NON-ASSOCIATIVE OPERATIONS ON VARIATIONS OF HODGE STRUCTURE

PANCHO BESHKOV, AZNIV KASPARIAN

MATHEMATIQUE
Géométrie algébrique

Abstract: Let $\pi : D \to S$ be the projection of a period domain $D = G/V$ onto a Riemannian symmetric space $S = G/K$ of non-compact type with $K \supseteq V$. The totally geodesic variations of Hodge structure $U \subset D$ are exactly the equivariantly embedded Hermitian symmetric subspaces U of D, which map diffeomorphically onto totally geodesic subspaces $\pi(U) \subset S$. The article shows that a variation of Hodge structure $U \subset D$ is totally geodesic exactly when $\pi(U)$ is a left quasi-subgroup of a left quasi-group with right neutral element $(S, \oplus_\sigma, K)$, induced by a real analytic section $\sigma : S \to G$ of $\pi_K : G \to G/K = S$. It establishes that $U \subset D$ is totally geodesic exactly when $(\pi(U), \cdot)$ is a Loos-symmetric subspace of the Loos-symmetric space $(S, \cdot)$. We introduce the notion of a Loos-Hermitian symmetric space of non-compact type and prove that $U \subset D$ is a totally geodesic variation of Hodge structure if and only if there is a Loos-Hermitian symmetric structure $(\pi(U), *)$ of non-compact type, whose square $(\pi(U), *^2)$ is a Loos-symmetric subspace of $(S, \cdot)$

Key words: Totally geodesic submanifold, variation of Hodge structure, left quasi-group, Loos-symmetric space, Loos-Hermitian symmetric space.

2000 Mathematics Subject Classification: 14D07, 53C30, 20N05

The study of the totally geodesic variations of Hodge structure is motivated by the presence of families of Calabi-Yau manifolds, whose Teichmüller spaces have totally geodesic images under the period map (cf.[1]). On the other hand, [2] shows that any irreducible Hermitian symmetric space of non-compact type is realized as a totally geodesic variation of Hodge structure of Calabi-Yau type.

If G is a real linear algebraic group and $H < G$ is a compact subgroup then $\mathfrak{g} = \mathrm{Lie}(G)$ admits a non-degenerate Ad_H-invariant bilinear form $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ and $\mathfrak{M} := \mathrm{Lie}(H)^\perp \subset \mathfrak{g}$ is called a canonical lift of $T^{\mathbb{R}}_{\check{o}}(G/H) = \mathfrak{g}/\mathrm{Lie}(H)$ to $\mathfrak{g}$. In particular, if H is a maximal compact subgroup of G then the canonical lift of $T^{\mathbb{R}}_{\check{o}}(G/H)$ to $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$ is denoted by $\mathfrak{p}$.

Recall that a subspace $\mathfrak{s} \subset \mathfrak{g}$ is a Lie triple system if $[[\mathfrak{s}, \mathfrak{s}], \mathfrak{s}] \subseteq \mathfrak{s}$.

Proposition 1. *Let $\pi : D \to S$ be the projection of a period domain onto a Riemannian symmetric space of non-compact type, $U \subset D$ be a totally geodesic variation of Hodge structure and $\mathfrak{s} \subset \mathfrak{p}$ be canonical lifts of $T^{\mathbb{R}}_{\check{o}}\pi(U) \subset T^{\mathbb{R}}_{\check{o}}S$ to $\mathfrak{g} = \mathrm{Lie}(G)$. Then there is a Lie subgroup $G(U)$ of G with $\mathrm{Lie}(G(U)) = \mathfrak{s} \oplus [\mathfrak{s}, \mathfrak{s}]$, such that $U = G(U)V/V \subset D$ is an equivariantly embedded Hermitian symmetric space of non-compact type, $\pi(U) \subset S$ is totally geodesic and $\pi : U \to \pi(U)$ is a global diffeomorphism.*

Conversely, if $U \subset D$ is a variation of Hodge structure with totally geodesic $\pi(U) \subset S$, then $U \subset D$ is totally geodesic.

