated with the long exact sequence

$$0 \to S^n N_X^* \to \ldots \to \Omega_Y^{n-1} \otimes N_X^* \to \Omega_Y^n \otimes \mathcal{O}_X \to \Omega_X^n \to 0.$$

□

The rest of the article derives sufficient conditions for $X = X_{d_1,\ldots,d_p} \subset Y = G/P_l$ to satisfy $K_X \geq 0$, (6) and (7).

Lemma 6. *Let $X = X_{d_1,\ldots,d_p}$ be a smooth complete intersection of $\dim_{\mathbb{C}} X = n$ with non-negative canonical bundle $K_X = \mathcal{O}_X\left(-k(Y) + \sum_{s=1}^{p} d_s\right)$, $\sum_{s=1}^{p} d_s \geq k(Y)$ in a Kähler C-space $Y = G/P_l$ with $b_2(Y) = 1$, such that*

$$d_p \geq d_{p-1} \geq \ldots \geq d_2 \geq d_1,$$

$$H^i(X, \Omega_Y^j(-\lambda)) = 0 \quad \text{for} \quad \forall i < n, \quad \forall 0 \leq j \leq n, \quad \forall \lambda > n_1 - 1, \tag{10}$$

$$H^i(X, \Omega_Y^j(-\mu)) = 0 \quad \text{for} \quad \forall i < n, \quad \forall 1 \leq j \leq n, \quad \forall \mu > (p+1)d_1 - k(Y) - 1. \tag{11}$$

Then Infinitesimal Torelli Theorem holds for X.

Proof. It suffices to establish that (10) implies (6) and (11) suffices for (7). To this end, note that for any $m \in \mathbb{N}$ the m-th symmetric tensor product

$$S^m N_X^* = \oplus_{\substack{k=(k_1,\ldots,k_p)\in(\mathbb{Z}^{\geq 0})^p \\ k_1+\ldots+k_p=n-1}} \mathcal{O}_X\left(-\sum_{s=1}^{p} k_s d_s\right)$$

corresponds to a completely reducible P_l-module, whose irreducible components have dominant weights $-\left(\sum_{s=1}^{p} k_s d_s\right)\omega_l$. Since

$$\sum_{s=1}^{p} k_s d_s \geq d_1\left(\sum_{s=1}^{p} k_s\right) = d_1 m \geq d_1,$$

(10) suffices for (6). Similarly,

$$\sum_{s=1}^{p} k_s d_s - k(Y) + \sum_{s=1}^{p} d_s \geq d_1 \left(\sum_{s=1}^{p} k_s \right) - k(Y) + pd_1 = md_1 - k(Y) + pd_1 \geq -k(Y) + (p+1)d_1$$

reveals that if (11) then (7).

□

The next lemma provides sufficient vanishing conditions on the cohomologies of Y, in order to have vanishing cohomologies on X.

Lemma 7. *Let $X = X_{d_1,\dots,d_p}$ be a smooth complete intersection in a compact complex manifold Y, $\nu_o \in \mathbb{Z}$, $0 \leq j \leq \dim_{\mathbb{C}} Y$. If*

$$H^i(Y, \Omega^j_Y(-\nu)) = 0 \quad \textit{for} \quad \forall i < \dim_{\mathbb{C}} Y, \quad \forall \nu > \nu_o \tag{12}$$

then

$$H^i(X, \Omega^j_Y(-\nu)) = 0 \quad \textit{for} \quad \forall i < \dim_{\mathbb{C}} X, \quad \forall \nu > \nu_o. \tag{13}$$

Proof. Let us consider the series

$$Y = X_0 \supset X_1 \supset \dots \supset X_{s-1} \supset X_s \supset \dots \supset X_{p-1} \supset X_p = X$$

of hypersurfaces X_s of degree d_s in X_{s-1} for $\forall 1 \leq s \leq p$. By an induction on $1 \leq s \leq p$, we shall prove that

$$H^i(X_s, \Omega^j_Y(-\nu)) = 0 \quad \text{for} \quad \forall i < \dim_{\mathbb{C}} X_s, \forall \nu > \nu_o. \tag{14}$$

To this end, let us consider the short restriction sequence

$$0 \to \Omega^j_Y(-\nu - d_s)|_{X_{s-1}} \to \Omega^j_Y(-\nu)|_{X_{s-1}} \to \Omega^j_Y(-\nu)|_{X_s} \to 0$$

and its associated long cohomology sequence

$$\dots \to H^i(X_{s-1}, \Omega^j_Y(-\nu)) \to H^i(X_s, \Omega^j_Y(-\nu)) \to H^{i+1}(X_{s-1}, \Omega^j_Y(-\nu - d_s)) \to \dots \tag{15}$$

