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VARIATIONS OF HODGE STRUCTURE, EXPRESSED BY MEROMORPHIC DIFFERENTIALS ON THE PROJECTIVE PLANE

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The tautological variations of Hodge structure over Siegel upper half space, the open quadric and the generalized ball are expressed explicitely by the variations of Hodge structure of Weil hypersurfaces in projective spaces. That realizes all the abelian-motivic variations of Hodge structure by families of Jacobians of plane curves, which are known to be described by meromorphic differentials on the projective plane. As a consequence, the geometric origin of a maximal dimensional variation of Hodge structure turns to be sufficient for expressing it by meromorphic differentials on the projective plane.

Keywords: tautological variations of Hodge structure and J-Hodge structure, abelian-motivic and hypersurface variations

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For a smooth projective manifold X, defined over a field $k \subset \mathbb{C}$ of finite type, Hodge has conjectured that $H^{2w}(X,\mathbb{Q}) \cap H^{w,w}$, $w \leq \dim_{\mathbb{C}} X$, consists of the $\mathbb{Q}$-linear combinations of the cohomology classes of the algebraic submanifolds of X. Let $\overline{k}$ be the algebraic closure of k and $H^{2w}_{sc}(X,\mathbb{Q}_l) \subset H^{2w}(X,\mathbb{Q}_l)$ be the subspace of l-adic cohomologies, over which the action of the Galois group $Gal(\overline{k}/k)$ reduces to multiplication by scalars. Tate has conjectured that $H^{2w}_{sc}(X,\mathbb{Q}_l)$ is generated by the cohomologies of the algebraic submanifolds of X. For abelian varieties X, Tate conjecture is known to imply Hodge conjecture (cf. Deligne [5]), but neither of them is proved. Let X be a separable scheme of finite type over $\mathbf{F}_q$ and $\overline{X}$ be the scheme obtained from X by extension of the scalars to $\overline{\mathbf{F}_q}$. For primes $l \neq q$, the l-adic cohomologies $H^w_c(\overline{X},\mathbb{Q}_l)$ with compact support are acted by Frobenius automorphism $\varphi \in Gal(\overline{\mathbf{F}_q}/\mathbf{F}_q)$, $\varphi(x) = x^q$. Weil conjecture asserts that the characteristic roots of φ^{-1} on $H^w_c(\overline{X}, \mathbb{Q}_l)$ are of absolute value $q^{-\frac{w}{2}}$. It is verified for the abelian varieties and therefore, for any X whose cohomologies are expressed through linear algebra constructions (by the cohomologies) of abelian varieties. Hodge, Tate and Weil conjectures motivate the interest in the abelian varieties A and their Hodge structures $H^w(A, \mathbb{Z}) = \wedge^w H^1(A, \mathbb{Z}) \simeq \wedge^w A$.

The present work concerns the variations of Hodge structure which are expressed by a special kind of abelian varieties, namely, by Jacobians of plane curves. The members of a family $\mathcal{J} \to S$ of Jacobians of plane curves, as well as the infinitesimal variations $T^{1,0}_s S$, $s \in S$, can be identified with subspaces of meromorphic differentials on $\mathbf{P}_2$. We exhibit explicit embeddings of the so-called tautological variations over Siegel upper half space $\mathcal{S}(p)$, the open quadric $\mathcal{Q}(p)$ and the generalized ball $\mathcal{B}(p,q)$, in the variations of Hodge structure of Weil hypersurfaces $X \subset \mathbf{P}_N$. Shermenev shows in [11] that the Hodge structures of X are expressed by meromorphic differentials on $\mathbf{P}_2$. So far, Kuga and Satake [8], Deligne [4], Carlson and Simpson [3] have established that the aforementioned tautological variations are expressed by abelian varieties. The abelian varieties are known to be from the tensor category of the Jacobians of all curves (cf. [12]). On the other hand, Rapoport [10] has classified the complete intersections Y, whose variations of Hodge structure are exactly the tautological variations over $\mathcal{S}(p)$ or $\mathcal{Q}(p)$. All such Y turn to be of Hodge level 1 or 2. Our X are of arbitrary Hodge level, equal to $\dim_{\mathbb{C}} X$, and we realize the tautological variations as proper subfamilies of the variations of X.

The provided construction reveals that all the abelian-motivic variations of Hodge structure are expressed by meromorphic differentials on $\mathbf{P}_2$. As another consequence, the geometrically arising variations of maximum dimension turn to be realized in the tensor category of Jacobians of plane curves.

1. PRELIMINARIES

1.1. TAUTOLOGICAL VARIATIONS OF HODGE STRUCTURE AND J-HODGE STRUCTURE

Hodge structure on a $\mathbb{C}$-vector space $V \doteq V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ defined over $\mathbb{Q}$ consists of Hodge decomposition $V = \sum_{i=0}^{w} V^{w-i,i}$, compatible with the complex conjugation $\overline{V^{w-i,i}} = V^{i,w-i}$, and a non-degenerate bilinear polarization form $\Psi : V_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}} \to \mathbb{Q}$, which is symmetric for an even weight w or skew-symmetric for an odd w. Hodge decomposition is orthogonal with respect to the Hermitian form $\Phi(a,b) := \Psi(a, \overline{b})$ for $a, b \in V$, and $\Phi|_{V^{w-2i,2i}} > 0$, $\Phi|_{V^{w-2i-1,2i+1}} < 0$. J-Hodge structure on $V = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ is Hodge structure with an endomorphism $J : V_{\mathbb{Q}} \to V_{\mathbb{Q}}$, $J^2 = -Id$, such that J is orthogonal with respect to Ψ, unitary with respect to Φ and compatible with Hodge decomposition $J : V^{w-i,i} \to V^{w-i,i}$.

The classifying space of Hodge structures on V with fixed bilinear form Ψ, Hermitian form Φ, and Hodge numbers $h^i := \dim_{\mathbb{C}} V^{w-i,i} > 0$ is the homogeneous space

$$D(V,\Psi,\Phi)=O(V,\Psi)\cap U(V,\Phi)/\prod_{i=0}^{[\frac{w}{2}]-1}U(h^i)\times(1-\varepsilon(w))O(h^{[\frac{w}{2}]},\mathbb{R}),$$

where $O(V,\Psi)$ is the orthogonal group of V with respect to Ψ, $O(h^{[\frac{w}{2}]},\mathbb{R})=O\left(V^{[\frac{w}{2}],[\frac{w}{2}]},\Psi|_{V^{[\frac{w}{2}],[\frac{w}{2}]}}\right)$, $U(V,\Phi)$ is the unitary group of V with respect to Φ, $U(h^i)=U\left(V^{w-i,i},\Phi|_{V^{w-i,i}}\right)$, and $\varepsilon(w):=w-2[\frac{w}{2}]$ stands for the parity of the weight w.