Proof. For an arbitrary variation of Hodge structure $U \subset D$, for every $o \in U$ and for $\check{o} = \pi(o)$, the differential $(d\pi)_o : T^{\mathbb{R}}_o U \to T^{\mathbb{R}}_{\check{o}}\pi(U)$ is an $\mathbb{R}$-linear isomorphism and there is a canonical lift $T^{\mathbb{R}}_o U = T^{\mathbb{R}}_{\check{o}}\pi(U) = \mathfrak{s}$. Moreover, $\mathfrak{A} := T^{1,0}_o U$ is an abelian Lie subalgebra of $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, so that $\mathfrak{s} = (\mathfrak{A} \oplus \overline{\mathfrak{A}}) \cap \mathfrak{g}$ satisfies $[\mathfrak{s}, \mathfrak{s}] \subseteq [\mathfrak{A}, \overline{\mathfrak{A}}] \cap \mathfrak{g} \subseteq \mathrm{Lie}(V)$.

If $U \subset D$ is totally geodesic then $\pi(U) \subset S$ is totally geodesic because π maps the D-geodesics onto the S-geodesics. By [3], $\mathfrak{s}$ is a Lie triple system, there is a Lie subgroup $G(U) \leq G$ with $\mathrm{Lie}(G(U)) = \mathfrak{s} \oplus [\mathfrak{s}, \mathfrak{s}]$ and an equivariantly embedded totally geodesic $U(\mathfrak{s}) := G(U)V/V \subset D$ with $T^{\mathbb{R}}_o U(\mathfrak{s}) = \mathfrak{s} = T^{\mathbb{R}}_o U$. Since D is complete and $U, U(\mathfrak{s})$ are geodesic at o, the coincidence $T^{\mathbb{R}}_o U(\mathfrak{s}) = T^{\mathbb{R}}_o U$ suffices for $U(\mathfrak{s}) = U$. Similarly, [3] reveals that $W = G(U)K/K \subset S$ is an equivariantly embedded, totally geodesic Riemannian symmetric subspace of non-compact type. By $\mathrm{Lie}(G(U) \cap K) = \mathrm{Lie}(G(U)) \cap \mathrm{Lie}(K) = [\mathfrak{s}, \mathfrak{s}] = \mathrm{Lie}(G(U) \cap V)$ it follows $G(U) \cap K = G(U) \cap V$ and $\pi : U \to W$ turns to be a global diffeomorphism. As a result, U is a Riemannian symmetric space of non-compact type and the holomorphy of the geodesic isometry $s_o : U \to U$ at $o \in U$ implies that U is Hermitian symmetric.

If $W := \pi(U) \subset S$ is totally geodesic then $T^{\mathbb{R}}_o U = T^{\mathbb{R}}_{\check{o}} W = \mathfrak{s}$ is a Lie triple system of $\mathfrak{g}$, there is a Lie subgroup $G(W) \leq G$ with $\mathrm{Lie}(G(W)) = \mathfrak{s} \oplus [\mathfrak{s}, \mathfrak{s}]$ and $U = G(W)V/V \subset D$ is an equivariantly embedded, totally geodesic submanifold. □

From now on, for a binary operation $Q \times Q \to Q$ and $a, b \in Q$, let us denote by $x_o(a, b), y_o(a, b) \in Q$ the solutions of the equations $ax = b$, respectively, $ya = b$, if they exist. In 1935 Moufang defines a quasigroup Q as a set with a binary operation $Q \times Q \to Q$, with respect to which there exist unique $x_o(a, b), y_o(a, b) \in Q$ for for all $a, b \in Q$.

Definition 2. *The pair* $(\mathcal{L}, \oplus)$ *with* $\oplus : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ *is a left quasi-group if for any* $a, b \in \mathcal{L}$ *there exists unique* $x_o(a,b) \in \mathcal{L}$. *An element* $e_r \in \mathcal{L}$ *is right neutral with respect to* $\oplus$ *if* $a \oplus e_r = a$ *for all* $a \in \mathcal{L}$.

