Tenth International Conference on
Geometry, Integrability and Quantization
June 6–11, 2008, Varna, Bulgaria
Ivaïlo M. Mladenov, Gaetano Vilasi
and Akira Yoshioka, Editors
**Avangard Prima**, Sofia 2009, pp 1–27

# MODULAR FORMS ON BALL QUOTIENTS OF NON-POSITIVE KODAIRA DIMENSION

AZNIV KASPARIAN

*Department of Mathematics and Informatics, Kliment Ohridski University of Sofia, Sofia 1164, Bulgaria*

**Abstract.** The Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma}$ of an arithmetic ball quotient admits projective embeddings by $\Gamma$-modular forms of sufficiently large weight. We are interested in the target and the rank of the projective map $\Phi$, determined by $\Gamma$-modular forms of weight 1. The note concentrates on the finite $H$-Galois quotients $\mathbb{B}/\Gamma_H$ of a specific $\mathbb{B}/\Gamma_{-1}^{(6,8)}$, birational to an abelian surface $A_{-1}$. Any compactification of $\mathbb{B}/\Gamma_H$ has non-positive Kodaira dimension. The rational maps $\Phi^H$ of $\widehat{\mathbb{B}/\Gamma_H}$ are studied by the means of $H$-invariant abelian functions on $A_{-1}$.

The modular forms of sufficiently large weight are known to provide projective embeddings of the arithmetic quotients of the 2-ball

$$\mathbb{B} = \{z = (z_1, z_2) \in \mathbb{C}^2 \ ; \ |z_1|^2 + |z_2|^2 < 1\} \simeq SU(2,1)/S(U_2 \times U_1)$$

The present work studies the projective maps, given by the modular forms of weight one on certain Baily-Borel compactifications $\widehat{\mathbb{B}/\Gamma_H}$ of Kodaira dimension $\kappa(\widehat{\mathbb{B}/\Gamma_H}) \leq 0$. More precisely, we start with a fixed smooth Picard modular surface $A'_{-1} = \left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)'$ with abelian minimal model $A_{-1} = E_{-1} \times E_{-1}$, $E_{-1} = \mathbb{C}/\mathbb{Z} + \mathbb{Z}i$. Any automorphism group of $A'_{-1}$, preserving the toroidal compactifying divisor $T' = \left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)' \setminus \left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)$ acts on $A_{-1}$ and lifts to a ball lattice $\Gamma_H$, normalizing $\Gamma_{-1}^{(6,8)}$. The ball quotient compactification $A'_{-1}/H = \overline{\mathbb{B}/\Gamma_H}$ is birational to $A_{-1}/H$. We study the $\Gamma_H$-modular forms $[\Gamma_H, 1]$ of weight one by realizing them as $H$-invariants of $[\Gamma_{-1}^{(6,8)}, 1]$. That allows to transfer $[\Gamma_H, 1]$ to the $H$-invariant abelian functions, in order to determine $\dim_{\mathbb{C}}[\Gamma_H, 1]$ and the transcendence dimension of the graded $\mathbb{C}$-algebra, generated by $[\Gamma_H, 1]$. The last one is exactly the rank of the projective map $\Phi : \widetilde{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, 1])$.

## 1. The Transfer of Modular Forms to Meromorphic Functions is Inherited by the Finite Galois Quotients

**Definition 1.** *Let $\Gamma < SU(2,1)$ be a lattice, i.e., a discrete subgroup, whose quotient $SU(2,1)/\Gamma$ has finite invariant measure. A $\Gamma$-modular form of weight $n$ is a holomorphic function $\delta : \mathbb{B} \to \mathbb{C}$ with transformation law*

$$\gamma(\delta)(z) = \delta(\gamma(z)) = [\det \mathrm{Jac}(\gamma)]^{-n}\delta(z) \quad \textit{for } \forall \gamma \in \Gamma, \ \ \forall z \in \mathbb{B}.$$

Bearing in mind that a biholomorphism $\gamma \in \mathrm{Aut}(\mathbb{B})$ acts on a differential form $\mathrm{d}z_1 \wedge \mathrm{d}z_2$ of top degree as a multiplication by the Jacobian determinant $\det \mathrm{Jac}(\gamma)$, one constructs the linear isomorphism

$$j_n : [\Gamma, n] \longrightarrow H^0(\mathbb{B}, (\Omega^2_{\mathbb{B}})^{\otimes n})^{\Gamma}$$
$$j_n(\delta) = \delta(z)(dz_1 \wedge dz_2)^n$$

with the $\Gamma$-invariant holomorphic sections of the canonical bundle $\Omega^2_{\mathbb{B}}$ of $\mathbb{B}$. Thus, the graded $\mathbb{C}$-algebra of the $\Gamma$-modular forms can be viewed as the tensor algebra of the $\Gamma$-invariant volume forms on $\mathbb{B}$. For any $\delta_1, \delta_2 \in [\Gamma, n]$ the quotient $\frac{\delta_1}{\delta_2}$ is a correctly defined holomorphic function on $\mathbb{B}/\Gamma$. In such a way, $[\Gamma, n]$ and $j_n[\Gamma, n]$ determine a projective map

$$\Phi_n : \mathbb{B}/\Gamma \longrightarrow \mathbb{P}([\Gamma, n]) = \mathbb{P}(j_n[\Gamma, n]).$$

The $\Gamma$-cusps $\partial_\Gamma \mathbb{B}/\Gamma$ are of complex co-dimension 2, so that $\Phi_n$ extends to the Baily-Borel compactification

$$\Phi_n : \widehat{\mathbb{B}/\Gamma} \longrightarrow \mathbb{P}([\Gamma, n]).$$

If the lattice $\Gamma < SU_{2,1}$ is torsion-free then the toroidal compactification $X' = (\mathbb{B}/\Gamma)'$ is a smooth surface. Denote by $\rho : X' = (\mathbb{B}/\Gamma)' \to \widehat{X} = \widehat{\mathbb{B}/\Gamma}$ the contraction of the irreducible components $T'_i$ of the toroidal compactifying divisor $T'$ to the $\Gamma$-cusps $\kappa_i \in \partial_\Gamma \mathbb{B}/\Gamma$. The tensor product $\Omega^2_{X'}(T')$ of the canonical bundle $\Omega^2_{X'}$ of $X'$ with the holomorphic line bundle $\mathcal{O}(T')$, associated with the toroidal compactifying divisor $T'$ is the logarithmic canonical bundle of $X'$. In [2] Hemperly has observes that

$$H^0(X', \Omega^2_{X'}(T')^{\otimes n}) = \rho^* j_n[\Gamma, n] \simeq [\Gamma, n].$$

Let $K_{X'}$ be the canonical divisor of $X'$,

$$\mathcal{L}_{X'}(nK_{X'} + nT') = \{f \in \mathfrak{Mer}(X') \ ; \ (f) + nK_{X'} + nT' \geq 0\}$$

be the linear system of the divisor $n(K_{X'} + T')$ and $s$ be a global meromorphic section of $\Omega^2_{X'}(T')$. Then

$$s^{\otimes n} : \mathcal{L}_{X'}(nK_{X'} + nT') \longrightarrow H^0(X', \Omega^2_{X'}(T')^{\otimes n})$$

is a $\mathbb{C}$-linear isomorphism. Let $\xi : X' \to X$ be the blow-down of the $(-1)$-curves on $X' = (\mathbb{B}/\Gamma)'$ to its minimal model $X$. The Kobayashi hyperbolicity of $\mathbb{B}$ requires $X'$ to be the blow-up of $X$ at the singular locus $T^{\rm sing}$ of $T = \xi(T')$. The canonical divisor $K_{X'} = \xi^* K_X + L$ is the sum of the pull-back of $K_X$ with the exceptional divisor $L$ of $\xi$. The birational map $\xi$ induces an isomorphism $\xi^* : \mathfrak{Mer}(X) \to \mathfrak{Mer}(X')$ of the meromorphic function fields. In order to translate the condition $\xi^*(f) + nK_{X'} + nT' \geq 0$ in terms of $f \in \mathfrak{Mer}(X)$, let us recall the notion of a multiplicity of a divisor $D \subset X$ at a point $p \in X$. If $D = \sum\limits_i n_i D_i$ is the decomposition of $D$ into irreducible components then $m_p(D) = \sum\limits_i n_i m_p(D_i)$, where

$$m_p(D_i) = \begin{cases} 1 & \text{for } p \in D_i \\ 0 & \text{for } p \notin D_i. \end{cases}$$