The semisimple linear automorphisms $J|_{V^{w-i,i}}$ split $V^{w-i,i}=V_+^{w-i,i}\oplus V_-^{w-i,i}$ in $\pm\sqrt{-1}$-eigenspaces with $\overline{V_+^{w-i,i}}=V_-^{i,w-i}$. As a result, Hodge decomposition of the Ψ-isotropic $V_+:=\sum_{i=0}^{w}V_+^{w-i,i}$ determines completely Hodge decomposition of $V=V_+\oplus\overline{V_+}$, and the classifying space of J-Hodge structures on V turns to be

$$D(V,\Psi,\Phi,J)\simeq U(h_+^0+h_+^2+\cdots,h_+^1+h_+^3+\cdots)/U(h_+^0)\times\cdots\times U(h_+^w),$$

where

$$h_+^i:=\dim_{\mathbb{C}}V_+^{w-i,i},$$
$$U(h_+^0+h_+^2+\cdots,h_+^1+h_+^3+\cdots)=U\left(V_+,\Phi|_{V_+}\right),$$
$$U(h_+^i)=U\left(V_+^{w-i,i},\Phi|_{V_+^{w-i,i}}\right).$$

The classifying spaces $D=D(V,\Psi,\Phi)$, respectively, $D=D(V,\Psi,\Phi,J)$ are open subsets of quotients $\check{D}=O(V,\Psi)/P(V)$, respectively, $\check{D}=GL(V_+,\mathbb{C})/P(V_+)$ of reductive complex algebraic groups $G^{\mathbb{C}}$ by parabolic subgroups P, stabilizing Hodge filtrations $F^i:=\sum_{j\geq i}V^{j,w-j}$, respectively, $F_+^i:=\sum_{j\geq i}V_+^{j,w-j}$. Hodge decompositions of V, V_+ induce weight zero Hodge decompositions $\mathfrak{g}^{\mathbb{C}}=LieG^{\mathbb{C}}=\sum_{i=-w}^{w}\mathfrak{g}^{i,-i}$ with $\mathfrak{g}^{i,-i}=\{\tau\in\mathfrak{g}^{\mathbb{C}}|\tau(V^{j,w-j})\subseteq V^{i+j,w-i-j}$ for all $0\leq j\leq w\}$. The parabolic subalgebras $LieP=\sum_{i\geq 0}\mathfrak{g}^{i,-i}$. The holomorphic tangent bundle $T^{1,0}D=T^{1,0}\check{D}|_D=[G^{\mathbb{C}}\times_P(LieG^{\mathbb{C}}/LieP)]_D$ contains an equivariant subbundle $T^hD:=\left[G^{\mathbb{C}}\times_P(\mathfrak{g}^{-1,1}+LieP)/LieP\right]_D$, associated with a non-integrable distribution and called horizontal. As far as an arbitrary family of Hodge structures with fixed Ψ, Φ, h^i is induced by the tautological family over D, there is no loss in regarding the base S of this family as a complex analytic subspace of $\Gamma\setminus D$ for some discrete subgroup Γ of the biholomorphism group G of $D=G/G\cap P$. Variation of Hodge structure is a family $\mathcal{V}\to S$, whose base S is locally tangent to the horizontal distribution T^hD. The complete tautological families of Hodge structures over $D(V,\Psi,\Phi)$ or, respectively, the complete tautological families of J-Hodge structures over $D(V,\Psi,\Phi,J)$, which are variations of Hodge structure are referred to as tautological variations.

Lemma 1. *All the tautological variations of Hodge structure are*

(i) $\mathcal{V}_{\mathcal{S}(p)}=\sum_{i=0}^{1}\mathcal{V}_{\mathcal{S}(p)}^{1-i,i}$ *of* $rank\mathcal{V}_{\mathcal{S}(p)}^{1-i,i}=p$ *over Siegel upper half spaces* $\mathcal{S}(p)=Sp(p,\mathbb{R})/U(p)$ *and*

(ii) $\mathcal{V}_{\mathcal{Q}(p)} = \sum_{i=0}^{2} \mathcal{V}_{\mathcal{Q}(p)}^{2-i,i}$ *of* $rank\mathcal{V}_{\mathcal{Q}(p)}^{2,0} = rank\mathcal{V}_{\mathcal{Q}(p)}^{0,2} = 1$, $rank\mathcal{V}_{\mathcal{Q}(p)}^{1,1} = p$ *over open quadrics* $\mathcal{Q}(p) = SO(2,p)/SO(2) \times SO(p)$.

All the tautological variations of J-Hodge structure are

(iii) $\mathcal{V}_{J,p,q}^{w} = \sum_{i=0}^{1} \left(\mathcal{V}_{+}^{w-i,i} + \overline{\mathcal{V}_{+}^{w-i,i}} \right)$ *of weight* $1 \leq w \leq 3$ *and* $rank\mathcal{V}_{+}^{w,0} = p$, $rank\mathcal{V}_{+}^{w-1,1} = q$ *over generalized balls* $\mathcal{B}(p,q) = U(p,q)/U_p \times U_q$.

The corresponding polarizations Ψ_D, $\Psi_{J,p,q}^{w}$ *are*

$$\begin{pmatrix} & I_p \\ -I_p & \end{pmatrix}, \quad \begin{pmatrix} & & 1 \\ & -I_p & \\ 1 & & \end{pmatrix}, \quad \begin{pmatrix} & & & I_p \\ & & (-1)^{w-1} I_q & \\ & -I_q & & \\ (-1)^{w} I_p & & & \end{pmatrix}.$$

Proof. The existence of tautological variations is equivalent to $T^{1,0}D = T^h D$ for the corresponding classifying space D. In the case of $w = 2k+1 \geq 3$ the symplectic Lie algebra $\mathfrak{g}^{\mathbb{C}} = o(V,\Psi)$ has $\mathfrak{g}^{-2,2} \neq 0$. For an even weight $w = 2k \geq 2$, the indefinite orthogonal Lie algebra $\mathfrak{g}^{\mathbb{C}} = o(V,\Psi) = \mathfrak{g}^{-1,1} + LieP$ if and only if $w = 2$ and $h^0 = 1$. That justifies the classification of the tautological variations of Hodge structure. In the case of J-Hodge structures one can assume that $h_{+}^{0} \neq 0$, after eventual shift $(w-i,i) \mapsto (i,w-i)$ of Hodge indices. Then $T^{1,0}D(V,\Psi,\Phi,J) = T^h D(V,\Psi,\Phi,J)$ holds only when $D(V,\Psi,\Phi,J) \simeq U(h_{+}^{0}, h_{+}^{1})/U(h_{+}^{0}) \times U(h_{+}^{1})$. The weights $w \leq 3$, since otherwise for $2 \leq j \leq w-2$ there follow $h_{+}^{j} = 0$ and $h_{-}^{j} = 0$, contrary to the assumption $h^j \neq 0$. □

