Weak form of Holzapfel's Conjecture

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Abstract

Let $\mathbb{B} \subset \mathbb{C}^2$ be the unit ball and Γ be a lattice of $SU(2,1)$. Bearing in mind that all compact Riemann surfaces are discrete quotients of the unit disc $\Delta \subset \mathbb{C}$, Holzapfel conjectures that the discrete ball quotients $\mathbb{B}/\Gamma$ and their compactifications are widely spread among the smooth projective surfaces. There are known ball quotients $\mathbb{B}/\Gamma$ of general type, as well as rational, abelian, K3 and elliptic ones. The present note constructs three non-compact ball quotients, which are birational, respectively, to a hyper-elliptic, Enriques or a ruled surface with an elliptic base. As a result, we establish that the ball quotient surfaces have representatives in any of the eight Enriques classification classes of smooth projective surfaces.

1 Introduction

In his monograph [4] Rolf-Peter Holzapfel states as a working hypothesis or a philosophy that : " ... up to birational equivalence and compactifications, all complex algebraic surfaces are ball quotients." By a complex algebraic surface is meant a smooth projective surface over $\mathbb{C}$. These have smooth minimal models, which are classified by Enriques in eight types - rational, ruled of genus ≥ 1, abelian, hyperelliptic, K3, Enriques, elliptic and of general type. The compact torsion free ball quotients $\mathbb{B}/\Gamma$ are smooth minimal surfaces of general type. Ishida (cf.[10]), Keum (cf.[11], [12]) and Dzambic (cf.[1]) obtain elliptic surfaces, which are minimal resolutions of the isolated cyclic quotient singularities of compact ball quotients. Hirzebruch (cf.[2]) and then Holzapfel (cf.[3], [9], [7]) construct torsion free ball quotient compactifications with abelian minimal models. In [9] Holzapfel provides a ball quotient compactification, which is birational to the Kummer surface of an abelian surface, i.e., to a smooth minimal K3 surface. Rational ball quotient surfaces are explicitly recognized and studied in [6], [8]. The present work constructs smooth ball quotients with a hyperelliptic or, respectively, a ruled model with an elliptic base. It provides also a ball quotient with one double point, which is birational to an Enriques surface. All of them are finite Galois quotients of a non-compact torsion free $\mathbb{B}/\Gamma_{-1}^{(6,8)}$, constructed by Holzapfel in [9] and having abelian minimal model of the toroidal compactification. As a result, we establish the following

Theorem 1. (Weak Form of Holzapfel's Conjecture) *Any of the eight Enriques classification classes of complex projective surfaces contains a ball quotient surface.*

2 Ball Quotient Compactifications with Abelian Minimal Models

Let us recall that the complex 2-ball

$$\mathbb{B} = \{(z_1, z_2) \in \mathbb{C}^2 \, ; \, |z_1|^2 + |z_2|^2 < 1\} = \mathrm{SU}(2,1)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$$

is an irreducible non-compact Hermitian symmetric space. The discrete biholomorphism groups $\Gamma \subset \mathrm{SU}(2,1)$ of $\mathbb{B}$, whose quotients $\mathbb{B}/\Gamma$ have finite $\mathrm{SU}(2,1)$-invariant measure are called ball lattices. The present section studies the image T of the toroidal compactifying divisor $T' = (\mathbb{B}/\Gamma)' \setminus (\mathbb{B}/\Gamma)$ on the minimal model A of $(\mathbb{B}/\Gamma)'$, whenever A is an abelian surface. It establishes that for any subgroup $H \subseteq Aut(A,T)$ there is a ball quotient $\mathbb{B}/\Gamma_H$, birational to A/H.

Lemma 2. *If a ball quotient $\mathbb{B}/\Gamma$ is birational to an abelian surface A then $\mathbb{B}/\Gamma$ is smooth and non-compact.*

Proof. Assume that $\mathbb{B}/\Gamma$ is singular. For a compact $\mathbb{B}/\Gamma$ set $U = \mathbb{B}/\Gamma$. If $\mathbb{B}/\Gamma$ is non-compact, let $U = (\mathbb{B}/\Gamma)'$ be the toroidal compactification of $\mathbb{B}/\Gamma$. In either case U is a compact surface with isolated cyclic quotient singularities. Consider the minimal resolution $\varphi : Y \to U$ of $p_i \in U^{\mathrm{sing}}$ by Hirzebruch-Jung strings $E_i = \sum_{t=1}^{\nu_i} E_i^t$. The irreducible components E_i^t of E_i are smooth rational curves of self-intersection $(E_i^t)^2 \leq -2$. The birational morphism $Y \dashrightarrow A$ transforms E_i^t onto rational curves on A. It suffices to observe that an abelian surface A does not support rational curves C, in order to conclude that $\mathbb{B}/\Gamma$ is smooth. The compact smooth ball quotients are known to be of general type, so that $\mathbb{B}/\Gamma$ is to be non-compact.

Assume that there is a rational curve $C \subset A$. Its desingularization $f : \widetilde{C} \to C$ can be viewed as a holomorphic map $F : \widetilde{C} \to A$. Homotopy lifting property applies to F and provides a holomorphic immersion $\widetilde{F} : \widetilde{C} \to \widetilde{A} = \mathbb{C}^2$ in the universal cover $\widetilde{A}$ of A, due to simply connectedness of the smooth rational curve $\widetilde{C}$. Its image $\widetilde{F}(\widetilde{C})$ is a compact complex-analytic subvariety of $\mathbb{C}^2$, which maps to compact complex-analytic subvarieties $\mathrm{pr}_i(\widetilde{F}(\widetilde{C})) \subset \mathbb{C}$ by the canonical projections $\mathrm{pr}_i : \mathbb{C}^2 \to \mathbb{C}$, $1 \leq i \leq 2$. Thus, $\mathrm{pr}_i(\widetilde{F}(\widetilde{C}))$ and, therefore, $\widetilde{F}(\widetilde{C})$ are finite. The contradiction justifies the non-existence of rational curves on A.