Let $L = \sum\limits_{p \in T^{\rm sing}} L(p)$ for $L(p) = \xi^{-1}(p)$ and $f \in \mathfrak{Mer}(X)$. The condition $\xi^*(f) + n\xi^* K_X + nL + nT' \geq 0$ is equivalent to $m_p(f) + n \geq 0$ for $\forall p \in T^{\rm sing}$. Thus, $\mathcal{L}_{X'}(nK_{X'} + nT')$ turns to be the pull-back of the subset

$$\mathcal{L}_X(nK_X + nT, nT^{\rm sing}) =$$

$$= \{ f \in \mathfrak{Mer}(X) \ ; \ (f) + nK_X + nT \geq 0, \ \ m_p(f) + n \geq 0, \ \ \forall p \in T^{\rm sing} \}$$

of the linear system $\mathcal{L}_X(nK_X + nT)$. The $\mathbb{C}$-linear isomorphism

$$\mathrm{Trans}_n := (\xi^*)^{-1} s^{\otimes(-n)} j_n : [\Gamma, n] \longrightarrow \mathcal{L}_X(nK_X + nT, nT^{\rm sing}),$$

introduced by Holzapfel in [3], is called transfer of modular forms.

Bearing in mind Hemperly's result $H^0(X', \Omega^2_{X'}(T')^{\otimes n}) = \rho^* j_1[\Gamma, n]$ for a fixed point free $\Gamma$, we refer to

$$\Phi^H_n : \widehat{\mathbb{B}/\Gamma_H} \longrightarrow \mathbb{P}([\Gamma_H, n]) = \mathbb{P}(j_n[\Gamma_H, n])$$

as the $n$-th logarithmic-canonical map of $\widehat{\mathbb{B}/\Gamma_H}$, regardless of the ramifications of $\mathbb{B} \to \mathbb{B}/\Gamma_H$.

The next lemma explains the transfer of modular forms on finite Galois quotients $\mathbb{B}/\Gamma_H$ of $\mathbb{B}/\Gamma$ to meromorphic functions on $X/H$. In general, the toroidal compactification $X'_H = (\mathbb{B}/\Gamma_H)'$ is a normal surface. The logarithmic-canonical bundle is not defined on a singular $X'_H$, but there is always a logarithmic-canonical Weil divisor on $X'_H$.

**Lemma 1.** *Let $A' = (\mathbb{B}/\Gamma)'$ be a neat toroidal compactification with an abelian minimal model $A$ and $H$ be a subgroup of $G = \mathrm{Aut}(A,T) = \mathrm{Aut}(A',T')$. Then:*
*(i) the transfer $\mathrm{Trans}_n := (\xi^*)^{-1} s^{\otimes(-n)} j_n : [\Gamma, n] \longrightarrow \mathcal{L}_A(nT, nT^{\mathrm{sing}})$ of $\Gamma$-modular forms to abelian functions induces a linear isomorphism*

$$\mathrm{Trans}_n^H : [\Gamma_H, n] \longrightarrow \mathcal{L}_A(nT, nT^{\mathrm{sing}})^H,$$

*of $\Gamma_H$-modular forms with rational functions on $A/H$, called also a transfer;*
*(ii) the projective maps*

$$\Phi_n^H : \widehat{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, n]), \quad \Psi_n^H : A/H \dashrightarrow \mathbb{P}(\mathcal{L}_A(nT, nT^{\mathrm{sing}})^H)$$

*coincide on an open Zariski dense subset.*

**Proof:** (i) Note that $j_n[\Gamma_H, n] = j_n[\Gamma, n]^H$. The inclusion $j_n[\Gamma_H, n] \subseteq j_n[\Gamma, n]$ follows from $\Gamma \leq \Gamma_H$. If $\Gamma_H = \cup_{j=1}^n \gamma_j \Gamma$ is the coset decomposition of $\Gamma_H$ modulo $\Gamma$, then $H = \{h_i = \gamma_i \Gamma \ ; \ 1 \leq i \leq n\}$. A $\Gamma$-modular form $\omega \in j_n[\Gamma, n]$ is $\Gamma_H$-modular exactly when it is invariant under all $\gamma_i$, which amounts to the invariance under all $h_i$.
One needs a global meromorphic $G$-invariant section $s$ of $\Omega^2_{A'}(T')$, in order to obtain a linear isomorphism

$$(\xi^*)^{-1} s^{\otimes(-n)} = \mathrm{Trans}_n^H j_n^{-1} : j_n[\Gamma_H, n] = j_n[\Gamma, n]^H \to \mathcal{L}_A(nT, nT^{\mathrm{sing}})^H.$$

The global meromorphic sections of the logarithmic-canonical line bundle $\Omega^2_{A'}(T')$ are in a bijective correspondence with the families $(f_\alpha, U_\alpha)_{\alpha \in S}$ of local meromorphic defining equations $f_\alpha : U_\alpha \to \mathbb{C}$ of the logarithmic-canonical divisor $L + T'$. We construct local meromorphic $G$-invariant equations $g_\alpha : V_\alpha \to \mathbb{C}$ of $T$ and pull-back to $(f_\alpha = \xi^* g_\alpha, U_\alpha = \xi^{-1}(V_\alpha))_{\alpha \in S}$. Let $F_A : \widetilde{A} = \mathbb{C}^2 \to A$ be the universal covering map of $A$. Then for any point $p \in A$ choose a lifting $\widetilde{p} \in F_A^{-1}(p)$ and a sufficiently small neighborhood $\widetilde{W}$ of $\widetilde{p}$ on $\widetilde{A}$, which is contained in the interior of a $\pi_1(A)$-fundamental domain on $\widetilde{A}$, centered at $\widetilde{p}$. The $G$-invariant open neighborhood $W = \cap_{g \in G} g\widetilde{W}$ of $\widetilde{p}$ on $\widetilde{A}$ intersects $F_A^{-1}(T)$ in lines with local equations $l_j(u,v) = a_j(\widetilde{p})u + b_j(\widetilde{p})v + c_j(\widetilde{p}) = 0$. The holomorphic function $g(u,v) = \prod_{g \in G} \prod_j (l_j(u,v))$ on $W$ is $G$-invariant and can be viewed as a $G$-invariant local defining equation of $T$ on $V = F_A(W)$. Note that $F_A$ is locally biholomorphic, so that $V \subset A$ is an open subset, after an eventual shrinking of $\widetilde{W}$. The family $(g, V)_{p \in A}$ of local $G$-invariant defining equations of $T$ pulls-back to a family $(f = \xi^* g, U = \xi^{-1}(V))_{p \in A}$ of local $G$-invariant sections of $\Omega^2_A(T')$.
(ii) Towards the coincidence $\Psi_n^H|_{[(A \setminus T)/H]} \equiv \Phi_n^H|_{[(\mathbb{B}/\Gamma_H) \setminus (L/H)]}$, let us fix a basis $\{\omega_i \ ; \ 1 \leq i \leq d\}$ of $j_n[\Gamma_H, n]$ and apply (i), in order to conclude that the set $\{f_i = \mathrm{Trans}_n^H j_n^{-1}(\omega_i) \ ; \ 1 \leq i \leq d\}$ is a basis of $\mathcal{L}_A(nT, nT^{\mathrm{sing}})^H$. Tensoring

by $s^{\otimes(-n)}$ does not alter the ratios $\frac{\omega_i}{\omega_j}$. The isomorphism $\xi : \mathfrak{Mer}(A) \to \mathfrak{Mer}(A')$ is identical on $(A \setminus T)/H$.