1.2. THE TAUTOLOGICAL VARIATIONS ARE ABELIAN-MOTIVIC

A variation is said to be abelian-motivic or expressed by abelian varieties if it is a direct summand of a tensor polynomial with $\mathbb{N}$-coefficients of variations of Hodge structure of abelian varieties. All the tautological variations are expressed by abelian varieties. More precisely:

Theorem 2. (i) (obvious) *The tautological variation* $\mathcal{V}_{\mathcal{S}(p)}$ *is the variation of Hodge structure of a polarized abelian variety* $A \simeq H^1(A,\mathbb{C})$ *of* $\dim_{\mathbb{C}} A = p$.

(ii) (Kuga and Satake [8], Deligne [4]) *Let* $C^{+}(V,\Psi_{\mathcal{Q}(p)})$ *be the even part of the Clifford algebra* $C(V,\Psi_{\mathcal{Q}(p)})$ *of the reference Hodge structure* $(V,\Psi_{\mathcal{Q}(p)}) \in \mathcal{V}_{\mathcal{Q}(p)}$. *Then there is a family* $\mathcal{A} \to \mathcal{Q}(p)$ *of* 2^{p+1}*-dimensional abelian varieties such that the variation of* $C^{+}(V,\Psi_{\mathcal{Q}(p)})$ *is*

$$C^{+}\left(\mathcal{V}_{\mathcal{Q}(p)}, \Psi_{\mathcal{Q}(p)}\right) = End_{C^{+}(V,\Psi_{\mathcal{Q}(p)})}(\mathcal{A}).$$

(iii) (Carlson and Simpson [3]) *The tautological variation of J-Hodge structure* $\mathcal{V}_{J,p,q}^{1}$ *is the restriction of* $\mathcal{V}_{\mathcal{S}(p+q)}$ *to a holomorphically and equivariantly embedded* $\mathcal{B}(p,q) \hookrightarrow \mathcal{S}(p+q)$. *Let E be the elliptic curve* $\mathbb{C}/(\mathbb{Z}+\sqrt{-1}\mathbb{Z})$, *J be the endomorphism of* $H^1(E,\mathbb{C})$ *induced from the multiplication by* $\sqrt{-1}$ *on E,* $\mathbf{E} = \mathbf{E}^{1,0} + \mathbf{E}^{0,1}$ *be the constant family of the aforementioned J-Hodge structure and* $\mathbf{E}^{r,s}(m)$ *be the* *m-th tensor power of* $\mathbf{E}^{r,s}$. *Then*

$$\mathcal{V}_{J,p,q}^{w} = \mathcal{V}_{+}^{1} \otimes \mathbf{E}^{1,0}(w-1) \oplus \overline{\mathcal{V}_{+}^{1}} \otimes \mathbf{E}^{0,1}(w-1)$$

of weight $w = 2$ *or* 3 *are expressed by* $\mathcal{V}_{J,p,q}^{1} = \mathcal{V}_{+}^{1} \oplus \overline{\mathcal{V}_{+}^{1}}$.

1.3. COMPLETE INTERSECTIONS WITH TAUTOLOGICAL VARIATIONS OF HODGE STRUCTURE

Let $X_n^{d_1,\dots,d_k} \subset \mathbf{P}_{n+k}$ be a complete intersection of hypersurfaces of degree $d_1, \dots, d_k$. The primitive cohomologies $H^*(X_n^{d_1,\dots,d_k}, \mathbb{C})_o$, i.e., the cohomologies which are not dual to intersections of $X_n^{d_1,\dots,d_k}$ with subspaces $\mathbf{P}_m \subset \mathbf{P}_{n+k}$, have only nonzero components $H^n(X_n^{d_1,\dots,d_k}, \mathbb{C})_o$. From now on, under a variation of Hodge structure of a complete intersection $X_n^{d_1,\dots,d_k}$ we mean the variation of Hodge structure on $H^n(X_n^{d_1,\dots,d_k}, \mathbb{C})_o$. If $h^j := \dim_{\mathbb{C}} H^{n-j,j}(X_n^{d_1,\dots,d_k})_o$ vanish for all $j < i$ and $j > n-i$, $h^i = h^{n-i} \neq 0$, then the integer $n - 2i$ (which is one less that the number of the non-trivial Hodge components of $H^n(X_n^{d_1,\dots,d_k}, \mathbb{C})_o$) is called level of $X_n^{d_1,\dots,d_k}$ or of its Hodge structure.

Theorem 3 (Rapoport [10]). (i) *All the complete families* $\mathcal{X}_{n,\mathcal{S}(p)}^{d_1,\dots,d_k}$ *of* $X_n^{d_1,\dots,d_k} \subset \mathbf{P}_{n+k}$, *whose associated variations of Hodge structure are discrete quotients of level one tautological variations over Siegel upper half spaces* $\mathcal{S}(p)$, *are* $\mathcal{X}_{2n-1,\mathcal{S}(n)}^{2,2}$, $\mathcal{X}_{2n-1,\mathcal{S}(2n^2+3n)}^{2,2,2}$, $\mathcal{X}_{3,\mathcal{S}(5)}^{3}$, $\mathcal{X}_{3,\mathcal{S}(20)}^{2,3}$, $\mathcal{X}_{5,\mathcal{S}(21)}^{3}$, $\mathcal{X}_{3,\mathcal{S}(30)}^{4}$.