□

The next lemma lists some immediate properties of the image T of the toroidal compactifying divisor T' of $A' = (\mathbb{B}/\Gamma)'$ on its abelian minimal model A.

Lemma 3. *Let $A' = (\mathbb{B}/\Gamma)'$ be a smooth toroidal ball quotient compactification, $\xi : A' \to A$ be the blow-down of the (-1)-curves $L = \sum_{j=1}^{s} L_j$ on A' to an abelian surface A*

and T'_i, $1 \leq i \leq h$ *be the disjoint smooth elliptic irreducible components of the toroidal compactifying divisor* $T' = (\mathbb{B}/\Gamma)' \setminus (\mathbb{B}/\Gamma)$. *Then:*

(i) $T_i = \xi(T'_i)$ *are smooth irreducible elliptic curves on* A;

(ii) $T^{\rm sing} = \sum\limits_{1 \leq i < j \leq h} T_i \cap T_j = \xi(L)$;

(iii) $T_i \cap T^{\rm sing} \neq \emptyset$ *and the restrictions* $\xi : T'_i \to T_i$ *are bijective for all* $1 \leq i \leq h$.

Proof. (i) According to the birational invariance of the genus, the curves $T_i = \xi(T'_i)$ have smooth elliptic desingularizations. It suffices to show that any curve $C \subset A$ of genus 1 is smooth. If C is singular then its desingularization $\widetilde{C}$ is a smooth elliptic curve. Therefore, the composition $\widetilde{C} \to C \hookrightarrow A$ of the desingularization map with the identical inclusion of C is a morphism of abelian varieties. In particular, it is unramified, which is not the case for $\widetilde{C} \to C$. Therefore any curve $C \subset A$ of genus 1 is smooth.

(ii) The inclusion $T^{\rm sing} \subseteq \sum\limits_{1 \leq i < j \leq h} T_i \cap T_j$ follows from (i). For the opposite inclusion, note that $\xi|_{A' \setminus L} = {\rm Id}_{(A' \setminus L)} : A' \setminus L \to A \setminus \xi(L)$ guarantees $T_i = \xi(T'_i) \neq \xi(T'_j) = T_j$ and different elliptic curves on an abelian surface intersect transversally at any of their intersection points. Thus, $T^{\rm sing} = \sum\limits_{1 \leq i < j \leq h} T_i \cap T_j$. The disjointness of T'_i yields $\sum\limits_{1 \leq i < j \leq h} T_i \cap T_j \subseteq \xi(L)$. Conversely, the Kobayashi hyperbolicity of $\mathbb{B}/\Gamma$ requires ${\rm card}(L_j \cap T') \geq 2$ for all $1 \leq j \leq s$. However, ${\rm card}(L_j \cap T'_i) \leq 1$ by the smoothness of $T_i = \xi(T'_i)$, so that there exist at least two $T'_i \neq T'_k$ with ${\rm card}(L_j \cap T'_i) = {\rm card}(L_j \cap T'_k) = 1$. In other words, the point $\xi(L_j) \in T_i \cap T_k$. That verifies the inclusion $\xi(L) \subseteq \sum\limits_{1 \leq i < j \leq h} T_i \cap T_j$, whereas the coincidence $\xi(L) = \sum\limits_{1 \leq i < j \leq h} T_i \cap T_j$.

(iii) If $T_i \cap \xi(L) = \emptyset$ then the intersection numbers $(T'_i)^2 = T_i^2$ coincide. By the Adjunction Formula,

$$0 = -e(T_i) = T_i^2 + K_A.T_i = T_i^2 + \mathcal{O}_A.T_i = T_i^2,$$

so that $(T'_i)^2 = 0$. That contradicts the contractibility of T'_i to the corresponding cusp of $\mathbb{B}/\Gamma$ and justifies $T_i \cap T^{\rm sing} \neq \emptyset$ for $\forall 1 \leq i \leq h$.

Note that $\xi|_{T'_i \setminus L} = {\rm Id}|_{T'_i \setminus L} : T'_i \setminus L \to T_i \setminus \xi(L)$ is bijective. In order to define $\xi^{-1} : T_i \cap \xi(L) \to T'_i \cap L$, let us recall that for any $p \in \xi(L)$ the smooth rational curve $\xi^{-1}(p)$ has ${\rm card}(\xi^{-1}(p) \cap T'_i) \leq 1$. More precisely, ${\rm card}(\xi^{-1}(p) \cap T'_i) = 1$ if and only if $p \in T_i$, so that for any $p \in T_i \cap \xi(L)$ there is a unique point $\{q(p)\} = T'_i \cap \xi^{-1}(p)$. That provides a regular morphism $\xi^{-1}(p) = q(p)$ for all $p \in T_i \cap \xi(L)$.

□

According to Lemma 3, the image $T = \xi(T')$ of the toroidal compactifying divisor $T' = (\mathbb{B}/\Gamma)' \setminus (\mathbb{B}/\Gamma)$ under the blow-down $\xi : (\mathbb{B}/\Gamma)' \to A$ of the (-1)-curves is a multi-elliptic divisor, i.e., $T = \sum\limits_{i=1}^{h} T_i$ has smooth elliptic irreducible components T_i, which intersect transversally. Note also that (A, T) determines uniquely $(\mathbb{B}/\Gamma)'$ as the blow-up of A at $T^{\rm sing}$.