Let G be a group, $H \leq G$ be a subgroup and let $\pi : G \to G/H$ be the canonical homomorphism defined by $\pi(a) = aH$, $a \in G$. A map $\sigma : G/H \to G$ is a section if $\pi\sigma = \mathrm{Id}_{G/H}$. Any σ induces a left quasi-group $\oplus_\sigma : G/H \times G/H \to G/H$, $aH \oplus_\sigma bH := \pi(\sigma(aH)\sigma(bH)) = \sigma(aH)bH$ for all $aH, bH \in G/H$ with right neutral element H. Let $(\mathcal{L}, \oplus, e_r)$ be a left quasi-group with right neutral element and $L : \mathcal{L} \to \mathrm{Sym}(\mathcal{L})$, $L_a(x) := a \oplus x$ for all $x \in \mathcal{L}$. Rephrasing [4], note that L is injective and the subgroup $G_{\mathcal{L}} := \langle L(\mathcal{L}) \rangle \leq \mathrm{Sym}(\mathcal{L})$ is isomorphic to $L(\mathcal{L}) \times H_{\mathcal{L}}$ with $H_{\mathcal{L}} := \mathrm{Stan}_{G_{\mathcal{L}}}(e_r)$ as a set. Thus, $\sigma_{\mathcal{L}} : G_{\mathcal{L}}/H_{\mathcal{L}} \to G_{\mathcal{L}}$, $\sigma_{\mathcal{L}}(L_a H_{\mathcal{L}}) = L_a$ is a section of $\pi : G_{\mathcal{L}} \to G_{\mathcal{L}}/H_{\mathcal{L}}$ and $\pi L : (\mathcal{L}, \oplus, e_r) \to (G_{\mathcal{L}}/H_{\mathcal{L}}, \oplus_{\sigma_{\mathcal{L}}}, H_{\mathcal{L}})$ is an isomorphism of left quasi-groups with right neutral elements. For a group G and a subgroup $H \leq G$ let $\Lambda : G \to \mathrm{Sym}(G/H)$ be the group homomorphism, associated with the G-action $G \times G/H \to G/H$, $(a, bH) \mapsto abH$. For any section $\sigma : \mathfrak{L} := G/H \to G$ of $\pi : G \to G/H$, the subgroup $\widetilde{G_{\mathfrak{L}}} := \langle \sigma(G/H) \rangle \leq G$ acts transitively on $\mathfrak{L}$ and $G_{\mathfrak{L}} = \Lambda(\widetilde{G_{\mathfrak{L}}})$, due to $L_{aH} = \Lambda(\sigma(aH))$, $L_{aH}\Lambda(\sigma(aH)^{-1}) = \mathrm{Id}_{G/H}$. Moreover, $H_{\mathfrak{L}} = G_{\mathfrak{L}} \cap \Lambda(H) = \Lambda(\widetilde{G_{\mathfrak{L}}} \cap H)$ by $\ker \Lambda = \cap_{b \in G}(bHb^{-1})$.

Definition 3. *If* $(\mathcal{L}, \oplus)$ *is a left quasi-group, then* $\mathcal{L}_1 \subset \mathcal{L}$ *is a left quasi-subgroup if the inclusions* $a \oplus b, x_o(a,b) \in \mathcal{L}_1$ *hold for all* $a, b \in \mathcal{L}_1$.

Theorem 4. *Let* $o \in U \subset D$ *be a variation of Hodge structure with a closed image* $W := \pi(U)$ *under* $\pi : D = G/V \to G/K = S$. *Then* $U \subset D$ *is totally geodesic if and only if there is a real analytic section* $\sigma : S \to G$ *of* $\pi_K : G \to S$ *with* $\sigma(\check{o}) = e \in G$, *such that* $(W, \oplus_\sigma)$ *is a left quasi-subgroup of* $(S, \oplus_\sigma, K)$.

Proof. Let $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g} = \mathrm{Lie}(G)$ be the Cartan decomposition and $\exp : \mathfrak{g} \to G$. Then $\exp_{\check{o}} : \mathfrak{p} \to S$ is a global diffeomorphism and $\sigma := \exp \exp_{\check{o}}^{-1} : S \to G$ is a real analytic section of π_K. If $U \subset D$ and $W \subset S$ are totally geodesic then $W = G(W)K/K$ for $G(W) \leq G$ and $\exp_{\check{o}}(\alpha) \oplus_\sigma \exp_{\check{o}}(\beta) = \exp(\alpha)\exp_{\check{o}}(\beta) \in W$, $x_o(\exp_{\check{o}}(\alpha), \exp_{\check{o}}(\beta)) = \exp(\alpha)^{-1}\exp_{\check{o}}(\beta) \in W$ for all $\alpha, \beta \in T^{\mathbb{R}}_{\check{o}}W$. Thus, $(W, \oplus_\sigma, K)$ is a left quasi-subgroup of $(S, \oplus_\sigma, K)$.