(ii) *The complete families* $\mathcal{X}_{n,\mathcal{Q}(p)}^{d_1,\dots,d_k}$ *of* $X_n^{d_1,\dots,d_k} \subset \mathbf{P}_{n+k}$, *whose associated variations of Hodge structure are discrete quotients of level two tautological variations over open quadrics* $\mathcal{Q}(p)$, *are depleted by the families* $\mathcal{X}_{2,\mathcal{Q}(19)}^{2,2,2}$, $\mathcal{X}_{2,\mathcal{Q}(19)}^{2,3}$, $\mathcal{X}_{2,\mathcal{Q}(19)}^{4}$ *of* $K3$ *surfaces and the family* $\mathcal{X}_{4,\mathcal{Q}(20)}^{3}$ *of cubic fourfolds.*

2. EXPLICIT CONSTRUCTIONS

Let us fix some standard notations. The Hodge structure on the second cohomology group $H^2(\mathbf{P}_1, \mathbb{C}) = H^{1,1}(\mathbf{P}_1)$ of the projective line $\mathbf{P}_1$ or, equivalently, on the cup product $\wedge^2 H^1(E, \mathbb{C}) = H^{1,0}(E) \wedge H^{0,1}(E)$ of the first cohomology group of the elliptic curve $E = \mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$ is called Tate Hodge structure. The constant family of Tate Hodge structures (over an arbitrary base) is denoted by $\mathbb{Q}(1)$. If $m \in \mathbb{N}$, then $\mathbb{Q}(m)$ and $\mathbb{Q}(-m)$ designate the m-th tensor powers of $\mathbb{Q}(1)$ and, respectively, $\mathbb{Q}(-1) = Hom(\mathbb{Q}(1), \mathbb{C})$. The polarization $\Psi^{\mathbf{E}} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ of the constant family $\mathbf{E}$ of J-Hodge structures on E induces the polarization $\Psi^{\mathbb{Q}(1)} = \Psi^{\mathbf{E}} \otimes \Psi^{\mathbf{E}} = 1$. In other words, all $\Psi^{\mathbb{Q}(m)}$, $m \in \mathbb{Z}$, coincide with the multiplication by complex numbers.

Theorem 4. *For the Hermitian symmetric spaces*

$$D = \mathcal{S}(p) = \{z \in Mat_{p,p}(\mathbb{C}) \mid {}^t z = z, z^t\overline{z} < I_p\} \text{ or}$$

$$D = \mathcal{B}(p,q) = \{z \in Mat_{p,q}(\mathbb{C}) \mid z^t\overline{z} < I_p\}, \quad p \leq q,$$

let $\delta = p+1$ *or, respectively,* $\delta = q+1$ *and consider the hypersurfaces* $\mathcal{X}_D(z) \subset \mathbf{P}_{2p+2}$, $z \in D$, *determined by the homogeneous equations*

$$\sum_{i=-1}^{0} x_i^{4\delta} + \sum_{i=1}^{p} \left(x_{2i-1}^{4\delta} + \frac{1}{2\delta} \sum_{j=1}^{\delta-1} z_{ij} x_{2i-1}^{4\delta-4j} x_{2i}^{4j} + x_{2i}^{4\delta} \right) + x_{2p+1}^{4\delta} = 0$$

of degree $d = 4\delta$. *Over the open quadric*

$$D = \mathcal{Q}(p) = \{ z \in Mat_{p,1}(\mathbb{C}) \mid |{}^t zz| < 1, 2^t\overline{z}z < 1 + |{}^t zz|^2 \}$$

define the family $\mathcal{X}_{\mathcal{Q}(p)}$ *of hypersurfaces*

$$\mathcal{X}_{\mathcal{Q}(p)}(z) = \left\{ x \in \mathbf{P}_{2p-1} \;\middle|\; \sum_{i=0}^{p-1} \left(x_{2i-1}^{4p} + \frac{1}{2p} z_i x_{2i-1}^{2p} x_{2i}^{2p} + x_{2i}^{4p} \right) = 0 \right\}$$

of degree $d = 4p$. *Let us denote by* $\mathcal{H}_D$ *the variations of Hodge structure of* $\mathcal{X}_D$ *with polarizations* $\Psi^{\mathcal{H}_D}$. *Put* $w_{\mathcal{S}(p)} := 1$, $w_{\mathcal{Q}(p)} := 2$ *for the weights* w_D *of the tautological variations of Hodge structure* $\mathcal{V}_D$, δ_j^i *for Kronecker's delta and introduce* $m_D := p - \delta_{w_D}^2 w_D$. *Then the components of the tautological variations of Hodge structure are the subbundles*

$$\mathcal{V}_D^{i,w_D-i} \subset \mathcal{H}_D^{m_D+i,m_D+w_D-i} \otimes \mathbb{Q}(-m_D)$$

of abelian-motivic variations, expressed by meromorphic differentials on $\mathbf{P}_2$, *and* $\Psi_D = \Psi^{\mathcal{H}_D} \otimes \Psi^{\mathbb{Q}(-m_D)}|_{\mathcal{V}_D}$. *The tautological variations of J-Hodge structure have*

$$\mathcal{V}_{J,p,q}^{i,w-i} \subset \sum_{j=0}^{1} \left(\sum_{k=0}^{1} \delta_{j(w-1)+k}^i \right) \mathcal{H}_{\mathcal{B}(p,q)}^{p+i-j(w-1),p+1-i+j(w-1)} \otimes \mathbf{E}^{j,1-j}(w-1) \otimes \mathbb{Q}(-p)$$

for $1 \le w \le 3$, $0 \le i \le w$, *expressed by meromorphic differentials on* $\mathbf{P}_2$, *and* $\Psi_{J,p,q}^w = \Psi^{\mathcal{H}_{\mathcal{B}(p,q)}} \otimes \Psi^{\mathbf{E}(w-1)} \otimes \Psi^{\mathbb{Q}(-p)}\Big|_{\mathcal{V}_{J,p,q}^w}$.

The proof is subdivided into several steps and presented by the rest of the section.

2.1. A SMOOTH FAMILY $\mathcal{X}$ OF HYPERSURFACES OVER A PRODUCT OF BALLS

Lemma 5. *All the hypersurfaces* $\mathcal{X}(z) = \{x \in \mathbf{P}_{n+1} \mid f_z(x) = 0\}$ *with homogeneous equations*

$$f_z(x) = \sum_{i=0}^{[\frac{n}{2}]} \left(x_{2i-1}^d + \frac{2}{d} \sum_{j=1}^{d-1} z_{ij} x_{2i-1}^{d-j} x_{2i}^j + x_{2i}^d \right) + \varepsilon(n) x_n^d = 0$$

of degree $d > n+2 \ge 3$, *parametrized by the product of balls*

$$\mathcal{B}(1,d-1)^{[\frac{n}{2}]+1} = \left\{ z \in Mat_{[\frac{n}{2}]+1,d-1}(\mathbb{C}) \;\middle|\; \sum_{j=1}^{d-1} |z_{ij}|^2 < 1, \quad \forall i \; 0 \leq i \leq \left[\frac{n}{2}\right] \right\},$$

are smooth.