Definition 4. *A pair* (A,T) *of an abelian surface* A *and a divisor* $T \subset A$ *is an abelian ball quotient model if there exists a torsion free toroidal ball quotient compactification* $(\mathbb{B}/\Gamma)'$, *such that the blow-down* $\xi : (\mathbb{B}/\Gamma)' \to A$ *of the* (-1)*-curves on* $(\mathbb{B}/\Gamma)'$ *maps the pair* $\left((\mathbb{B}/\Gamma)', T' = (\mathbb{B}/\Gamma)' \setminus (\mathbb{B}/\Gamma)\right)$ *onto* (A,T).

The next lemma explains the construction of non-compact ball quotients, which are finite Galois quotients of torsion free non-compact $\mathbb{B}/\Gamma$, birational to abelian surfaces.

Lemma 5. *Let* $A' = (\mathbb{B}/\Gamma)' = (\mathbb{B}/\Gamma) \cup T'$ *be a torsion free ball quotient compactification by a toroidal divisor* T', $\xi : A' \to A$ *be the blow-down of the* (-1)*-curves on* A' *to the abelian minimal model* A *and* $T = \xi(T')$. *Then*

(i) $Aut(A,T) = Aut(A',T')$ *is a finite group;*

(ii) any subgroup $H \subseteq Aut(A,T)$ *lifts to a ball lattice* Γ_H, *such that* Γ *is a normal subgroup of* Γ_H *with quotient group* $\Gamma_H/\Gamma = H$ *and* $\mathbb{B}/\Gamma_H$ *is a non-compact ball quotient, birational to* $X = A/H$.

Moreover, if $X = A/H$ *is a smooth surface then* $\mathbb{B}/\Gamma_H$ *is a smooth ball quotient.*

Proof. (i) If $G = Aut(A,T)$, then Lemma 3(ii) implies the G-invariance of $\xi(L)$. By the means of an arbitrary automorphism of the smooth projective line $\mathbb{P}^1$, one extends the G-action to L and, therefore, to

$$A' = (A' \setminus L) \cup L = (A \setminus \xi(L)) \cup L.$$

The G-invariance of $T' = \sum_{i=1}^{h} T'_i$ follows from Lemma 3(iii). That justifies the inclusion $G \subseteq Aut(A',T')$. For the opposite inclusion, note that the union L of the (-1)-curves is invariant under an arbitrary automorphism of A'. As a result, there arises a G-action on $\xi(L)$ and $A = (A \setminus \xi(L)) \cup \xi(L) = (A' \setminus L) \cup \xi(L)$. The multi-elliptic divisor $T = \sum_{i=1}^{h} T_i$ is G-invariant according to Lemma 3(iii). Consequently, $Aut(A',T') \subseteq G$, whereas $G = Aut(A',T')$.

In order to show that G is finite, let us consider the natural representation

$$\varphi : G \longrightarrow \mathrm{Sym}(T_1, \dots, T_h) \simeq Sym_h$$

in the permutation group of the irreducible components T_i of T. It suffices to prove that the kernel kerφ is finite, in order to assert that G is finite. For any $g = \tau_p g_o \in \ker \varphi \subset Aut(A)$ with linear part $g_o \in Gl_2(\mathbb{C})$ and translation part τ_p, $p \in A$, we show that g_o and τ_p take finitely many values. Note that the identical inclusions $T_i \subset A$ are morphisms of abelian varieties. Thus, for any choice of an origin $\check{o}_A \in T_i$ there is a $\mathbb{C}$-linear embedding $\mathcal{E}_i : \widetilde{T_i} = \mathbb{C} \hookrightarrow \mathbb{C}^2 = \widetilde{A}$ of the corresponding universal covers. If $\mathcal{E}_i(1) = (a_i, b_i)$ then

$$T_i = E_{a_i,b_i} = \{(a_i t, b_i t)(\mathrm{mod}\, \pi_1(A))\ ;\ t \in \mathbb{C}\} \subset A.$$

If the origin $\check{o}_A \not\in T_i$, then for any point $(P_i, Q_i) \in T_i$ the elliptic curve $T_i = E_{a_i,b_i} + (P_i, Q_i)$. In either case, all $v_i = (a_i, b_i)$ are eigenvectors of the linear part g_o of $g = \tau_p g_o \in$

$\ker\varphi$. We claim that there are at least three pairwise non-proportional v_i. Indeed, if all v_i were parallel, then $T^{\rm sing} = \emptyset$, which contradicts $T_i \cap T^{\rm sing} \neq \emptyset$ for $1 \leq i \leq h$ by Lemma 3 (iii). Suppose that among $v_1, \ldots, v_h$ there are two non-parallel and all other v_i are proportional to one of them. Then after an eventual permutation there is $1 \leq k \leq h-1$, such that v_1, v_k are linearly independent, $v_i = \mu_i v_1$ for $\mu_i \in \mathbb{C}$, $2 \leq i \leq k$ and $v_i = \mu_i v_{k+1}$ for $\mu_i \in \mathbb{C}$, $k+2 \leq i \leq h$. Holzapfel has proved in [9] that any abelian ball quotient model (A, T) is subject to $\sum_{i=1}^{h} {\rm card}(T_i \cap T^{\rm sing}) = 4{\rm card}(T^{\rm sing})$. In the case under consideration

$$ {\rm card}(T^{\rm sing}) = \sum_{i=1}^{k} \sum_{j=k+1}^{h} {\rm card}(T_i \cap T_j), $$

$$ {\rm card}(T_i \cap T^{\rm sing}) = \sum_{j=k+1}^{h} {\rm card}(T_i \cap T_j) \quad \text{for} \quad 1 \leq i \leq k \quad \text{and} $$

$$ {\rm card}(T_j \cap T^{\rm sing}) = \sum_{i=1}^{k} {\rm card}(T_i \cap T_j) \quad \text{for} \quad k+1 \leq j \leq h. $$