By the inductional hypothesis,

$$H^i(X_{s-1}, \Omega^j_Y(-\nu)) = 0 \quad \text{for} \quad \forall i < \dim_{\mathbb{C}} X_s = \dim_{\mathbb{C}} X_{s-1} - 1 < \dim_{\mathbb{C}} X_{s-1}, \quad \forall \nu > \nu_o$$

and

$$H^{i+1}(X_{s-1}, \Omega^j_Y(-\nu - d_s)) = 0 \quad \text{for} \quad \forall i+1 < \dim_{\mathbb{C}} X_s + 1 = \dim_{\mathbb{C}} X_{s-1}, \forall \nu + d_s > \nu_o + d_s > \nu_o.$$

Now (15) provides

$$H^i(X_s, \Omega^j_Y(-\nu)) = 0 \quad \text{for} \quad \forall i < \dim_{\mathbb{C}} X_s, \quad \forall \nu > \nu_o$$

and concludes the proof of the lemma. □

Lemma 8. *Let $Y = G/P_l$ be a Kähler C-space with $b_2(Y) = 1$. Then there exists a sufficiently large natural number $d_1 \in \mathbb{N}$, such that*

$$H^i(Y, \Omega^j_Y(-\lambda)) = 0 \quad \text{for} \quad \forall i < \dim_{\mathbb{C}} Y, \ \ \forall 0 \leq j \leq n, \ \ \forall \lambda > d_1 - 1 \quad \text{and} \tag{16}$$

$$H(Y, \Omega^j_Y(-\mu)) = 0 \quad \text{for} \quad \forall i < \dim_{\mathbb{C}} Y, \ \ \forall 1 \leq j \leq n, \ \ \forall \mu > (p+1)d_1 - k(Y) - 1. \tag{17}$$

Proof. Note that holomorphic bundles $\Omega^j_Y \to Y = G/P_l$ are not associated with completely reducible P_l-modules but there is an appropriate filtration on Ω^j_Y, whose successive quotients correspond to irreducible P_l-modules E_w (cf. [5]). The dominant weights w of these quotients are of the form $w = -\beta_1 - \ldots - \beta_j$ for distinct positive complementary roots

$$\beta_r \in \Delta^+_l := \left\{ \sum_{t=1}^r n_t \alpha_t \ \Big| \ n_t \geq 0, \ \ n_l > 0 \right\}$$

of $Lie(P_l) = \mathfrak{h} + \sum\limits_{\alpha \in \Delta^+} \mathfrak{g}_\alpha + \sum\limits_{\alpha \Delta^-_{\widehat{l}}} \mathfrak{g}_\alpha$ with

$$\Delta^-_{\widehat{l}} = \left\{ - \sum_{t=1}^r n_t \alpha_t \ \Big| \ n_t \geq 0, \ \ n_l = 0 \right\}.$$

It suffices to establish the existence of a sufficiently large natural number $d_1 \in \mathbb{N}$, such that

$$\begin{gathered} H^i(Y, E_{w - \lambda\omega_l}) = 0 \quad \text{for} \quad \forall i < \dim_{\mathbb{C}} Y = n + p, \\ \forall w = -\beta_1 - \ldots - \beta_j, \ \ \beta_t \in \Delta^+_l, \ \ \forall 0 \leq j \leq n, \ \ \forall \lambda > d_1 - 1 \end{gathered} \tag{18}$$

and

$$\begin{gathered} H^i(Y, E_{w - \mu\omega_l}) = 0 \quad \text{for} \quad \forall i < \dim_{\mathbb{C}} Y = n + p, \\ \forall w = -\beta_1 - \ldots - \beta_j, \ \ \beta_t \in \Delta^+_l, \ \ \forall 1 \leq j \leq n, \ \ \forall \mu > (p+1)d_1 - k(Y) - 1, \end{gathered} \tag{19}$$

where $E_{w-\lambda\omega_l}$, $E_{w-\mu\omega_l}$ denote the homogeneous vector bundles on $Y = G/P_l$, induced by the irreducible representations of P_l with dominant weights $w - \lambda\omega_l$, respectively, $w - \mu\omega_l$. Recall that ω_l stands for the fundamental weight, associated with the simple root α_l of $\mathfrak{g} = Lie(G)$, i.e., the Killing form $(\omega_l, \alpha_t) = 0$ vanishes for $1 \leq l \neq t \leq r$ and $(\omega_l, \alpha_l) = \frac{(\alpha_l, \alpha_l)}{2}$. Let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha = \sum_{t=1}^r \omega_t$$

be the half-sum of the positive roots of $\mathfrak{g} = Lie(G)$ or the sum of the fundamental weights of $\mathfrak{g}$. We say that the weight $w - \nu\omega_l + \rho$ is singular if there exists a positive root $\alpha \in \Delta^+$ with $(w - \nu\omega_l + \rho, \alpha) = 0$. When $(w - \nu\omega_l + \rho, \alpha) \neq 0$ for $\forall \alpha \in \Delta^+$ and there are exactly i positive roots α with $(w - \nu\omega_l + \rho, \alpha) < 0$, the weight $w - \nu\omega_l + \rho$ is called regular of index i. Borel-Weil-Bott Theorem (cf. [1], [8]) asserts that if the weight $w - \nu\omega_l + \rho$ is singular, then $H^i(Y, E_{w-\nu\omega_l}) = 0$ for $\forall i \geq 0$. If $w - \nu\omega_l + \rho$ is a regular weight of index i_o, then $H^i(Y, E_{w-\nu\omega_l}) = 0$ for all $i \neq i_o$. In order to have