Proof. One has to justify that the system of the polynomial equations $\frac{\partial f_z}{\partial x_{2i-1}} = \frac{\partial f_z}{\partial x_{2i}} = 0$ with $0 \leq i \leq \left[\frac{n}{2}\right]$ and $\frac{\partial f_z}{\partial x_n} = 0$ for an odd n has only the trivial solution $x_{-1} = x_0 = x_1 = \dots = x_n = 0$. As far as the pairs $\frac{\partial f_z}{\partial x_{2i-1}}, \frac{\partial f_z}{\partial x_{2i}}$ depend only on x_{2i-1}, x_{2i}, this system splits in $\left[\frac{n}{2}\right] + 1$ parts of two equations with two variables. For x_{2i-1}, x_{2i} with $|x_{2i-1}| \leq |x_{2i}| \neq 0$ one puts $y := \frac{x_{2i-1}}{x_{2i}}$ in order to express $\frac{\partial f_z}{\partial x_{2i}}$ as $\frac{2}{d}\sum_{j=1}^{d-1} j z_{ij} y^{d-j} + d = 0$. According to Cauchy–Schwarz inequality,

$$\left(\frac{d^2}{2}\right)^2 = \left|\sum_{j=1}^{d-1} z_{ij}(jy^{d-j})\right|^2 \leq \left(\sum_{j=1}^{d-1} |z_{ij}|^2\right)\left(\sum_{j=1}^{d-1} j^2 |y|^{2(d-j)}\right).$$

Bearing in mind that $z \in \mathcal{B}(1,d-1)^{[\frac{n}{2}]+1}$ and $|y| \leq 1$, one infers that

$$\left(\frac{d^2}{2}\right)^2 < \sum_{j=1}^{d-1} j^2 = \frac{(d-1)d(2d-1)}{6} < \frac{2d^3}{6},$$

which contradicts $d \geq 3$. Similarly, for x_{2i-1}, x_{2i} with $|x_{2i}| \leq |x_{2i-1}| \neq 0$ one introduces $t := \frac{x_{2i}}{x_{2i-1}}$. Then converting the equation $\frac{\partial f_z}{\partial x_{2i-1}} = 0$ into the form $d + \frac{2}{d}\sum_{k=1}^{d-1} k z_{id-k} t^{d-k} = 0$, one gets an absurd. $\square$

2.2. EXPRESSING THE VARIATION OF HODGE STRUCTURE OF $\mathcal{X}$ BY MEROMORPHIC DIFFERENTIALS ON $\mathbf{P}_2$

Suppose that X_n of $\dim_{\mathbb{C}} X_n = n \geq 2$ is a hypersurface from the constructed smooth family, X_{n-1} is the intersection of X_n with the hyperplane $x_{-1} = 0$, and Y_1 is the plane curve with homogeneous equation $y_1^d = y_{-1}^d + \frac{2}{d}\sum_{j=1}^{d-1} z_{0j} y_{-1}^{d-j} y_0^j + y_0^d$. The presence of a rational map $X_{n-1} \times Y_1 \to X_n$,

$$((x_0 : x_1 : \dots : x_n), (y_{-1} : y_0 : y_1)) \mapsto (x_0 y_{-1} : x_0 y_0 : x_1 y_1 : \dots x_i y_1 : \dots : x_n y_1)$$

of degree d with singular locus $(X_{n-1} \cap \{x_0 = 0\}) \times (Y_1 \cap \{y_1 = 0\})$ justifies the next

Lemma 6 (Shermenev [11], Shioda and Katsura [13]). *In the notations, introduced in Lemma* 5, *let us fix a hypersurface* $X_n := \mathcal{X}(z)$ *of dimension* $n \geq 2$ *and consider the complete intersections* $X_{n-1} := X_n \cap \{x_{-1} = 0\}$, $X_{n-2} := X_{n-1} \cap \{x_0 = 0\}$, *the plane curve*

$$Y_1 := \left\{ y \in \mathbf{P}_2 \;\middle|\; y_{-1}^d + \frac{2}{d}\sum_{j=1}^{d-1} z_{0j} y_{-1}^{d-j} y_0^j + y_0^d - y_1^d = 0 \right\},$$

and the points $\{p_1, \ldots, p_d\} := Y_1 \cap \{y_1 = 0\}$. *Then* X_n *can be obtained by a blow up* $\beta_1 : Z_1 \to X_{n-1} \times Y_1$ *along* $X_{n-2} \times \{p_1, \ldots, p_d\} \simeq dX_{n-2}$, *a morphism* $\zeta : Z_1 \to Z_2$ *of degree* d *and a blow down* $\beta_2 : Z_2 \to X_n$, *contracting* $\zeta(X_{n-1} \times p_i) = \mathbf{P}_{n-1} \times p_i$ *to* p_i *and* $\zeta(X_{n-2} \times Y_1) \simeq X_{n-2} \times \mathbf{P}_1$ *to* X_{n-2}.

As an immediate consequence, the variation of Hodge structure of $\mathcal{X}$ is expressed by the variations of plane curves.

Corollary 7. *Given a Fermat hypersurface*

$$Z = \left\{ z \in \mathbf{P}_{[\frac{n}{2}]+\varepsilon(n)} \;\middle|\; \sum_{i=0}^{[\frac{n}{2}]+\varepsilon(n)} z_i^d = 0 \right\}$$

of dimension $k := \left[\frac{n}{2}\right] + \varepsilon(n) - 1$, *let us denote by* $\mathcal{H}^m_{Fermat}$ *the constant family of* $H^m(Z, \mathbb{C}) = H^m(\mathbf{P}_{k+1}, \mathbb{C})$ *for* $0 \leq m \leq k-1$ *and* $H^k(Z,\mathbb{C}) = H^k(\mathbf{P}_{k+1}, \mathbb{C}) + H^k(Z, \mathbb{C})_o$. *Then the variation of Hodge structure of the family of smooth hypersurfaces* $\mathcal{X} \to \mathcal{B}_0 \times \mathcal{B}_1 \times \ldots \times \mathcal{B}_{[\frac{n}{2}]}$, *defined in Lemma* 5, *is a direct summand of*

$$\sum_{0 \leq i_1 < \ldots < i_s \leq [\frac{n}{2}]} \eta_i \mathcal{H}^1(\mathcal{B}_{i_1}) \otimes \ldots \otimes \mathcal{H}^1(\mathcal{B}_{i_s}) \otimes \mathcal{H}^{s+\varepsilon(n)-2}_{Fermat} \otimes \mathbb{Q}\left(\left[\frac{n}{2}\right] + 1 - s\right),$$

where $\mathcal{H}^1(\mathcal{B}_t)$ *stands for the variation of the plane curves with homogeneous equations*

$$y_{-1}^d + \frac{2}{d}\sum_{j=1}^{d-1} z_{tj} y_{-1}^{d-j} y_0^j + y_0^d \pm y_1^d = 0,$$