Therefore $\sum_{i=1}^{h} {\rm card}(T_i \cap T^{\rm sing}) = 2{\rm card}(T^{\rm sing}) \neq 4{\rm card}(T^{\rm sing})$ and there are at least three pairwise non-proportional eigenvectors v_1, v_2, v_3 of g_o. Let λ_i be the corresponding eigenvalues of v_i and $v_3 = \rho_1 v_1 + \rho_2 v_2$ for some $\rho_1, \rho_2 \in \mathbb{C}^*$. Then $\lambda_3 v_3 = g_o(v_3) = \rho_1 \lambda_1 v_1 + \rho_2 \lambda_2 v_2$ implies that $\lambda_1 = \lambda_3 = \lambda_2$ and $g_o = \lambda_o I_2$ is a scalar matrix. On the other hand, $g(T_i) = g_o(T_i) + p = T_i$ for $\forall 1 \leq i \leq h$, so that g_o permutes among themselves the parallel elliptic curves among $T_1, \ldots, T_h$. Since T_i are finitely many, there is a natural number m, such that $g_o^m \in \ker\varphi$. Therefore, $\lambda_o^m \in {\rm End}(T_i)$ and $\lambda_o^{-m} \in {\rm End}(T_i)$ for all $1 \leq i \leq h$, due to $(g_o^m)^{-1} = g_o^{-m} \in \ker\varphi$. Recall that the units group ${\rm End}^*(T_i) = \mathbb{Z}^* = \{\pm 1\}$ for T_i without a complex multiplication. If the elliptic curve T_i has complex multiplication by an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$, $d \in \mathbb{N}$, then ${\rm End}(T_i)$ is a subring of the integers ring $\mathcal{O}_{-d}$ of $\mathbb{Q}(\sqrt{-d})$. The units groups $\mathcal{O}_{-1}^* = \langle i \rangle$, $\mathcal{O}_{-3}^* = \langle e^{\frac{2\pi i}{6}} \rangle$, and $\mathcal{O}_{-d}^* = \langle -1 \rangle$ for $\forall d \neq 1, 3$ are finite cyclic groups. As a subgroup of $\mathcal{O}_{-d}^*$, the units group ${\rm End}^*(T_i)$ is a finite cyclic group. Therefore $\lambda_o^m \in {\rm End}^*(T_i)$ and $g_o = \lambda_o I_2$ take finitely many values.

Concerning the translation part τ_p of $g \in \ker\varphi$, one can always move the origin $\check{o}_A$ of A at one of the singular points of T. Due to the G-invariance of $T^{\rm sing}$, there follows $g(\check{o}_A) = \tau_p g_o(\check{o}_a) = \tau_p(\check{o}_A) = p \in T^{\rm sing}$. Therefore p takes finitely many values and $\ker\varphi$ is finite.

(ii) Since $\Gamma \subset {\rm SU}(2,1)$ is a torsion free lattice, any subgroup H of

$$ G = Aut(A', T') \subseteq Aut(A' \setminus T') = Aut\,(\mathbb{B}/\Gamma) $$

lifts to a subgroup $\Gamma_H \subset Aut(\mathbb{B}) = {\rm SU}(2,1)$, which normalizes Γ and has quotient $\Gamma_H/\Gamma = H$. We claim that Γ_H is discrete. Indeed, $\Gamma_H = \cup_{i=1}^{k} \gamma_i \Gamma$ is a finite disjoint union of cosets, relative to Γ. Suppose that Γ_H is not discrete and there is a sequence $\{\nu_n\}_{n=1}^{\infty} \subset \Gamma_H$ with a limit point $\nu_o \in \gamma_{i_o}\Gamma$. Then pass to a subsequence $\{\nu_{m_n}\}_{n=1}^{\infty} \subset \gamma_{i_o}\Gamma$, converging to ν_o. As a result $\{\gamma_{i_o}^{-1}\nu_{m_n}\}_{n=1}^{\infty} \subset \Gamma$ converges to $\gamma_{i_o}^{-1}\nu_o \in \Gamma$ and contradicts the discreteness of Γ. Thus, $\Gamma_H \supseteq \Gamma$ is discrete and, therefore, a ball lattice. Straightforwardly,

$$A'/H = [(\mathbb{B}/\Gamma) \,/\, (\Gamma_H/\Gamma)] \cup (T'/H) = (\mathbb{B}/\Gamma_H) \cup (T'/H) = \overline{(\mathbb{B}/\Gamma_H)}$$

is the compactification of the ball quotient $\mathbb{B}/\Gamma_H$ by the divisor T'/H. The H-Galois covers $\zeta_H : A \to A/H$ and $\zeta'_H : A' \to \overline{(\mathbb{B}/\Gamma_H)}$ fit in a commutative diagram

$$\begin{array}{ccc} A & \xleftarrow{\ \xi\ } & A' \\ \downarrow{\scriptstyle \zeta_H} & & \downarrow{\scriptstyle \zeta'_H} \\ A/H & \xleftarrow{\ \xi_H\ } & \overline{(\mathbb{B}/\Gamma_H)} \end{array}$$

with the contraction ξ_H of L/H to $\xi(L)/H$.