□

## 2. Preliminaries

In order to specify $A'_{-1} = \left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)'$ let us note that the blow-down $\xi : A'_{-1} \to A_{-1}$ of the $(-1)$-curves maps $T'$ to a divisor $T = \xi(T')$ with smooth elliptic irreducible components $T_i$. Such $T$ are called multi-elliptic divisors. Any irreducible component $T_i$ of $T$ lifts to a $\pi_1(A_{-1})$-orbit of complex lines on the universal cover $\widetilde{A'_{-1}} = \mathbb{C}^2$. That allows to represent

$$T_j = \{(u(\text{mod } \mathbb{Z}+\mathbb{Z}i), v(\text{mod } \mathbb{Z}+\mathbb{Z}i)) \ ; \ a_j u + b_j v + c_j = 0\}.$$

If $T_j$ is defined over the field $\mathbb{Q}(\mathrm{i})$ of Gauss numbers, there is no loss of generality in assuming $a_j, b_j \in \mathbb{Z}[\mathrm{i}]$ to be Gaussian integers.

**Theorem 1.** (Holzapfel [4]) *Let $A_{-1} = E_{-1} \times E_{-1}$ be the Cartesian square of the elliptic curve $E_{-1} = \mathbb{C}/\mathbb{Z}+\mathbb{Z}\mathrm{i}$, $\omega_1 = \frac{1}{2}$, $\omega_2 = i\omega_1$, $\omega_3 = \omega_1 + \omega_2$ be half-periods,*

$$Q_0 = 0(\text{mod } \mathbb{Z}+\mathbb{Z}\mathrm{i}), \ Q_1 = \omega_1(\text{mod } \mathbb{Z}+\mathbb{Z}\mathrm{i}), \ Q_2 = \mathrm{i}Q_1, \ Q_3 = Q_1 + Q_2$$

*be the 2-torsion points on $E_{-1}$, $Q_{ij} = (Q_i, Q_j) \in A_{-1}^{2-\mathrm{tor}}$ and*

$$T_k = \{(u(\text{mod } \mathbb{Z}+\mathbb{Z}\mathrm{i}), v(\text{mod } \mathbb{Z}+\mathbb{Z}\mathrm{i}) \ ; \ u - i^k v = 0\} \quad \textit{with} \quad 1 \le k \le 4,$$

$$T_{4+m} = \{u(\text{mod}\,\mathbb{Z}+\mathbb{Z}\mathrm{i}), v(\text{mod}\,\mathbb{Z}+\mathbb{Z}\mathrm{i}) \ ; \ u - \omega_m = 0\} \ \textit{for} \ \ 1 \le m \le 2 \ \ \textit{and}$$

$$T_{6+m} = \{u(\text{mod}\,\mathbb{Z}+\mathbb{Z}\mathrm{i}), v(\text{mod}\,\mathbb{Z}+\mathbb{Z}\mathrm{i}) \ ; \ v - \omega_m = 0\} \ \textit{for} \ \ 1 \le m \le 2.$$

*Then the blow-up of $A_{-1}$ at the singular locus $\left(T_{-1}^{(6,8)}\right)^{\mathrm{sing}} = Q_{00} + Q_{33} + \sum_{i=1}^{2}\sum_{j=1}^{2} Q_{ij}$ of the multi-elliptic divisor $T_{-1}^{(6,8)} = \sum_{i=1}^{8} T_i$ is a neat toroidal ball quotient compactification $A'_{-1} = \left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)'$.*

**Theorem 2.** (Kasparian, Kotzev [6]) *The group $G_{-1} = \mathrm{Aut}(A_{-1}, T_{-1}^{(6,8)}) = \mathrm{Aut}(A'_{-1}, T')$ of order $64$ is generated by the translation $\tau_{33}$ with $Q_{33}$, the multiplications*

$$I = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \ \textit{respectively,} \ J = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

*with $\mathrm{i} \in \mathbb{Z}[\mathrm{i}]$ on the first, respectively, the second factor $E_{-1}$ of $A_{-1}$ and the transposition*

$$\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

*of these factors.*

Throughout, we use the notations from Theorem 1 and Theorem 2, without mentioning explicitly.

With a slight abuse of notation, we speak of Kodaira-Enriques classification type, irregularity and geometric genus of $A_{-1}/H$, $H \leq G_{-1}$, referring actually to a smooth minimal model $Y$ of $A_{-1}/H$.

**Theorem 3.** (Kasparian, Nikolova [7]) *Let*

$$\mathcal{L} : G_{-1} \to \mathrm{Gl}_2(\mathbb{Z}[\mathrm{i}]) = \{g \in \mathbb{Z}[\mathrm{i}]_{2\times 2} \ ; \ \det(g) \in \mathbb{Z}[\mathrm{i}]^* = \langle i \rangle\}$$

*be the homomorphism, associating to* $g \in G_{-1}$ *its linear part* $\mathcal{L}$ *and*

$$L_1(G_{-1}) = \{g \in G_{-1} \ ; \ \mathrm{rk}(\mathcal{L}(g) - I_2) = 1\} =$$

$$\{\tau_{33}^n I^k, \tau_{33}^n J^k, \tau_{33}^n I^l J^{-l}\theta \ ; \ 0 \leq n \leq 1, \ 1 \leq k \leq 3, \ 0 \leq l \leq 3\}.$$

*Then:*

*(i)* $A_{-1}/H$ *is an abelian surface for* $H = \langle \tau_{33} \rangle$*;*

*(ii)* $A_{-1}/H$ *is a hyperelliptic surface for* $H = \langle \tau_{33} I^2 \rangle$ *or* $H = \langle \tau_{33} J^2 \rangle$*;*

*(iii)* $A_{-1}/H$ *is a ruled surface with an elliptic base for*

$$H = \langle h \rangle, \ h \in L_1(G_{-1}) \backslash \{\tau_{33} I^2, \tau_{33} J^2\} \ \ \textit{or} \ \ H = \langle \tau_{33}, h_o \rangle, \ h_o \in \mathcal{L}(L_1(G_{-1}));$$

*(iv)* $A_{-1}/H$ *is a K3 surface for* $\langle \tau_{33}^n \rangle \neq H \leq K = \ker \det \mathcal{L}$*, where*

$$K = \{\tau_{33}^n I^k J^{-k}, \tau_{33}^n I^k J^{2-k}\theta \ ; \ 0 \leq n \leq 1, \ 0 \leq k \leq 3\};$$

*(v)* $A_{-1}/H$ *is an Enriques surface for* $H = \langle I^2 J^2, \tau_{33} I^2 \rangle$*;*

*(vi)* $A_{-1}/H$ *is a rational surface for*

$$\langle h \rangle \leq H, \ h \in \{\tau_{33}^n IJ, \tau_{33}^n I^2 J, \tau_{33}^n IJ^2 \ ; \ 0 \leq n \leq 1\} \ \ \textit{or} \ \ \langle \tau_{33}^n I^2 J^2, h_1 \rangle \leq H,$$

$$h_1 \in \{I^{2m} J^{2-2m}, \tau_{33}^m I, \tau_{33}^m J, \tau_{33}^m I^l J^{-l}\theta \ ; \ 0 \leq m \leq 1, \ 0 \leq l \leq 3\}, \ 0 \leq n \leq 1.$$

The following lemma specifies some known properties of Weierstrass $\sigma$-function over Gaussian integers $\mathbb{Z}[\mathrm{i}]$.