$(z_{t1}, \ldots, z_{td-1}) \in \mathcal{B}_t$, *and* η_i, $i = (i_1, \ldots, i_s)$, *are natural numbers.*

Proof. According to [5], if $N \to M$ is a finite map of equi-dimensional connected manifolds, then $H^*(M, \mathbb{C})$ is a direct summand of $H^*(N, \mathbb{C})$, and if M' is a blow up of M along a closed submanifold T of codimension c, then $H^*(M', \mathbb{C}) = H^*(M, \mathbb{C}) + \sum_{i=1}^{c-1} H^*(T, \mathbb{C}) \otimes \mathbb{Q}(i)$. By Künneth formula, the variation $\mathcal{H}^n(Z_2)$ of $Z_2 = \zeta\beta_1^{-1}(X_{n-1} \times Y_1)$ is a direct summand of $\mathcal{H}^{n-1}(\mathcal{B}_1 \times \ldots \times \mathcal{B}_{[\frac{n}{2}]}) \otimes \mathcal{H}^1(\mathcal{B}_0) + d\mathcal{H}^{n-2}(\mathcal{B}_1 \times \ldots \times \mathcal{B}_{[\frac{n}{2}]}) \otimes \mathbb{Q}(1)$. On the other hand, $Z_2 = \beta_2^{-1}(X_n)$ implies the equality $\mathcal{H}^n(Z_2) = \mathcal{H}^n(\mathcal{B}_0 \times \mathcal{B}_1 \times \ldots \times \mathcal{B}_{[\frac{n}{2}]}) + \mathcal{H}^{n-2}(\mathcal{B}_1 \times \ldots \times \mathcal{B}_{[\frac{n}{2}]}) \otimes \mathbb{Q}(1)$ for the variation $\mathcal{H}^n(\mathcal{B}_0 \times \mathcal{B}_1 \times \ldots \times \mathcal{B}_{[\frac{n}{2}]})$ of $\mathcal{X}$. Consequently, $\mathcal{H}^n(\mathcal{B}_0 \times \ldots \times \mathcal{B}_{[\frac{n}{2}]})$ turns to be a direct summand of $\mathcal{H}^1(\mathcal{B}_0) \otimes \mathcal{H}^{n-1}(\mathcal{B}_1 \times \ldots \times \mathcal{B}_{[\frac{n}{2}]}) + (d-1)\mathcal{H}^{n-2}(\mathcal{B}_1 \times \ldots \times \mathcal{B}_{[\frac{n}{2}]}) \otimes \mathbb{Q}(1)$. The proof is proceeded by induction on $\left[\frac{n}{2}\right]$. □

2.3. EXPLICIT REALIZATIONS OF THE TAUTOLOGICAL VARIATIONS BY THE VARIATION OF $\mathcal{X}$

Let $\mathcal{X}$ be the family of smooth hypersurfaces, constructed in Lemma 5. Restricting to the bounded symmetric realizations of

$$D = \mathcal{S}(p) \subset \mathcal{B}(1,p)^p \subset \mathcal{B}(1,4p+3)^{p+1},$$

$$D = \mathcal{B}(p,q) \subset \mathcal{B}(1,q)^p \subset \mathcal{B}(1,4q+3)^{p+1},$$

$$D = \mathcal{Q}(p) \subset \mathcal{B}(p,1) \subset \mathcal{B}(1,1)^p \subset \mathcal{B}(1,4p-1)^p,$$

specified in Theorem 4, one obtains the families $\mathcal{X}_D = \cup_{z \in D}\{x \in \mathbf{P}_{n+1} \mid f_z(x) = 0\}$. Let $\mathbf{S}_D$ be the trivial family of polynomial rings $S = \mathbb{C}[x_{-1}, x_0, x_1, \ldots, x_n]$ over D, $\mathbf{J}_D$ be the family of Jacobian ideals $\mathbf{J}_z := \left\langle \frac{\partial f_z}{\partial x_i} \mid -1 \leq i \leq n \right\rangle \subset \mathbf{S}_z$, $z \in D$, and $\mathcal{R}_D := \mathbf{S}_D/\mathbf{J}_D$ be the family of Jacobian rings. Denote by $\mathbf{f}$ the sheaf of the equations f_z of $\mathcal{X}_D(z)$, $d := \deg f_z$, $\Delta(i) := -(n+2) + d(n+1-i)$, and $\Omega = \sum_{i=-1}^{n}(-1)^i x_i dx_{-1} \wedge dx_0 \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$. Griffiths [6] shows that the residue map

$$Res_D : \mathcal{R}_D^{\Delta(i)} \frac{\Omega}{\mathbf{f}^{n+1-i}} \to \mathcal{H}_D^{i,n-i} \simeq \mathcal{F}_D^i / \mathcal{F}_D^{i+1}$$

is an isomorphism. Carlson and Griffiths [1] establish that the non-degenerate pairing $\Psi^{\mathcal{H}_D} : \mathcal{H}_D^{i,n-i} \times \mathcal{H}_D^{n-i,i} \to \mathcal{H}_D^{2n}$, i.e., the Serre duality map with values in the constant family of $H^{2n}(\mathcal{X}_D(z), \mathbb{C}) = \mathbb{C}$, can be naturally identified with the ring multiplication $\mathcal{R}_D^{\Delta(i)} \times \mathcal{R}_D^{\Delta(n-i)} \to \mathcal{R}_D^{(d-2)(n+2)}$.