Note that $X = A/H$ is smooth exactly when H has no isolated fixed points on A. The blow-up $\xi : A' \to A$ replaces an arbitrary $p_j = \xi(L_j)$ with stabilizer $Stab_H(p_j)$ by a smooth rational curve L_j with $Stab_H(q) = Stab_H(p_j)$ for all $q \in L_j$. Therefore the blow-up ξ does not create isolated H-fixed points on A' and $A'/H = \overline{(\mathbb{B}/\Gamma_H)}$ is a smooth compactification. Its open subset $\mathbb{B}/\Gamma_H$ is smooth.

□

3 Explicit Constructions

The present section applies Lemma 5 to a specific abelian ball quotient model over the Gauss numbers $\mathbb{Q}(ß)$, in order to provide ball quotient compactifications, which are birational to a hyperelliptic, Enriques or a ruled surface with an elliptic base.

Theorem 6. *(*Holzapfel [9]) Let us consider the elliptic curve $E_{-1} = \mathbb{C}/(\mathbb{Z} + ß\mathbb{Z})$ with complex multiplication by the Gauss numbers $\mathbb{Q}(ß)$, its 2-torsion points

$$Q_0 = 0(\mathrm{mod}\,\mathbb{Z} + i\mathbb{Z}), \quad Q_1 = \frac{1}{2}(\mathrm{mod}\,\mathbb{Z} + i\mathbb{Z}), \quad Q_2 = ßQ_1, \quad Q_3 = Q_1 + Q_2,$$

the abelian surface $A_{-1} = E_{-1} \times E_{-1}$, the points

$$Q_{ij} = (Q_i, Q_j) \in A_{2-tor} \subset A_{-1}$$

and the divisor $T_{-1}^{(6,8)} = \sum_{i=1}^{8} T_i$ with smooth elliptic irreducible components

$$T_k = E_{ß^k,1} \quad \text{for} \quad 1 \leq k \leq 4,$$

$$T_{m+4} = Q_m \times E_{-1}, \quad T_{m+6} = E_{-1} \times Q_m \quad \text{for} \quad 1 \leq m \leq 2.$$

Then $\left(A_{-1}, T_{-1}^{(6,8)}\right)$ is an abelian model of an arithmetic ball quotient $\mathbb{B}/\Gamma_{-1}^{(6,8)}$, defined over $\mathbb{Q}(ß)$.

Corollary 7. *(*Holzapfel [9]) (i) In the notations from Theorem 6, the multiplications $I = \begin{pmatrix} ß & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 1 & 0 \\ 0 & ß \end{pmatrix}$ by $ß \in \mathbb{Z}[ß] = \mathrm{End}(E_{-1})$ on the first, respectively, the second elliptic factor E_{-1} of A_{-1} are automorphisms of $\left(A_{-1}, T_{-1}^{(6,8)}\right)$.

(ii) If $\Gamma_{K3,-1}^{(6,8)}$ is the ball lattice, containing $\Gamma_{-1}^{(6,8)}$ as a normal subgroup with quotient $\Gamma_{K3,-1}^{(6,8)}/\Gamma_{-1}^{(6,8)} = \langle -I_2 = I^2 J^2 \rangle \subset Aut\left(A_{-1}, T_{-1}^{(6,8)}\right)$, then the ball quotient $\mathbb{B}/\Gamma_{K3,-1}^{(6,8)}$ is birational to the Kummer surface X_{K3} of A_{-1}.

(iii) If $\Gamma_{Rat,-1}^{(6,8)}$ is the ball lattice, containing $\Gamma_{-1}^{(6,8)}$ as a normal subgroup with quotient $\Gamma_{Rat,-1}^{(6,8)}/\Gamma_{-1}^{(6,8)} = \langle I, J \rangle \subseteq Aut\left(A_{-1}, T_{-1}^{(6,8)}\right)$, then the ball quotient $\mathbb{B}/\Gamma_{Rat,-1}^{(6,8)}$ is a rational surface.

The next lemma obtains the entire automorphism group $G_{-1}^{(6,8)} = Aut\left(A_{-1}, T_{-1}^{(6,8)}\right)$.

Lemma 8. *In the notations from Theorem 6, the group* $G_{-1}^{(6,8)} = Aut\left(A_{-1}, T_{-1}^{(6,8)}\right)$ *is generated by* $I = \begin{pmatrix} ß & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 1 & 0 \\ 0 & ß \end{pmatrix}$, *the transposition* $\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *of the elliptic factors* E_{-1} *of* A_{-1} *and the translation* τ_{33} *by* Q_{33}*. The aforementioned generators are subject to the relations*

$$I^4 = \mathrm{Id}, \quad J^4 = \mathrm{Id}, \quad \theta^2 = \mathrm{Id}, \quad \tau_{33}^2 = \mathrm{Id},$$

$$IJ = JI, \quad \theta I = J\theta, \quad \theta J = I\theta,$$

$$I\tau_{33} = \tau_{33} I, \quad J\tau_{33} = \tau_{33} J, \quad \theta\tau_{33} = \tau_{33}\theta.$$

and $G_{-1}^{(6,8)}$ *is of order* 64.