Conversely, suppose that $(W, \oplus_\sigma, K)$ is a left quasi-subgroup of $(S, \oplus_\sigma, K)$. Then $W = \widetilde{G_o}K/K$ for the subgroup $\widetilde{G_o} := \langle \sigma(W) \rangle \leq G$. Since $\sigma(W) \subset G$ is closed and $\widetilde{G_o} \cap K \leq G$ is a compact subgroup,

$$\sigma(W)(\widetilde{G_o} \cap K) := \{\sigma(p)k \mid p \in W, \, k \in \widetilde{G_o} \cap K\}$$

is a closed submanifold of G. Clearly, $\sigma(W)(\widetilde{G_o} \cap K) \subseteq \widetilde{G_o}$. Any $a \in \widetilde{G_o}$ has $\pi_K(a) = aK = \sigma(aK)K \in \widetilde{G_o}K/K = W$, so that $a = \sigma(aK)k_o$ for some $k_o \in \widetilde{G_o} \cap K$ and $\widetilde{G_o} \subseteq \sigma(W)(\widetilde{G_o} \cap K)$. The coincidence of manifolds $\widetilde{G_o} = \sigma(W)(\widetilde{G_o} \cap K)$ implies that $\widetilde{G_o}$ is a closed and, therefore, a Lie subgroup of G. The diffeomorphism $\sigma : W \to \sigma(W)$ induces $(d\sigma)_o = \mathrm{Id} : T^{\mathbb{R}}_{\check{o}} W \to T^{\mathbb{R}}_e \sigma(W)$ for $T^{\mathbb{R}}_{\check{o}} W \subset \mathfrak{p}$. If $v \in T^{\mathbb{R}}_{\check{o}} W$ then the S-geodesic $\gamma^v_{\check{o}}(t) : \mathbb{R} \to S$ with $\gamma^v_{\check{o}}(0) = \check{o}$, $\left.\frac{d\gamma^v_{\check{o}}(t)}{dt}\right|_{t=0} = v$ has factorization $\gamma^v_{\check{o}}(t) = \pi_K \exp(tv)$ and takes values in $\pi_K \widetilde{G_o} = W$. Thus, $W \subset S$ is geodesic at $\check{o} \in W$ and, therefore, totally geodesic. □

Definition 5. *A complete manifold S with a smooth binary operation $S \times S \to S$, $(x, y) \mapsto x \cdot y$ is a Loos-symmetric space if it satisfies the axioms:*
(A1) *$x \cdot x = x$ for all $x \in S$;*
(A2) *$x \cdot (x \cdot y) = y$ for all $x, y \in S$;*
(A3) *$x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$ for all $x, y, z \in S$;*
(A4) *Every $x \in S$ has an open neighborhood $\mathcal{U}_x \subset S$ such that $x \cdot y = y$ for $y \in \mathcal{U}_x$ implies $y = x$.*

Theorem 6. (Loos [5], cf. also [6]) *Any Riemannian symmetric space $S = G/K$ is a Loos symmetric space $(S, \cdot)$ with respect to $x \cdot y := s_x(y)$ for the involutive isometry $s_x : S \to S$ with isolated fixed point $x \in S$. Any Loos-symmetric space $(S, \cdot)$ is supported by a Riemannian symmetric space S.*

Any Loos-symmetric space $(S, \cdot)$ is a left quasi-group with $x_o(a, b) = a \cdot b$ for all $a, b \in S$ by (A2) from Definition 5.