**Lemma 2.** *Let* $\sigma(z) = z \prod_{\lambda \in \mathbb{Z}[\mathrm{i}] \backslash \{0\}} \left(1 - \frac{z}{\lambda}\right)^{\frac{z}{\lambda} + \frac{1}{2}\left(\frac{z}{\lambda}\right)^2}$ *be Weierstrass* $\sigma$*-function, associated with the lattice* $\mathbb{Z}[\mathrm{i}]$ *of* $\mathbb{C}$*. Then:*

*(i)* $\sigma(\mathrm{i}^k z) = \mathrm{i}^k \sigma(z)$ *for* $\forall z \in \mathbb{C}$, $\forall 0 \leq k \leq 3$*;*

*(ii)* $\frac{\sigma(z+\lambda)}{\sigma(z)} = \varepsilon(\lambda) e^{-\pi \overline{\lambda} z - \frac{\pi}{2}|\lambda|^2}$ *for* $\forall z \in \mathbb{C}$, $\forall \lambda \in \mathbb{Z}[\mathrm{i}]$*, where*

$$\varepsilon(\lambda) = \begin{cases} -1 & \textit{if}\ \lambda \in \mathbb{Z}[\mathrm{i}] \setminus 2\mathbb{Z}[\mathrm{i}] \\ 1 & \textit{if}\ \lambda \in 2\mathbb{Z}[\mathrm{i}] \end{cases}.$$

**Proof:** (i) follows from

$$\prod_{\lambda\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}}\left(1-\frac{i^k z}{\lambda}\right)^{\frac{i^k z}{\lambda}+\frac{1}{2}\left(\frac{i^k z}{\lambda}\right)^2}=\prod_{\mu=\frac{\lambda}{i^k}\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}}\left(1-\frac{z}{\mu}\right)^{\frac{z}{\mu}+\frac{1}{2}\left(\frac{z}{\mu}\right)^2}.$$

(ii) According to Lang's book [8],

$$\frac{\sigma(z+\lambda)}{\sigma(z)}=\varepsilon(\lambda)\mathrm{e}^{\eta(\lambda)\left(z+\frac{\lambda}{2}\right)}\text{ for }\forall z\in\mathbb{C},\ \forall\lambda\in\mathbb{Z}[\mathrm{i}],$$

where $\eta:\mathbb{Z}[\mathrm{i}]\to\mathbb{C}$ is the homomorphism of $\mathbb{Z}$-modules, related to Weierstrass $\zeta$-function $\zeta(z)=\frac{\sigma'(z)}{\sigma(z)}$ by the identity $\zeta(z+\lambda)=\zeta(z)+\eta(\lambda)$. It suffices to establish that $\eta(\lambda)=-\pi\overline{\lambda}$ for $\forall\lambda\in\mathbb{Z}[\mathrm{i}]$. Recall from [8] Legendre's equality $\eta(\mathrm{i})-\mathrm{i}\eta(1)=2\pi\mathrm{i}$, in order to derive

$$\eta(\lambda)=\frac{\lambda+\overline{\lambda}}{2}\eta(1)+\frac{\lambda-\overline{\lambda}}{2\mathrm{i}}\eta(\mathrm{i})=(\eta(1)+\pi)\lambda-\pi\overline{\lambda}\quad\text{for}\quad\forall\lambda\in\mathbb{Z}[\mathrm{i}].$$

Combining with homogeneity $\eta(\mathrm{i}\lambda)=\frac{1}{\mathrm{i}}\eta(\lambda)$ for $\forall\lambda\in\mathbb{Z}[\mathrm{i}]$ (cf.[8]), one obtains

$$(\eta(1)+\pi)\mathrm{i}\lambda+\pi\mathrm{i}\overline{\lambda}=\eta(\mathrm{i}\lambda)=-\mathrm{i}\eta(\lambda)=-(\eta(1)+\pi)\mathrm{i}\lambda+\pi\mathrm{i}\overline{\lambda}\quad\text{for}\quad\forall\lambda\in\mathbb{Z}[\mathrm{i}].$$

Therefore $\eta(1)=-\pi$ and $\eta(\lambda)=-\pi\overline{\lambda}$ for $\forall\lambda\in\mathbb{Z}[\mathrm{i}]$.

□

**Corollary 1.**

$$\frac{\sigma(z+\omega_m)}{\sigma(z-\omega_m)}=-\mathrm{e}^{2(-1)^m\omega_m\pi z},$$

$$\frac{\sigma(z+\omega_m+2\varepsilon\omega_{3-m})}{\sigma(z-\omega_m)}=(-1)^{m+1}\varepsilon\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}+2(-1)^{m+1}\varepsilon\omega_{3-m}\pi z+2(-1)^m\omega_m\pi z},$$

$$\frac{\sigma(z-\omega_m+2\varepsilon\omega_{3-m})}{\sigma(z-\omega_m)}=(-1)^{m+1}\varepsilon\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}+2(-1)^{m+1}\varepsilon\omega_{3-m}\pi z}.$$

*for the half-periods* $\omega_1=\frac{1}{2}$, $\omega_2=\mathrm{i}\omega_1$ *and* $\varepsilon=\pm1$.

**Corollary 2.**

$$\frac{\sigma(z+2\varepsilon\omega_m)}{\sigma(z-1)}=\mathrm{e}^{-\pi z+(-1)^m2\varepsilon\pi\omega_m z},$$

$$\frac{\sigma(z+(-1)^m\omega_m+\varepsilon(-1)^m\omega_{3-m})}{\sigma(z-(-1)^m\omega_m+(-1)^m\omega_{3-m})}=-\mathrm{i}^{(-1)^m\frac{(1+\varepsilon)}{2}}\mathrm{e}^{2\omega_m\pi z+(1-\varepsilon)\omega_{3-m}\pi z}.$$

*for the half-periods* $\omega_1=\frac{1}{2}$, $\omega_2=\mathrm{i}\omega_1$ *and* $\varepsilon=\pm1$.

Corollaries 1 and 2 follow from Lemma 2(ii) and $\overline{\omega_m} = (-1)^{m+1}\omega_m$, $\omega_m^2 = \frac{(-1)^{m+1}}{4}$.

In [5] the map $\Phi : \widetilde{\mathbb{B}/\Gamma_{-1}^{(6,8)}} \to \mathbb{P}([\Gamma_{-1}^{(6,8)}, 1])$ is shown to be a regular embedding. This is done by constructing a $\mathbb{C}$-basis of $\mathcal{L} = \mathcal{L}_{A_{-1}}\left(T_{-1}^{(6,8)}, \left(T_{-1}^{(6,8)}\right)^{\text{sing}}\right)$, consisting of binary parallel or triangular $\sigma$-quotients. An abelian function $f_{\alpha,\beta} \in \mathcal{L}$ is binary parallel if the pole divisor $(f_{\alpha,\beta})_\infty = T_\alpha + T_\beta$ consists of two disjoint smooth elliptic curves $T_\alpha$ and $T_\beta$. A $\sigma$-quotient $f_{i,\alpha,\beta} \in \mathcal{L}$ is triangular if $T_i \cap T_\alpha \cap T_\beta = \emptyset$ and any two of $T_i$, $T_\alpha$ and $T_\beta$ intersect in a single point.