Proof. Any $g \in G_{-1}^{(6,8)}$ leaves invariant

$$\left(T_{-1}^{(6,8)}\right)^{\text{sing}} = \sum_{1 \leq i < j \leq 8} T_i \cap T_j = \sum_{m=1}^{2} \sum_{n=1}^{2} Q_{mn} + Q_{00} + Q_{33}.$$

Thus, $g(T_i) = T_j$ implies $s_i = \mathrm{card}(T_i \cap T^{\text{sing}}) = \mathrm{card}(T_j \cap T^{\text{sing}}) = s_j$, according to the bijectiveness of g. In the case under consideration, $s_1 = s_2 = s_3 = s_4 = 4$ and $s_5 = s_6 = s_7 = s_8 = 2$, so that $G_{-1}^{(6,8)}$ permutes separately $T_1, \ldots, T_4$ and $T_5, \ldots, T_8$. In particular, the intersection $\cap_{i=1}^{4} T_i = \{Q_{00}, Q_{33}\}$ is $G_{-1}^{(6,8)}$-invariant and any $g = \tau_{(U,V)} g_o \in G_{-1}^{(6,8)}$ transforms the origin $\check{o}_{A_{-1}} = Q_{00}$ into $g(\check{o}_{A_{-1}}) = (U_1, U_2) \in$

$\{Q_{00}, Q_{33}\}$. Straightforwardly, $\tau_{33}(T_i) = T_i$ for $1 \leq i \leq 4$ and $\tau_{33}(T_{m+2n}) = T_{3-m+2n}$ for $1 \leq m \leq 2$, $2 \leq n \leq 3$ imply that $\tau_{33} \in G^{(6,8)}_{-1}$. Therefore $G^{(6,8)}_{-1}$ is generated by $G^{(6,8)}_{-1} \cap Gl_2(\mathrm{End}(E_{-1})) = G^{(6,8)}_{-1} \cap Gl_2(\mathbb{Z}[i])$ and τ_{33}. Note that $\theta \in Aut(A_{-1})$ acts on $T^{(6,8)}_{-1}$ and induces the permutation $(T_1, T_3)(T_5, T_7)(T_6, T_8)$ of its irreducible components. Therefore $\theta \in G^{(6,8)}_{-1}$ and $\langle I, J, \theta\rangle$ is a subgroup of $G^{(6,8)}_{-1} \cap Gl_2(\mathbb{Z}[i])$. On the other hand, any $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G^{(6,8)}_{-1} \cap \mathrm{GL}_2(\mathbb{Z}[ß])$ acts on $T_5, \dots, T_8$ and, therefore, on the set $\{\widetilde{T_5} = \widetilde{T_6} = 0 \times \mathbb{C}, \widetilde{T_7} = \widetilde{T_8} = \mathbb{C} \times 0\}$ of the corresponding universal covers. If $g(0 \times \mathbb{C}) = 0 \times \mathbb{C}$, $g(\mathbb{C} \times 0) = \mathbb{C} \times 0$ then $\beta = \gamma = 0$, so that $\alpha, \delta \in \mathrm{End}(E_{-1}) = \mathbb{Z}[ß]$ and $\det(g) = \alpha\delta \in \mathrm{End}^*(E_{-1}) = \langle ß\rangle = \mathbb{C}_4$ imply $g = I^k J^l$ for some $0 \leq k, l \leq 3$. Similarly, for $g(0 \times \mathbb{C}) = \mathbb{C} \times 0$, $g(\mathbb{C} \times 0) = 0 \times \mathbb{C}$ one has $\alpha = \delta = 0$, whereas $\beta, \gamma \in \mathbb{Z}[ß]$, $\beta\gamma \in \mathbb{Z}[ß]^* = \langle ß\rangle$ and $g = I^k J^l \theta$ for some $0 \leq k, l \leq 3$. Consequently, $G^{(6,8)}_{-1} \cap Gl_2(\mathbb{Z}[ß]) = \langle I, J, \theta\rangle$ and $G^{(6,8)}_{-1} = \langle I, J, \theta, \tau_{33}\rangle$. The announced relations among τ_{33}, I, J, θ imply that

$$G^{(6,8)}_{-1} = \{\tau_{33}^n I^k J^l \theta^m \quad | \quad 0 \leq k, l \leq 3, \quad 0 \leq m, n \leq 1\}$$

is of order 64.

□

Theorem 9. *In the notations from Lemma 5, Theorem 6 and Lemma 8, let us consider the subgroups* $H_{HE} = \langle \tau_{33} J^2\rangle$, $H_{Enr} = \langle -I_2, \tau_{33} I^2\rangle$, $H_{Rul} = \langle J^2\rangle$ *of* $G^{(6,8)}_{-1} = Aut\left(A_{-1}, T^{(6,8)}_{-1}\right)$, *their liftings* $\Gamma^{(6,8)}_{HE,-1}$, $\Gamma^{(6,8)}_{Enr,-1}$, $\Gamma^{(6,8)}_{Rul,-1}$ *to ball lattices and the blow-up* $A_{\widehat{2-tor}}$ *of* A_{-1} *at the 2-torsion points* A_{2-tor}*. Then*

(i) $\mathbb{B}/\Gamma^{(6,8)}_{HE,-1}$ *is a smooth ball quotient, birational to the smooth hyperelliptic surface* A_{-1}/H_{HE}*;*

(ii) $\mathbb{B}/\Gamma^{(6,8)}_{Enr,-1}$ *is a ball quotient with one double point* $Orb_{H_{Enr}}(Q_{03})$*, which is birational to the smooth Enriques surface* $A_{\widehat{2-tor}}/H_{Enr}$*;*

(iii) $\mathbb{B}/\Gamma^{(6,8)}_{Rul,-1}$ *is a smooth ball quotient, birational to the smooth trivial ruled surface* $A_{-1}/H_{Rul} = E_{-1} \times \mathbb{P}^1$ *with an elliptic base* E_{-1}*.*