Lemma 8. *In the notations from Theorem 4, let*

$$\mathcal{V}_{\mathcal{B}(p,q)} := \mathcal{V}^1_{J,p,q}, \quad w_{\mathcal{B}(p,q)} := 1,$$

$$k(i,j) := ij + (1-i)(4\delta - 2 - j), \quad l(i,j) := i(2\delta - 2 - j) + (1-i)(2\delta + j).$$

Consider the holomorphic subbundles

$$\mathcal{W}_D^{m_D+i, m_D+w_D-i} \subset \mathcal{H}_D^{m_D+i, m_D+w_D-i}, \quad 0 \leq i \leq w_D,$$

generated by the global sections

$$Res_D \left\{ \left[w_j^{D,i} + \mathbf{J}_D^{\Delta(m_D+i)} \right] \frac{\Omega}{\mathbf{f}^{m_D+w_D-i+1}} \right\},$$

where

$$w_j^{D,i} := x_{-1}^{k(i,j)} x_0^{l(i,j)} \left(\prod_{t=1}^{2p+1} x_t \right)^{2\delta-1} \qquad \text{for } 1 \le j \le p, D = \mathcal{S}(p) \text{ or } \mathcal{B}(p,q);$$

$$w_{j+p}^{\mathcal{B}(p,q),i} := x_{-1}^{k(i,j)} x_{2p+1}^{l(i,j)} \left(\prod_{t=0}^{2p} x_t \right)^{2\delta-1} \qquad \text{for } 1 \le j \le q;$$

$$w^{\mathcal{Q}(p),i} := \left(\prod_{t=-1}^{2p-2} x_t \right)^{2p+1-2i} \qquad \text{for } i = 0 \text{ or } 2;$$

$$w_j^{\mathcal{Q}(p),1} := \left(\frac{1}{2}\right)^{\frac{1}{2}} (x_{2j-1}^2 + x_{2j}^2) \left(\prod_{t \ne 2j-1, 2j} x_t \right) \left(\prod_{t=-1}^{2p-2} x_t \right)^{2p-2} \qquad \text{for } 0 \le j \le p-1.$$

Then the bundles $\mathcal{W}_D = \sum_{i=0}^{w_D} \mathcal{W}_D^{m_D+i, m_D+w_D-i}$ *admit polarization preserving isomorphisms*

$$\varphi_D : \mathcal{W}_D \to \mathcal{V}_D \otimes \mathbb{Q}(m_D),$$
$$\Psi^{\mathcal{H}_D}(w_1, w_2) = \Psi_D \otimes \Psi^{\mathbb{Q}(m_D)}(\varphi_D(w_1), \varphi_D(w_2))$$

for sections w_1, w_2 *of* $\mathcal{W}_D$.

Proof. In the cases of $D = \mathcal{S}(p)$ or $\mathcal{B}(p,q)$ the claim is a straightforward consequence of $x_i^s \in \mathbf{J}_D(z)$ for $s \ge 4\delta - 1$, $i \in \{-1, 0, 2p+1\}$, $z \in D$, and the fact that the line bundles $\mathcal{R}_D^{(4\delta-2)(2p+3)}$ are associated with the sheaves of sections

$$\left(\prod_{j=-1}^{2p+1} x_j \right)^{4\delta-2} + \mathbf{J}_D^{(4\delta-2)(2p+3)}.$$

For $D = \mathcal{Q}(p)$, $r \in \{2j-1, 2j\}$ let us note that

$$\frac{\partial f_z}{\partial x_r} = 4p x_r^{4p-1} + z_j x_r^{2p-1} x_{4j-1-r}^{2p} \in \mathbf{J}_{\mathcal{Q}(p)}(z)$$

and the line bundle $\mathcal{R}_{\mathcal{Q}(p)}^{2p(4p-2)}$ is generated by

$$\sigma = \left(\prod_{t=-1}^{2p-2} x_t \right)^{4p-2} + \mathbf{J}_{\mathcal{Q}(p)}^{2p(4p-2)}.$$

Applying repeatedly the aforementioned relations of the Jacobian rings, one computes for $j \ne k$ that

$$w_j^{\mathcal{Q}(p),1} w_k^{\mathcal{Q}(p),1} = \frac{1}{2} \prod_{l=j,k} \left[(x_{2l-1}^2 + x_{2l}^2) (x_{2l-1} x_{2l})^{4p-3} \right] \left(\prod_{t, [\frac{t+1}{2}] \ne j,k} x_t \right)^{4p-2}$$

$$= \frac{1}{2} \prod_{l=j,k} \left[\frac{-z_l}{4p} \left(x_{2l-1}^{2p-1} x_{2l}^{6p-3} + x_{2l-1}^{6p-3} x_{2l}^{2p-1} \right) \right] \left(\prod_{t,[\frac{t+1}{2}] \neq j,k} x_t \right)^{4p-2},$$

$$= \frac{z_j^2 z_k^2}{512p^4} \prod_{l=j,k} \left(x_{2l-1}^{4p-1} x_{2l}^{4p-3} + x_{2l-1}^{4p-3} x_{2l}^{4p-1} \right) \left(\prod_{t,[\frac{t+1}{2}] \neq j,k} x_t \right)^{4p-2},$$

i.e.,

$$\prod_{l=j,k} w_l^{\mathcal{Q}(p),1} = \frac{z_j^2 z_k^2}{256p^4} \left(\prod_{l=j,k} w_l^{\mathcal{Q}(p),1} \right).$$

However, $2^t\overline{z}z < 1 + |^t zz|^2 < 1 + 1$ for $z = {}^t(z_0, z_1, \dots, z_{p-1}) \in \mathcal{Q}(p)$ reveals that $|z_j|^2 \leq \sum_{i=0}^{p-1} |z_i|^2 < 1$, whereas $\left| \frac{z_j^2 z_k^2}{256p^4} \right| < 1$. In the torsion free Jacobian rings $\mathcal{R}_{\mathcal{Q}(p)}(z)$ that siffices for the vanishing of $\prod_{l=j,k} w_l^{\mathcal{Q}(p),1}$, $j \neq k$. Similarly, the expression

$$\left(w_j^{\mathcal{Q}(p),1} \right)^2 = \frac{1}{2} \left(x_{2j-1}^2 + x_{2j}^2 \right)^2 \left(x_{2j-1} x_{2j} \right)^{4p-4} \left(\prod_{t \neq 2j-1,2j} x_t \right)^{4p-2}$$

$$= \frac{-z_j}{8p} \left(x_{2j-1}^{2p} x_{2j}^{6p-4} + x_{2j-1}^{6p-4} x_{2j}^{2p} \right) \left(\prod_{t \neq 2j-1,2j} x_t \right)^{4p-2} + \left(\prod_{t=-1}^{2p-2} x_t \right)^{4p-2}$$

$$= \frac{z_j^2}{32p^2} \left(x_{2j-1}^{4p} x_{2j}^{4p-4} + x_{2j-1}^{4p-4} x_{2j}^{4p} \right) \left(\prod_{t \neq 2j-1,2j} x_t \right)^{4p-2} + \sigma$$

$$= \frac{z_j^2}{16p^2} \left[\left(w_j^{\mathcal{Q}(p),1} \right)^2 - \sigma \right] + \sigma$$

with $\left| \frac{z_j^2}{16p^2} \right| < 1$ forces $\left(w_j^{\mathcal{Q}(p),1} \right)^2 = \sigma$. □

That completes the proof of Theorem 4.