**Theorem 4.** (Kasparian, Kotzev [5]) *Let*

$$\Sigma_{12}(z) = \frac{\sigma(z-1)\sigma(z+\omega_1-\omega_2)}{\sigma(z-\omega_1)\sigma(z-\omega_2)}, \quad \Sigma_1 = \frac{\sigma(u-iv+\omega_3)}{\sigma(u-iv)},$$

$$\Sigma_2 = \frac{\sigma(u+v+\omega_3)}{\sigma(u+v)}, \ \Sigma_3 = \frac{\sigma(u+iv+\omega_3)}{\sigma(u+iv)}, \ \Sigma_4 = \frac{\sigma(u-v+\omega_3)}{\sigma(u-v)},$$

$$\Sigma_5 = \frac{\sigma(u-\omega_2)}{\sigma(u-\omega_1)}, \ \Sigma_6 = \frac{\sigma(u-\omega_1)}{\sigma(u-\omega_2)}, \ \Sigma_7 = \frac{\sigma(v-\omega_2)}{\sigma(v-\omega_1)}, \ \Sigma_8 = \frac{\sigma(v-\omega_1)}{\sigma(v-\omega_2)}.$$

*Then:*

*(i) the space* $\mathcal{L} = \mathcal{L}_{A_{-1}}\left(T_{\sqrt{-1}}^{(6,8)}, \left(T_{\sqrt{-1}}^{(6,8)}\right)^{\text{sing}}\right)$ *contains the binary parallel $\sigma$-quotients $f_{56}(u,v) = \Sigma_{12}(u)$, $f_{78}(u,v) = \Sigma_{12}(v)$ and the triangular $\sigma$-quotients*

$$\begin{array}{ll} f_{157} = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}+\pi u}\Sigma_1\Sigma_5\Sigma_7, & f_{168} = -\mathrm{e}^{-\pi-\pi\mathrm{i}u-\pi v-\pi\mathrm{i}v}\Sigma_1\Sigma_6\Sigma_8, \\ f_{357} = -\mathrm{e}^{-\pi+\pi u+\pi v+\pi\mathrm{i}v}\Sigma_3\Sigma_5\Sigma_7, & f_{368} = -\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}-\pi\mathrm{i}u}\Sigma_3\Sigma_6\Sigma_8, \\ f_{258} = \mathrm{e}^{-\pi+\pi u-\pi\mathrm{i}v}\Sigma_2\Sigma_5\Sigma_8, & f_{267} = \mathrm{e}^{-\pi-\pi\mathrm{i}u+\pi v}\Sigma_2\Sigma_6\Sigma_7, \\ f_{458} = -\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}+\pi u-\pi v}\Sigma_4\Sigma_5\Sigma_8, & f_{467} = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}-\pi\mathrm{i}u+\pi\mathrm{i}v}\Sigma_4\Sigma_6\Sigma_7 \end{array};$$

*(ii) a $\mathbb{C}$-basis of $\mathcal{L}$ is*

$$f_o := 1,\ f_1 := f_{157},\ f_2 := f_{258},\ f_3 := f_{368},\ f_4 := f_{467},\ f_5 := f_{56},\ f_6 := f_{78}.$$

## 3. Technical Preparation

The group $G_{-1} = \mathrm{Aut}\left(A_{-1}, T_{-1}^{(6,8)}\right)$ permutes the eight irreducible components of $T_{-1}^{(6,8)}$ and the $\Gamma_{-1}^{(6,8)}$-cusps. For any subgroup $H$ of $G_{-1}$, the $\Gamma_H$-cusps are the $H$-orbits of $\partial_{\Gamma_{-1}^{(6,8)}}\mathbb{B}/\Gamma_{-1}^{(6,8)} = \{\kappa_i\, ;\, 1 \leq i \leq 8\}$.

**Lemma 3.** *If $\varphi : G_{-1} \to S_8(\kappa_1, \ldots, \kappa_8)$ is the natural representation of $G_{-1} =$* $\mathrm{Aut}\left(A_{-1}, T_{-1}^{(6,8)}\right)$ *in the symmetric group of the $\Gamma_{-1}^{(6,8)}$-cusps, then*

$$\varphi(\tau_{33}) = (\kappa_5, \kappa_6)(\kappa_7, \kappa_8), \ \ \varphi(I) = (\kappa_1, \kappa_4, \kappa_3, \kappa_2)(\kappa_5, \kappa_6),$$

$$\varphi(J) = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)(\kappa_7, \kappa_8), \ \ \varphi(\theta) = (\kappa_1, \kappa_3)(\kappa_5, \kappa_7)(\kappa_6, \kappa_8).$$

**Proof:** The $\Gamma_{-1}^{(6,8)}$-cusps $\kappa_i$ are obtained by contraction of the proper transforms $T_i'$ of $T_i$ under the blow-up of $A_{-1}$ at $\left(T_{-1}^{(6,8)}\right)^{\rm sing}$. Therefore the representations of $G_{-1}$ in the permutation groups of $\{T_i \ ; \ 1 \leq i \leq 8\}$, $\{T_i' \ ; \ 1 \leq i \leq 8\}$ and $\{\kappa_i \ ; \ 1 \leq i \leq 8\}$ coincide.

According to $\tau_{33}(u - {\rm i}^k v) = u - {\rm i}^k v + (1 - {\rm i}^k)\omega_3 = u - {\rm i}^k v (\mathrm{mod} \ \ \mathbb{Z} + \mathbb{Z}{\rm i})$, the translation $\tau_{33}$ acts identically on $T_1$, $T_2$, $T_3$, $T_4$. Further, $\tau_{33}(u - \omega_m) = u + \omega_{3-m} \equiv u - \omega_{3-m} (\mathrm{mod} \ \ \mathbb{Z} + \mathbb{Z}{\rm i})$ reveals the permutation $(T_5, T_6)(T_7, T_8)$ of the last four components of $T_{-1}^{(6,8)}$.

Due to $I(u - {\rm i}^k v) = {\rm i}u - {\rm i}^k v = {\rm i}(u - {\rm i}^{k-1} v)$, the automorphism $I$ induces the permutation $(T_1, T_4, T_3, T_2)$ of the first four components of $T_{-1}^{(6,8)}$. Further, $I(u - \omega_m) = {\rm i}(u \pm \omega_{3-m})$ reveals that $I$ permutes $T_5$ with $T_6$. Note that $I$ acts identically on $v$ and fixes $T_7$, $T_8$.

In a similar vein, $J(u - {\rm i}^k v) = u - {\rm i}^{k+1} v$, $J(v - \omega_m) = {\rm i}(v \pm {\rm i}\omega_{3-m})$ determine that $\varphi(J) = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)(\kappa_7, \kappa_8)$. According to $\theta(u - {\rm i}^k v) = v - {\rm i}^k u = -{\rm i}^k (u - {\rm i}^{-k} v)$ and $\theta(u - \omega_m) = v - \omega_m$, one concludes that $\varphi(\theta) = (\kappa_1, \kappa_3)(\kappa_5, \kappa_7)(\kappa_6, \kappa_8)$.

□

The following lemma incorporates several arguments, which will be applied repeatedly towards determination of the target $\mathbb{P}([\Gamma_H, 1])$ and the rank of the logarithmic canonical map $\Phi^H$.