Proof. (i) Recall that the $\mathbb{Z}$-module $\pi_1(E_{-1}) = \mathbb{Z} + i\mathbb{Z} = \mathbb{Z} + (1+i)\mathbb{Z}$ is generated by $1, 1+i$ and $Q_3 = \frac{1+i}{2} (\mathrm{mod}\, \pi_1(E_{-1}))$. The translation $\tau_{Q_3} : E_{-1} \to E_{-1}$ is of order 2, as well as the morphism

$$\tau_{Q_3}(-1) : E_{-1} \longrightarrow E_{-1},$$

$$\tau_{Q_3}(-1)(P) = -P + Q_3$$

with four fixed points

$$\frac{1}{2}Q_3 + (E_{-1})_{2-tor} = \frac{1}{2}Q_3 + \{Q_i \,|\, 0 \leq i \leq 3\}.$$

According to [5], the quotient A_{-1}/H_{HE} by the cyclic group

$$H_{HE} = \langle \tau_{Q_3} \times \tau_{Q_3}(-1) \rangle$$

of order 2 is a smooth hyperelliptic surface. Lemma 5 (ii) implies that $\mathbb{B}/\Gamma^{(6,8)}_{HE,-1}$ is a smooth ball quotient, birational to A_{-1}/H_{HE}.

(ii) The quotient $X_{K3} = A_{\widehat{2-tor}}/\langle -I_2 \rangle$ is a smooth K3 surface, called the Kummer surface of A_{-1}. We claim that the involution $\tau_{33}I^2$ acts on $A_{\widehat{2-tor}}$ and determines an unramified double cover

$$\zeta : X_{K3} = A_{\widehat{2-tor}}/\langle -I_2 \rangle \rightarrow A_{\widehat{2-tor}}/\langle -I_2, \tau_{33}I^2 \rangle = A_{\widehat{2-tor}}/H_{Enr}.$$

More precisely, $\tau_{33}I^2 = \tau_{Q_3}(-1) \times \tau_{Q_3}$ leaves invariant the 2-torsion points $A_{2-tor} = \{Q_{ij} \,|\, 0 \leq i,j \leq 3\}$ and any choice of an automorphism of $\mathbb{P}^1$ extends $\tau_{33}I^2$ to an automorphism of $A_{\widehat{2-tor}}$. Note that $\tau_{33}I^2(-I_2) = (-I_2)\tau_{33}I^2$, so that $\tau_{33}I^2$ normalizes $\langle -I_2 \rangle$ and there is a well defined quotient group $H_{Enr}/\langle -I_2 \rangle = \langle \tau_{33}I^2 \rangle$ of order 2. That allows to define $\zeta : X_{K3} \rightarrow A_{\widehat{2-tor}}/H_{Enr}$ as an $H_{Enr}/\langle -I_2 \rangle$-Galois cover. We claim that $\tau_{33}I^2$ is a fixed point free involution on X_{K3}, in order to conclude that $A_{\widehat{2-tor}}/H_{Enr}$ is a smooth Enriques surface. More precisely, the fixed points of $\tau_{33}I^2$ on the set X_{K3} of the $\langle -I_2 \rangle$-orbits on $A_{\widehat{2-tor}}$ lift to ε-fixed points of $\tau_{33}I^2$ on $A_{\widehat{2-tor}}$ for $\varepsilon = \pm 1$. The ε-fixed points $(P,Q) \in A_{-1}$ are subject to

$$\left| \begin{array}{c} -P + Q_3 = \varepsilon P \\ Q + Q_3 = \varepsilon Q \end{array} \right.$$

For $\varepsilon = 1$ the equality $Q + Q_3 = Q$ has no solution $Q \in E_{-1}$, while for $\varepsilon = -1$ the equation $-P + Q_3 = -P$ on $P \in E_{-1}$ is inconsistent. Therefore $\tau_{33}I^2$ has no ε-fixed points on A_{-1}. By the very definition of the $\tau_{33}I^2$-action on $A_{\widehat{2-tor}}$, there are no ε-fixed points for $\tau_{33}I^2$ on $A_{\widehat{2-tor}}$ and $\tau_{33}I^2 : X_{K3} \rightarrow X_{K3}$ is a fixed point free involution. As a result, $A_{\widehat{2-tor}}/H_{Enr}$ is a smooth Enriques surface.

Recall that the exceptional divisor $\xi^{-1}_{2-tor}(A_{2-tor})$ of the blow-up

$$\xi_{2-tor} : A_{\widehat{2-tor}} \rightarrow A_{-1}$$

of A_{-1} at A_{2-tor} is H_{Enr}-invariant, so that ξ_{2-tor} descends to the contraction $\overline{\xi_{2-tor}} : A_{\widehat{2-tor}}/H_{Enr} \rightarrow A_{-1}/H_{Enr}$ of $\xi^{-1}_{2-tor}(A_{2-tor})/H_{Enr}$ to A_{2-tor}/H_{Enr}. In particular, the smooth Enriques surface $A_{\widehat{2-tor}}/H_{Enr}$ is birational to A_{-1}/H_{Enr}. The singular locus $(A_{-1}/H_{Enr})^{\text{sing}} \subseteq (A_{2-tor}/H_{Enr})$, according to the smoothness of $A_{\widehat{2-tor}}/H_{Enr}$. On the other hand, $\tau_{33}I^2$ has no fixed points on A_{2-tor}, so that A_{2-tor}/H_{Enr} consists of eight double points