3. CONSEQUENCES

3.1. ABELIAN-MOTIVIC VARIATIONS

Corollary 9. *The following three tensor categories are equivalent:*
(I) *the category $\mathcal{A}$ of the abelian-motivic variations of Hodge structure,*

(II) *the category $\mathcal{AH}$ of the abelian-motivic hypersurface variations of Hodge structure, and*

(III) *the category $\mathcal{JPC}$ of the variations of Hodge structure, expressed by Jacobians of plane curves.*

Proof. The inclusions $\mathcal{JPC} \subseteq \mathcal{AH} \subseteq \mathcal{A}$ are obvious. As far as $\mathcal{A}$ is generated by the tautological variations of Hodge structure $\mathcal{V}_{\mathcal{S}(p)}$ over Siegel upper half spaces $\mathcal{S}(p)$, Theorem 4 implies that $\mathcal{A} \subseteq \mathcal{JPC}$. □

3.2. MAXIMAL DIMENSIONAL VARIATIONS

For Hodge structure H of weight $w = 2k+1 > 1$ let

$$\mu_1^{odd} := \sum_{i \geq 0} h^{k-2i} h^{k-1-2i}$$

and

$$\mu_2^{odd} := \frac{1}{2} h^k (h^k + 1) + \sum_{i \geq 0} h^{k-1-2i} h^{k-2-2i}.$$

In the case of $w = 2k$ let

$$\mu_1^{even} := \sum_{i \geq 0} h^{k-1-2i} h^{k-2-2i},$$

$$\mu_2^{even} := h^k + (h^{k-1} - 1) h^{k-2} + \sum_{i \geq 0} h^{k-3-2i} h^{k-4-2i}$$

for $w \geq 4$,

$$\mu_3^{even} := \overline{\mu}_3 + \sum_{i \geq 0} h^{k-2-2i} h^{k-3-2i}$$

with $\overline{\mu_3} := h^k$ for $h^{k-1} = 1$, $\overline{\mu_3} := \frac{1}{2} h^k h^{k-1}$ for an even h^k and $h^{k-1} > 1$, $\overline{\mu_3} := \frac{1}{2}(h^k - 1) h^{k-1} + 1$ for an odd h^k and $h^{k-1} > 1$. According to [9] or [2], the maximum dimension μ of a variation of Hodge structure is $\mu^{odd} = \max(\mu_1^{odd}, \mu_2^{odd})$ if $w = 2k+1$, or $\mu^{even} = \max(\mu_1^{even}, \mu_2^{even}, \mu_3^{even})$ if $w = 2k$.

The summands $h^j h^{j-1}$, $j < \left[\frac{w}{2}\right]$, of μ, including $(h^{k-1} - 1) h^{k-2}$ from μ_2^{even} and $\left(\frac{1}{2} h^k\right) h^{k-1}$ from $\overline{\mu_3}$ with an even h^k, $h^{k-1} > 1$, are realized by appropriate shifts of the tautological variations of J-Hodge structure over the generalized balls $\mathcal{B}(h^j, h^{j-1})$, respectively, $\mathcal{B}(h^{k-1} - 1, h^{k-2})$, $\mathcal{B}(\frac{1}{2} h^k, h^{k-1})$. The tautological variation of Hodge structure $\mathcal{V}_{\mathcal{S}(h^k)}$ provides a variation of dimension $\frac{1}{2} h^k (h^k + 1)$ in the case of μ_2^{odd}. The tautological variation of Hodge structure $\mathcal{V}_{\mathcal{Q}(h^k)}$ is an example of dimension h^k for μ_2^{even} or $\overline{\mu_3}$ with $h^{k-1} = 1$. The non-symmetric domain

$$\Omega(h^{k-1}, \frac{1}{2}(h^k - 1)) \subset \mathcal{B}(1,1) \times \mathcal{B}(h^{k-1} - 1, \frac{1}{2}(h^k - 1)) \times \mathcal{B}(1, \frac{1}{2}(h^k - 1)),$$

cut by the inequality ${}^t\overline{Y}Y < (1 - |t|^2)^2(I_q - {}^t\overline{X}X)$ for $t \in \mathcal{B}(1,1)$, $X \in \mathcal{B}(h^{k-1} - 1, \frac{1}{2}(h^k - 1))$, $Y \in \mathcal{B}(1, \frac{1}{2}(h^k - 1))$, is an instance of a variation of Hodge structure of dimension $\frac{1}{2}(h^k - 1)h^{k-1} + 1$ in the case of $\overline{\mu_3}$ with an odd h^k, $h^{k-1} > 1$.

If all Hodge numbers of H are greater than 1, $h^{[\frac{w-1}{2}]} > 2$ and $h^{[\frac{w}{2}]} \geq 4$, then the results of [9] and [7] imply that all the maximal dimensional simply connected variations of Hodge structure are isomorphic to products of the aforementioned bounded domains. The lack of quasiprojective discrete quotients of $\Omega(h^{k-1}, \frac{1}{2}(h^k - 1))$ (cf. [7]) reveals that the maximal dimensional variations, covered by $\Omega(h^{k-1}, \frac{1}{2}(h^k - 1)) \times \prod_{i \geq 0} \mathcal{B}(h^{k-2-2i}, h^{k-3-2i})$, do not arise from geometry. All the other maximal dimensional variations are direct sums of tautological ones, so that Theorem 4 implies

Corollary 10. *The geometrically arising maximal dimensional variations of Hodge structure with sufficiently large Hodge numbers are expressed by Jacobians of plane curves.*

Let us observe that our main result provides a "new" symplectic representation of $SO(2,p)$. Indeed, the inclusion $\mathcal{V}_{\mathcal{Q}(p)} \subset C(\mathcal{V}_{\mathcal{Q}(n)}, \Psi_{\mathcal{Q}(p)})$ from Theorem 2 (ii) induces a symplectic representation $SO(2,p) \hookrightarrow Sp(2^{p+1}, \mathbb{R})$. Since a plane curve Y of degree $4p$ has genus $\frac{1}{2}(4p-1)(4p-2)$, Theorem 4 interprets as a realization of $SO(2,p)$ in a product of Mumford–Tate groups $Sp((4p-1)(2p-1), \mathbb{R})$ of Hodge structures of such Y.

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