**Lemma 4.** *In the notations from Theorem 4, for an arbitrary subgroup $H$ of $G_{-1} = \mathrm{Aut}\left(A_{-1}, T_{-1}^{(6,8)}\right)$ and any $f \in \mathcal{L} = \mathcal{L}_{A_{-1}}\left(T_{-1}^{(6,8)}, \left(T_{-1}^{(6,8)}\right)^{\rm sing}\right)$, let $R_H(f) = \sum_{h \in H} h(f)$ be the value of Reynolds operator $R_H$ of $H$ on $f$.*

*(i)The space $\mathcal{L}^H$ of the $H$-invariants of $\mathcal{L}$ is spanned by $\{R_H(f_i) \ ; \ 0 \leq i \leq 6\}$.*

*(ii) Let $T_i \subset (R_H(f_{i,\alpha_1,\beta_1}))_\infty, (R_H(f_{i,\alpha_2,\beta_2}))_\infty \subseteq \mathrm{Orb}_H(T_i) + \sum_{\alpha=5}^{8} T_\alpha$ for some $1 \leq i \leq 4$, $5 \leq \alpha_j \leq 6$, $7 \leq \beta_j \leq 8$. Then*

$$R_H(f_{i,\alpha_2,\beta_2}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{78}), R_H(f_{i,\alpha_1,\beta_1})).$$

*(iii) Let $\overline{\kappa_{i_1}}, \ldots, \overline{\kappa_{i_p}}$ with $1 \leq i_1 < \ldots < i_p \leq 4$ be different $\Gamma_H$-cusps,*

$$T_{i_j} \subset (R_H(f_{i_j}))_\infty \subseteq \mathrm{Orb}_H(T_{i_j}) + \sum_{\alpha=5}^{8} T_\alpha \quad \textit{for all} \quad 1 \leq j \leq p$$

*and $B$ be a $\mathbb{C}$-basis of $\mathcal{L}_2^H = \mathcal{L}_{A_{-1}}\left(\sum\limits_{\alpha=5}^{8} T_\alpha\right)^H$. Then the set*

$$\{R_H(f_{i_j,\alpha_j,\beta_j}) \ ; \ 1 \leq j \leq p\} \cup B$$

*is linearly independent over $\mathbb{C}$.*

*(iv) If $R_j = R_H(f_{j,\alpha_j,\beta_j}) \not\equiv \mathrm{const}$, $R_j|_{T_j} = \infty$ and $R_i = R_H(f_{i,\alpha_i,\beta_i})$ has $R_i|_{T_j} \not\equiv \mathrm{const}$ then for any subgroup $H_o$ of $H$ the projective maps*

$$\Psi^{H_o} : X/H_o \dashrightarrow \mathbb{P}(\mathcal{L}^{H_o}), \quad \Phi^{H_o} : \widehat{\mathbb{B}/\Gamma_{H_o}} \dashrightarrow \mathbb{P}(j_1[\Gamma_{H_o}, 1])$$

*are of rank $\mathrm{rk}\Phi^{H_o} = \mathrm{rk}\Psi^{H_o} = 2$.*

*(v) If the group $H'$ is obtained from the group $H$ by replacing all $\tau_{33}^n I^k J^l \theta^m \in H$ with $\tau_{33}^n I^l J^k \theta^m$, then the spaces of modular forms $j_1[\Gamma_{H'}, 1] \simeq j_1[\Gamma_H, 1]$ are isomorphic and the logarithmic-canonical maps have equal rank $\mathrm{rk}\Phi^H = \mathrm{rk}\Phi^{H'}$.*

**Proof:** (i) By Theorem 4(ii), $\mathcal{L} = \mathrm{Span}_{\mathbb{C}}(f_i \ ; \ 0 \leq 6)$. Therefore any $f \in \mathcal{L}$ is a $\mathbb{C}$-linear combination $f = \sum\limits_{i=0}^{6} c_i f_i$. Due to $H$-invariance of $f$ and the linearity of the representation of $H$ in $\mathrm{Aut}(\mathcal{L})$, Reynolds operator

$$|H| f = R_H(f) = \sum_{i=0}^{6} c_i R_H(f_i).$$

(ii) Let $\omega_s \in j_1\left[\Gamma_{-1}^{(6,8)}, 1\right]^H$ be the modular forms, which transfer to $R_H(f_{i,\alpha_s,\beta_s})$, $1 \leq s \leq 2$. Since $\omega_1(\kappa_i) \neq 0$, there exists $c_i \in \mathbb{C}$, such that $\omega_i' = \omega_2 - c_i\omega_1$ vanishes at $\kappa_i$. By the assumption $(R_H(f_{i,\alpha_s,\beta_s}))_\infty \subseteq \mathrm{Orb}_H(T_i) + \sum\limits_{\alpha=5}^{8} T_\alpha$, the transfer $F_i \in \mathcal{L}^H$ of $\omega_i'$ belongs to $\mathrm{Span}_{\mathbb{C}}(1, f_{56}, f_{78})^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{78}))$.

(iii) As far as the transfer $\mathrm{Trans}_1^H : j_1[\Gamma_H, 1] \to \mathcal{L}$ is a $\mathbb{C}$-linear isomorphism, it suffices to establish the linear independence of the corresponding modular forms $\{\omega_{i_j} \ ; \ 1 \leq j \leq p\} \cup \{\omega_b \ ; \ b \in B\}$. Evaluating the $\mathbb{C}$-linear combination $\sum\limits_{j=1}^{p} c_{i_j}\omega_{i_j} + \sum\limits_{b\in B} c_b\omega_b = 0$ at $\overline{\kappa_{i_1}}, \ldots, \overline{\kappa_{i_p}}$, one obtains $c_{i_j} = 0$, according to $\omega_{i_j}(\overline{\kappa_{i_s}}) = \delta_j^s = \begin{cases} 1 & \text{for } j = s \\ 0 & \text{for } j \neq s \end{cases}$ and $\omega_b(\overline{\kappa_{i_j}}) = 0, \forall b \in B, \forall 1 \leq j \leq p$. Then $\sum\limits_{b\in B} \omega_b = 0$ requires the vanishing of all $c_b$, due to the linear independence of $B$.

(iv) If $H_o$ is a subgroup of $H$ then $\mathcal{L}^H$ is a subspace of $\mathcal{L}^{H_o}$, $j_1[\Gamma_H, 1]$ is a subspace of $j_1[\Gamma_{H_o}, 1]$ and $\Psi^H = \mathrm{pr}^{\mathcal{L}}\Psi^{H_o}$, $\Phi^H = \mathrm{pr}^{\Gamma_H}\Phi^{H_o}$ for the projections $\mathrm{pr}^{\mathcal{L}} : \mathbb{P}(\mathcal{L}^{H_o}) \to \mathbb{P}(\mathcal{L}^H)$, $\mathrm{pr}^{\Gamma_H} : \mathbb{P}(j_1[\Gamma_{H_o}, 1]) \to \mathbb{P}(j_1[\Gamma_H, 1])$. That is why, it suffices to justify that $\mathrm{rk}\Phi^H = \mathrm{rk}\Psi^H = 2$ is maximal. Assume the opposite and consider $R_i, R_j : X/H \dashrightarrow \mathbb{P}^1$. The commutative diagram

$$\begin{array}{ccc} X/H & \xrightarrow{(R_i,R_j)} & \mathbb{P}^1 \times \mathbb{P}^1 \\ {\scriptstyle R_j}\downarrow & \swarrow {\scriptstyle \mathrm{pr}_2} & \\ \mathbb{P}^1 & & \end{array}$$

has surjective $R_j$, as far as $R_j \not\equiv \mathrm{const}$. If the image $C = (R_i, R_j)(X/H)$ is a curve, then the projection $\mathrm{pr}_2 : C \to \mathbb{P}^1$ has only finite fibers. In particular, $\mathrm{pr}_2^{-1}(\infty) = R_i((R_j)_\infty) \times \infty \supseteq R_i(T_j) \times \infty$ consists of finitely many points. However, $R_i(T_j) = \mathbb{P}^1$ as an image of the non-constant elliptic function $R_i : T_j \dashrightarrow \mathbb{P}^1$. The contradiction implies that $\dim_{\mathbb{C}} C = 2$ and $\mathrm{rk}\Psi^H = 2$.

(v) The transposition of the holomorphic coordinates $(u, v) \in \mathbb{C}^2$ affects non-trivially the constructed $\sigma$-quotients. However, one can replace the equations $u - \mathrm{i}^k v = 0$ of $T_k$, $1 \leq k \leq 4$ by $v - \mathrm{i}^{-k} u = 0$ and repeat the above considerations with interchanged $u, v$. The dimension of $j_1[\Gamma_H, 1]$ and the rank of $\Phi^H$ are invariant under the transposition of the global holomorphic coordinates on $\widetilde{A_{-1}} = \mathbb{C}^2$.