$$Orb_{H_{Enr}}(Q_{ij}) = Orb_{H_{Enr}}(Q_{3-i,3-j}), \quad 0 \leq i,j \leq 3$$

and $(A_{-1}/H_{Enr})^{\text{sing}} = A_{2-tor}/H_{Enr}$. Note that

$$\left(T^{(6,8)}_{-1} \right)^{\text{sing}} = \{ Orb_{H_{Enr}}(Q_{00}), Orb_{H_{Enr}}(Q_{11}), Orb_{H_{Enr}}(Q_{12}) \}$$

is contained in $(A_{-1}/H_{Enr})^{\rm sing}$ and the birational morphism

$$\xi_{H_{Enr}}:\overline{\left(\mathbb{B}/\Gamma^{(6,8)}_{Enr,-1}\right)}\to A_{-1}/H_{Enr}$$

resolves $\left(T^{(6,8)}_{-1}\right)^{\rm sing}$ by smooth rational curves of self-intersection (-2). Therefore $\overline{\left(\mathbb{B}/\Gamma^{(6,8)}_{Enr,-1}\right)}^{\rm sing}$ consists of the following five double point:

$$Orb_{H_{Enr}}(Q_{01}),\ Orb_{H_{Enr}}(Q_{10}),\ Orb_{H_{Enr}}(Q_{02}),\ Orb_{H_{Enr}}(Q_{20}),\ Orb_{H_{Enr}}(Q_{03}).$$

Since

$$Orb_{H_{Enr}}(Q_{0,m})\in\left[T_{m+6}\setminus\left(T^{(6,8)}_{-1}\right)^{\rm sing}\right]/H_{Enr}=\left(T'_{m+6}\setminus L\right)/H_{Enr},$$

$$Orb_{H_{Enr}}(Q_{m,0})\in\left[T_{m+4}\setminus\left(T^{(6,8)}_{-1}\right)^{\rm sing}\right]/H_{Enr}=\left(T'_{m+4}\setminus L\right)/H_{Enr}$$

for $\forall 1\leq m\leq 2$ belong to the compactifying divisor T'/H_{Enr}, the ball quotient $\mathbb{B}/\Gamma^{(6,8)}_{Enr,-1}$ has only one singular point

$$\left(\mathbb{B}/\Gamma^{(6,8)}_{Enr,-1}\right)^{\rm sing}=\{Orb_{H_{Enr}}(Q_{0,3})\}.$$

(iii) The quotient $X=A_{-1}/H_{Rul}=E_{-1}\times[E_{-1}/\langle(-1)\rangle]$ of A_{-1} by the reflection $J^2=1\times(-1)$ is a smooth surface, birational to the smooth ball quotient $\mathbb{B}/\Gamma^{(6,8)}_{Rul,-1}$. It is well known that $C=E_{-1}/\langle -1\rangle$ is a smooth projective curve. More precisely, if

$$\mathfrak{p}(t)=\frac{1}{t^2}+\sum_{\lambda\in(\mathbb{Z}+i\mathbb{Z})\setminus\{0\}}\left[\frac{1}{(t-\lambda)^2}-\frac{1}{\lambda^2}\right]$$

is the Weierstrass $\mathfrak{p}$-function, associated with the lattice $\mathbb{Z}+i\mathbb{Z}=\pi_1(E_{-1})$, then the map

$$\psi:E_{-1}\setminus\{\check{o}_{E_{-1}}\}\longrightarrow\mathbb{P}^2,$$

$$\psi(t+(\mathbb{Z}+i\mathbb{Z}))=[1:\mathfrak{p}(t+(\mathbb{Z}+i\mathbb{Z})):\mathfrak{p}'(t+(\mathbb{Z}+i\mathbb{Z}))]=[1:\mathfrak{p}(t):\mathfrak{p}'(t)]$$

extends by $\psi(\check{o}_{E_{-1}})=[0:0:1]=p_\infty$ to a projective embedding of E_{-1}. The image

$$\psi(E_{-1})=\left\{[z:x:y]\in\mathbb{P}^2\ ;\ zy^2=(x-\mathfrak{p}(Q_1))(x-\mathfrak{p}(Q_2))(x-\mathfrak{p}(Q_3))\right\}$$

is a cubic hypersurface in $\mathbb{P}^2$. As far as $\mathfrak{p}(t)$ is even and $\mathfrak{p}'(t)$ is an odd function of t, the multiplication μ_{-1} by -1 on E_{-1} acts on $\psi(E_{-1})$ by the rule

$$\mu_{-1}([z:x:y])=[z:x:-y].$$

The fixed points of this action are p_∞ and $\mathfrak{p}(Q_i)$ for $1\leq i\leq 3$. The fibres of the projection

$$\Pi:\psi(E_{-1})\setminus\{p_\infty\}\longrightarrow\mathbb{P}^1\setminus\{q_\infty=[0:1]\},$$

$$\Pi([z:x:y]) = [z:x]$$

are exactly the μ_{-1}-orbits on $\psi(E_{-1}) \setminus \{p_\infty\}$, so that its image

$$\mathbb{P}^1 \setminus \{q_\infty\} = \Pi(\psi(E_{-1}) \setminus \{p_\infty\}) = (\psi(E_{-1}) \setminus \{p_\infty\})/\langle \mu_{-1} \rangle$$

is the corresponding Galois quotient by the cyclic group $\langle \mu_{-1} \rangle$ of order 2. Thus,

$$\psi(E_{-1})/\langle \mu_{-1} \rangle = (\psi(E_{-1}) \setminus \{p_\infty\})/\langle \mu_{-1} \rangle \cup \{p_\infty\} = (\mathbb{P}^1 \setminus \{q_\infty\}) \cup \{p_\infty\} = \mathbb{P}^1.$$

□

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