Definition 7. *A Loos-symmetric space $(S, \cdot)$ is of non-compact type if for any $y \in S$ with $x \cdot y = y$ it follows $x = y$.*

Proposition 8. *The following conditions are equivalent for a Loos-symmetric space $(S, \cdot)$:*
(i) *S is a Riemannian symmetric space of non-compact type;*
(ii) *$(S, \cdot)$ is a Loos-symmetric space of non-compact type;*
(iii) *$(S, \cdot)$ is a quasi-group.*

Proof. (i) ⇒ (ii). Since $\exp_x : T^{\mathbb{R}}_x S \to S$ is a global diffeomorphism, $x.y = s_x \exp_x \exp_x^{-1}(y) = \exp_x(-\exp_x^{-1}(y)) = \exp_x(\exp_x(y)) = y$ is equivalent to $\exp_x^{-1}(y) = 0$ and holds only for $x = y$.

(ii) ⇒ (i). If the Riemannian symmetric space S is of compact type, there is a periodic geodesic $\gamma : \mathbb{R} \to S$ through $\gamma(0) = \check{o}$ with minimal period $t_o \in \mathbb{R}^{>0}$. Then $s_{\check{o}}\left(\gamma\left(\frac{t_o}{2}\right)\right) = \gamma\left(-\frac{t_o}{2}\right)$ implies $\gamma\left(\frac{t_o}{2}\right) = \check{o}$, whereas $\gamma\left(t + \frac{t_o}{2}\right) = \gamma(t)$ for all $t \in \mathbb{R}$, which is an absurd.

(ii) $\Rightarrow$ (iii). By assumption, $y \cdot a = a$ forces $y = a$. If $a \neq b$ and $\gamma_{y,a} : \mathbb{R} \to S$ is the unique geodesic with $\gamma_{y,a}(0) = y$, $\gamma_{y,a}(1) = a$ then $b = \gamma_{y,a}(-1)$ and the unique solution $y_o(a,b) \in S$ of $y \cdot a = b$ is the middle point of the geodesic segment from a to b.

(iii) $\Rightarrow$ (ii). If $y \cdot a = a = a \cdot a$ has unique solution in S then $y = a$. $\square$

Definition 9. *A submanifold S_1 of a Loos-symmetric space $(S, \cdot)$ of non-compact type is a Loos-symmetric subspace if for all $a, b \in S$ it holds $a \cdot b, x_o(a,b), y_o(a,b) \in S_1$.*

Definition 10. *A complete manifold S with a smooth binary operation $S \times S \to S$, $(x,y) \mapsto x * y$ is a Loos-Hermitian symmetric space of non-compact type if it satisfies the following axioms:*
(a1) *$x * x = x$ for all $x \in S$;*
(a2) *$x * \{x * [x * (x * y)]\} = y$ for all $x, y \in S$;*
(a3) *$x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in S$;*
(a4) *if $x * (x * y) = y$ for some $x, y \in S$ then $x = y$.*

Theorem 11. *A complete manifold S admits a Loos-Hermitian symmetric structure $(S, *)$ of non-compact type if and only if it is a Hermitian symmetric space of non-compact type. If so, then $(S, *^2)$ with $x *^2 y := x * (x * y)$ is the Loos-symmetric space of non-compact type, supported by S.*

Proof. The axioms (a1)-(a4) for $(S, *)$ imply that $(S, *^2)$ is a Loos-symmetric space of non-compact type. The integrable almost complex structures

$$J_x : T_x^{\mathbb{R}} S \to T_x^{\mathbb{R}} S, \quad J_x(u) := \exp_x^{-1}(x * \exp_x(u))$$

turn the Riemannian symmetric space S of non-compact type into a Hermitian symmetric space of non-compact type.

Let $S = G/K$ be a Hermitian symmetric space of non-compact type and $J_x : T_x^{\mathbb{R}} S \to T_x^{\mathbb{R}} S$ be the almost complex structure at $x \in S$. Then it is clear that $x * y = j_x(y) := \exp_x J_x \exp_x^{-1}(y)$ is subject to (a1), (a2), (a4) from Definition 10 and $(S, *^2)$ satisfies Definition 7. Towards (a3), note that $S = \cup_{v \in T_y^{\mathbb{R}} S} \gamma_y^v(\mathbb{R})$ is covered by the images of the geodesics $\gamma_y^v : \mathbb{R} \to S$ with $\gamma_y^v(0) = y$, $\left.\frac{d\gamma_y^v(t)}{dt}\right|_{t=0} = v$. Further,