□

With a slight abuse of notation, we write $g(f)$ instead of $g^*(f)$, for $g \in G_{-1}$, $f \in \mathcal{L} = \mathcal{L}_{A_{-1}}\left(T_{-1}^{(6,8)}, \left(T_{-1}^{(6,8)}\right)^{\mathrm{sing}}\right)$.

**Lemma 5.** *The generators $\tau_{33}, I, J, \theta$ of $G_{-1}$ act on the binary parallel and triangular $\sigma$-quotients from Corollary 4 as follows:*

$$\tau_{33}(f_{56}) = -f_{56}, \quad \tau_{33}(f_{78}) = -f_{78},$$
$$\tau_{33}(f_{157}) = -\mathrm{i}e^{\frac{\pi}{2}} f_{168}, \quad \tau_{33}(f_{168}) = \mathrm{i}e^{-\frac{\pi}{2}} f_{157}, \quad \tau_{33}(f_{357}) = -\mathrm{i}e^{-\frac{\pi}{2}} f_{368},$$
$$\tau_{33}(f_{368}) = \mathrm{i}e^{\frac{\pi}{2}} f_{357}, \quad \tau_{33}(f_{258}) = f_{267}, \quad \tau_{33}(f_{267}) = f_{258},$$
$$\tau_{33}(f_{458}) = -f_{467}, \quad \tau_{33}(f_{467}) = -f_{458},$$
$$I(f_{56}) = -\mathrm{i}f_{56}, \quad I(f_{78}) = f_{78},$$
$$I(f_{157}) = -\mathrm{i}f_{467}, \quad I(f_{168}) = -e^{-\frac{\pi}{2}} f_{458}, \quad I(f_{357}) = \mathrm{i}f_{267},$$
$$I(f_{368}) = -e^{\frac{\pi}{2}} f_{258}, \quad I(f_{258}) = \mathrm{i}f_{168}, \quad I(f_{267}) = -e^{-\frac{\pi}{2}} f_{157},$$
$$I(f_{458}) = -\mathrm{i}f_{368}, \quad I(f_{467}) = -e^{\frac{\pi}{2}} f_{357},$$
$$J(f_{56}) = f_{56}, \quad J(f_{78}) = -\mathrm{i}f_{78},$$
$$J(f_{157}) = -\mathrm{i}e^{\frac{\pi}{2}} f_{258}, \quad J(f_{168}) = f_{267}, \quad J(f_{357}) = \mathrm{i}e^{-\frac{\pi}{2}} f_{458},$$
$$J(f_{368}) = f_{467}, \quad J(f_{258}) = f_{357}, \quad J(f_{267}) = -\mathrm{i}e^{-\frac{\pi}{2}} f_{368},$$

$$J(f_{458}) = f_{157}, \quad J(f_{467}) = \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{168},$$
$$\theta(f_{56}) = f_{78}, \quad \theta(f_{78}) = f_{56},$$
$$\theta(f_{157}) = -\mathrm{e}^{\frac{\pi}{2}} f_{357}, \quad \theta(f_{168}) = -\mathrm{e}^{-\frac{\pi}{2}} f_{368}, \quad \theta(f_{357}) = -\mathrm{e}^{-\frac{\pi}{2}} f_{157},$$
$$\theta(f_{368}) = -\mathrm{e}^{\frac{\pi}{2}} f_{168}, \quad \theta(f_{258}) = f_{267}, \quad \theta(f_{267}) = f_{258},$$
$$\theta(f_{458}) = f_{467}, \quad \theta(f_{467}) = f_{458}.$$

**Proof:** Making use of Lemma 2 and Corollary 2, one computes that

$$\tau_{33}\sigma(u-1) = -\mathrm{e}^{\pi u + \pi \mathrm{i} u}\sigma(u+\omega_1-\omega_2), \;\; \tau_{33}\sigma(u+\omega_1-\omega_2) = \mathrm{e}^{-2\pi u}\sigma(u-1),$$
$$\tau_{33}\sigma(u-\omega_1) = -\mathrm{e}^{\pi \mathrm{i} u}\sigma(u-\omega_2), \;\; \tau_{33}\sigma(u-\omega_2) = -\mathrm{e}^{-\pi u}\sigma(u-\omega_1),$$
$$\tau_{33}(\Sigma_1) = -\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}}\Sigma_1, \;\; \tau_{33}(\Sigma_2) = \mathrm{e}^{-\pi}\Sigma_2, \;\; \tau_{33}(\Sigma_3) = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}}\Sigma_3, \;\; \tau_{33}(\Sigma_4) = \Sigma_4,$$
$$\tau_{33}(\Sigma_5) = \mathrm{e}^{-\pi u - \pi \mathrm{i} u}\Sigma_6, \;\; \tau_{33}(\Sigma_6) = \mathrm{e}^{\pi u + \pi \mathrm{i} u}\Sigma_5,$$
$$\tau_{33}(\Sigma_7) = \mathrm{e}^{-\pi v - \pi \mathrm{i} v}\Sigma_8, \;\; \tau_{33}(\Sigma_8) = \mathrm{e}^{\pi v + \pi \mathrm{i} v}\Sigma_7,$$
$$I\sigma(u-1) = \mathrm{i}\mathrm{e}^{-\pi u + \pi \mathrm{i} u}\sigma(u-1), \;\; I\sigma(u+\omega_1-\omega_2) = -\mathrm{e}^{\pi u}\sigma(u+\omega_1-\omega_2),$$
$$I\sigma(u-\omega_1) = -\mathrm{i}\mathrm{e}^{\pi \mathrm{i} u}\sigma(u-\omega_2), \;\; I\sigma(u-\omega_2) = \mathrm{i}\sigma(u-\omega_1),$$
$$I(\Sigma_1) = \mathrm{i}\mathrm{e}^{-\pi \mathrm{i} u + \pi \mathrm{i} v}\Sigma_4, \;\; I(\Sigma_2) = \mathrm{i}\mathrm{e}^{-\pi \mathrm{i} u - \pi v}\Sigma_1,$$
$$I(\Sigma_3) = \mathrm{i}\mathrm{e}^{-\pi \mathrm{i} u - \pi \mathrm{i} v}\Sigma_2, \;\; I(\Sigma_4) = \mathrm{i}\mathrm{e}^{-\pi \mathrm{i} u + \pi v}\Sigma_3,$$
$$I(\Sigma_5) = -\mathrm{e}^{-\pi \mathrm{i} u}\Sigma_6, \;\; I(\Sigma_6) = -\mathrm{e}^{\pi \mathrm{i} u}\Sigma_5, \;\; I(\Sigma_7) = \Sigma_7, \;\; I(\Sigma_8) = \Sigma_8,$$
$$J\sigma(v+\mu) = I\sigma(u+\mu) \quad \text{for} \quad \forall \mu \in \mathbb{C},$$
$$J(\Sigma_1) = \Sigma_2, \;\; J(\Sigma_2) = \Sigma_3, \;\; J(\Sigma_3) = \Sigma_4, \;\; J(\Sigma_4) = \Sigma_1,$$
$$J(\Sigma_5) = \Sigma_5, \;\; J(\Sigma_6) = \Sigma_6, \;\; J(\Sigma_7) = -\mathrm{e}^{-\pi \mathrm{i} v}\Sigma_8, \;\; J(\Sigma_8) = -\mathrm{e}^{\pi \mathrm{i} v}\Sigma_7,$$
$$\theta\sigma(u+\mu) = \sigma(v+\mu) \quad \text{for} \quad \forall \mu \in \mathbb{C},$$
$$\theta(\Sigma_1) = -\mathrm{i}\mathrm{e}^{\pi u + \pi \mathrm{i} v}\Sigma_3, \;\; \theta(\Sigma_2) = \Sigma_2,$$
$$\theta(\Sigma_3) = \mathrm{i}\mathrm{e}^{-\pi \mathrm{i} u - \pi v}\Sigma_1, \;\; \theta(\Sigma_4) = -\mathrm{e}^{\pi u - \pi \mathrm{i} u - \pi v + \pi \mathrm{i} v}\Sigma_4,$$
$$\theta(\Sigma_5) = \Sigma_7, \;\; \theta(\Sigma_6) = \Sigma_8, \;\; \theta(\Sigma_7) = \Sigma_5, \;\; \theta(\Sigma_8) = \Sigma_6.$$

□

The following lemma is an immediate consequence of Lemma 2 and Corollary 1.