$$H^i(Y, E_{w-\nu\omega_l}) = 0 \quad \text{for} \quad \forall i < \dim_{\mathbb{C}} Y,$$

it suffices to require

$$(w - \nu\omega_l + \rho, \beta) \leq 0 \quad \text{for} \quad \forall \beta \in \Delta_l^+ = \left\{ \sum_{t=1}^{r} n_t \alpha_t \;\middle|\; n_t \geq 0, \quad n_l > 0 \right\} \tag{20}$$

Indeed, if (20) holds for all positive complementary roots β, then either $w - \nu\omega_l + \rho$ is singular or $w - \nu\omega_l + \rho$ is regular of index

$$i_o \geq |\Delta_l^+| = \dim_{\mathbb{C}} Y,$$

as far as the holomorphic tangent space of $Y = G/P_l$ at the origin is

$$T^{1,0}Y = \sum_{\beta \in \Delta_l^+} \mathfrak{g}_\beta$$

and $|\Delta_l^+| = \dim_{\mathbb{C}} T^{1,0}Y = \dim_{\mathbb{C}} Y$.

An arbitrary complementary root $\beta \in \Delta_l^+$ decomposes into a linear combination $\beta = \sum_{t=1}^{r} b_t \alpha_t$ of the simmple roots α_t with non-negative coefficients $b_t \geq 0$ for $\forall 1 \leq t \leq r$ and $b_l > 0$. Then

$$(w - \nu\omega_l + \rho, \beta) = -\sum_{r=1}^{j} (\beta_r, \beta) - \nu b_l \frac{(\alpha_l, \alpha_l)}{2} + \frac{1}{2} \sum_{t=1}^{r} b_t (\alpha_t, \alpha_t) \leq 0$$

is equivalent to

$$nu \geq \frac{2}{b_l(\alpha_l, \alpha_l)} \left[\frac{1}{2} \sum_{t=1}^{r} b_t (\alpha_t, \alpha_t) - \sum_{r=1}^{j} (\beta_r, \beta) \right]$$

and holds for $\forall \nu \geq \nu_o$, provided

$$\nu_o \geq \frac{2}{b_l(\alpha_l, \alpha_l)} \left[\frac{1}{2} \sum_{t=1}^{r} b_t (\alpha_t, \alpha_t) - \sum_{r=1}^{j} (\beta_r, \beta) \right].$$

For any fixed $w = -\sum_{r=1}^{j} \beta_r$ and $\beta = \sum_{t=1}^{r} b_t \alpha_t \in \Delta_l^+$, let us denote

$$C(w, \beta) := \frac{2}{b_l(\alpha_l, \alpha_l)} \left[\frac{1}{2} \sum_{t=1}^{r} b_t (\alpha_t, \alpha_t) - \sum_{r=1}^{j} (\beta_r, \beta) \right].$$

It suffices to choose

$$d_1 \geq C_w := \max_{\beta \in \Delta_l^+} C(w, \beta)$$

in order to have $H^i(Y, E_{w - \lambda\omega_l}) = 0$ for $w = -\sum_{r=1}^{j} \beta_r$, $\forall i < \dim_{\mathbb{C}} Y$ and $\forall \lambda \geq d_1$. Similarly, for

$$C_w = \max_{\beta \in \Delta_l^+} C(w, \beta) \quad \text{and} \quad d_1 \geq \frac{C_w + k(Y)}{p + 1}$$

one has $H^i(Y, E_{w-\mu\omega_l}) = 0$ for $w = -\sum_{r=1}^{j} \beta_r$, $\forall i < \dim_{\mathbb{C}} Y$ and $\forall \mu \geq (p+1)d_1 - k(Y)$. For any $0 \leq j \leq n$ there are finitely many weights $w = -\sum_{r=1}^{j} \beta_r$ with different positive complementary roots $\beta_r \in \Delta_l^+$. If C is the maximum of C_w over the weights $w = -\sum_{r=1}^{j} \beta_r$ with different $\beta_1 \dots \beta_j \in \Delta_l^+$ and any $0 \leq j \leq n$, then the choice of

$$d_1 \geq \max\left(C, \frac{C + k(Y)}{p+1}\right)$$

suffices for (18) and (19).

□

Combining Lemmas 6, 7 and 8, one obtains the following

Theorem 9. *For any Kähler C-space $Y = G/P_l$ with $b_2(Y) = 1$ there exist sufficiently large natural numbers $d_p \geq d_{p-1} \geq \ldots \geq d_2 \geq d_1$, such that Infinitesimal Torelli Theorem holds for any smooth complete intersection $X = X_{d_1,\ldots,d_p}$ in $Y = G/P_l$.*

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