$$j_y \gamma_y^v(t) = j_y \exp_y(tv) = \exp_y(t J_y(v)) = \gamma_y^{J_y(v)}(t)$$

and $j_x\gamma_y^v(t) = \gamma_{j_x(y)}^{(dj_x)_y v}(t)$ imply that $j_x j_y \gamma_y^v(t) = \gamma_{j_x(y)}^{(dj_x)_y J_y(v)}(t)$. The differentials of the holomorphic isometries $j_x : S \to S$ are subject to $(dj_x)_y J_y = J_{j_x(y)}(dj_x)_y$. As a result,

$$j_x j_y \gamma_y^v(t) = \gamma_{j_x(y)}^{J_{j_x(y)}(dj_x)_y(v)}(t) = j_{j_x(y)}\gamma_{j_x(y)}^{(dj_x)_y(v)}(t) = j_{j_x(y)} j_x \gamma_y^v(t),$$

whereas $j_x j_y = j_{j_x(y)} j_x$ and (a3). □

Corollary 12. *The following conditions are equivalent for a variation of Hodge structure $U \subset D$ and the projection $\pi : D \to S$ onto a Riemannian symmetric space S of non-compact type:*

(i) *$U \subset D$ is totally geodesic;*

(ii) *$(\pi(U), \cdot)$ is a Loos-symmetric subspace of $(S, \cdot)$;*

(iii) *there is such a Loos-Hermitian symmetric structure $(\pi(U), *)$ of non-compact type that $(\pi(U), *^2)$ with $x *^2 y := x * (x * y)$ is a Loos-symmetric subspace of $(S, \cdot)$.*

Proof. For a Riemannian symmetric space R of non-compact type and $x \in R$, let $s_x^R : R \to R$ be the geodesic isometry with unique fixed point x.

(i) $\Rightarrow$ (ii). Any totally geodesic subspace $W \subset S$ is Riemannian symmetric and, therefore, Loos-symmetric with

$$x \cdot y = s_x^W(y) = \exp_x^W(-(\exp_x^W)^{-1}(y)) = \exp_x^S(-(\exp_x^S)^{-1}(y)) = s_x^S(y)$$

for all $x, y \in W$.

(ii) $\Rightarrow$ (i). By Proposition 8, $W = G(W)/K(W) \subset S$ is an equivariantly embedded Riemannian symmetric subspace of non-compact type, as far as $G(W) = \langle s_x^W = s_x^S|_W \,\Big|\, x \in W \rangle$ is a subgroup of G.

(i) $\Rightarrow$ (iii). follows from the Hermitian symmetry of the totally geodesic $U \subset D$ and (iii) $\Rightarrow$ (ii) is obvious. □

Acknowledgements: Research partially supported by Contract 144/ 2015 with the Scientific Foundation of Kliment Ohridski University of Sofia.

REFERENCES

[1] Liu K., Ch. Yin, Quantum correction and the moduli spaces of Calabi-Yau manifolds, arXiv:1411.0069v1[mathDG].

[2] Friedman R., R. Laza (2013) Duke Math. J. 162, No. 12, 2077-2148.

[3] Helgason S. (1978) *Differential Geometry, Lie Groups, and Symmetric Spaces,* London, Academic Press, pp. 224-226.

[4] Sabinin L.(1999) *Smooth Quasigroups and Loops,* Nordrecht, Kluwer Academic Publishers, pp. 40-41.

[5] Loos, O. (1969) *Symmetric Spaces. I: General Theory. II: Compact Spaces and Classification,* New York-Amsterdam, W. A. Benjamin, Inc., pp. 45-51 in the Russian translation, Moscow, Nauka, 1985.

[6] Nagy P., K. Strambach (2002) *Loops in Groups Theory and Lie Theory,* Berlin-New York, Walter de Gruyter, pp. 80-82.

Department of Mathematics and Informatics
Kliment Ohridski University of Sofia
5 James Bouchier Blvd., 1164 Sofia, Bulgaria
e-mail: pbeshkov@gmail.com, kasparia@fmi.uni-sofia.bg