**Lemma 6.**

$$\left.\frac{f_{157}}{\Sigma_1}\right|_{T_1} = -\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}}, \;\; \left.\frac{f_{168}}{\Sigma_1}\right|_{T_1} = \mathrm{e}^{-\pi}, \;\; \left.\frac{f_{258}}{\Sigma_2}\right|_{T_2} = \mathrm{e}^{-\pi}, \;\; \left.\frac{f_{267}}{\Sigma_2}\right|_{T_2} = \mathrm{e}^{-\pi},$$
$$\left.\frac{f_{357}}{\Sigma_3}\right|_{T_3} = \mathrm{e}^{-\pi}, \;\; \left.\frac{f_{368}}{\Sigma_3}\right|_{T_3} = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}}, \;\; \left.\frac{f_{458}}{\Sigma_4}\right|_{T_4} = -\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}}, \;\; \left.\frac{f_{467}}{\Sigma_4}\right|_{T_4} = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}},$$
$$\left.\frac{f_{157} + \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{357}}{\Sigma_5}\right|_{T_5} = 0, \;\; \left.\frac{f_{258} - \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}} f_{458}}{\Sigma_5}\right|_{T_5} = 0.$$

**Lemma 7.**

$$[(f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}) + c(f_{357} - \mathrm{i}e^{-\frac{\pi}{2}} f_{368})]|_{T_2} = \mathrm{i}e^{-\frac{\pi}{2}-\pi v}\left(1 + c e^{-\frac{\pi}{2}}\right)$$

$$\frac{\sigma((1+\mathrm{i})v + \omega_3)}{\sigma((1+\mathrm{i})v)}\left[e^{(1+\mathrm{i})\pi v}\frac{\sigma(v-\omega_2)^2}{\sigma(v-\omega_1)^2} + e^{-(1+\mathrm{i})\pi v}\frac{\sigma(v-\omega_1)^2}{\sigma(v-\omega_2)^2}\right]$$

*is non-constant for all* $c \in \mathbb{C} \setminus \{-e^{\frac{\pi}{2}}\}$.

**Proof:** Note that

$$f(v) = [(f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}) + c(f_{357} - \mathrm{i}e^{-\frac{\pi}{2}} f_{368})]|_{T_2} =$$

$$\left[\mathrm{i}e^{-\frac{\pi}{2}-\pi v}\Sigma_1(-v,v) - c e^{-\pi+\pi \mathrm{i} v}\Sigma_3(-v,v)\right][\Sigma_5(-v)\Sigma_7(v) + \Sigma_6(-v)\Sigma_8(v)] =$$

$$= \mathrm{i}e^{-\frac{\pi}{2}-\pi v}\left(1 + c e^{-\frac{\pi}{2}}\right)\frac{\sigma((1+\mathrm{i})v - \omega_3)}{\sigma((1+\mathrm{i})v)}$$

$$\left[e^{(1+\mathrm{i})\pi v}\frac{\sigma(v-\omega_2)^2}{\sigma(v-\omega_1)^2} + e^{-(1+\mathrm{i})\pi v}\frac{\sigma(v-\omega_1)^2}{\sigma(v-\omega_2)^2}\right],$$

making use of Lemma 2 and Corollary 1. Obviously, $f(v)$ has no poles outside $\mathbb{Q}(\mathrm{i})$. It suffices to justify that $\lim_{v\to 0} f(v) = \infty$, in order to conclude that $f(v) \not\equiv$ const. To this end, use $\sigma(\omega_2) = \mathrm{i}\sigma(\omega_1)$ to observe that

$$f(v)\sigma((1+\mathrm{i})v)\Big|_{v=0} = -\mathrm{i}e^{-\frac{\pi}{2}}\left(1 + c e^{-\frac{\pi}{2}}\right)\sigma(\omega_3)\left(\mathrm{i}^2 + \frac{1}{\mathrm{i}^2}\right) \neq 0,$$

whenever $c \neq -e^{\frac{\pi}{2}}$, while $\sigma((1+\mathrm{i})v)|_{v=0} = 0$.

□

## 4. Basic Results

**Lemma 8.** *For* $H = \langle IJ^2, \tau_{33}J^2\rangle$, $\langle I^2J, \tau_{33}I^2\rangle$ *with rational* $A_{-1}/H$ *and any* $-\mathrm{Id} \in H \leq G_{-1}$, *the map* $\Phi^H : \widehat{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, 1])$ *is constant.*

**Proof:** By Lemma 4 (iv), the assertion for $\langle I^2J, \tau_{33}I^2\rangle$ is a consequence of the one for $\langle IJ^2, \tau_{33}J^2\rangle$. In the case of $H = \langle IJ^2, \tau_{33}J^2\rangle$, the space $\mathcal{L}^H$ is spanned by Reynolds operators

$$R_H(f_{56}) = 0, \quad R_H(f_{78}) = 0,$$

$R_H(f_{157}) = f_{157} + \mathrm{i}e^{\frac{\pi}{2}} f_{168} + e^{\frac{\pi}{2}} f_{267} - e^{\frac{\pi}{2}} f_{258} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} - f_{368} + \mathrm{i}f_{467} + \mathrm{i}f_{458}$. The $\Gamma_H$-cusps are $\overline{\kappa_1} = \overline{\kappa_2} = \overline{\kappa_3} = \overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_6}$ and $\overline{\kappa_7} = \overline{\kappa_8}$. By Lemma 6, $\frac{f_{157} + \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\Big|_{T_1} = 0$, so that $R_H(f_{157})|_{T_1} \neq \infty$. Therefore $R_H(f_{157}) \in \mathcal{L}^H_2 = \mathbb{C}$ and $\mathrm{rk}\Phi^H = 0$.

It suffices to observe that $-\operatorname{Id}$ changes the signs of the $\mathbb{C}$-basis

$$f_{56},\ f_{78},\ f_{157},\ f_{258},\ f_{368},\ f_{467} \tag{1}$$

of $\mathcal{L} = \mathcal{L}_{A_{-1}}\left(T_{-1}^{(6,8)}, \left(T_{-1}^{(6,8)}\right)^{\text{sing}}\right)$. Then for $H_o = \langle -\operatorname{Id}\rangle$ the space $\mathcal{L}^{H_o}$ is generated by $R_{H_o}(1) = 1$. Any subgroup $H_o \leq H \leq G_{-1}$ decomposes into cosets $H = \cup_{i=1}^{k} h_i H_o$ and $R_H = \sum_{i=1}^{k} h_i R_{H_o}$ vanishes on (1). Thus, $\mathcal{L}^H = \mathbb{C}$ and $\operatorname{rk}\Phi^H = 0$.

□

Note that $A_{-1}/\langle -\operatorname{Id}\rangle$ has 16 double points, whose minimal resolution is the Kummer surface $X_{-1}$ of $A_{-1}$. Thus, $H \ni -\operatorname{Id}$ exactly when the minimal resolution $Y$ of the singularities of $A_{-1}/H$ is covered by a smooth model of $X_{-1}$. More precisely, all $A_{-1}/H$ with $-\operatorname{Id} \in H$ have vanishing irregularity $0 \leq q(A_{-1}/H) \leq q(X_{-1}) = 0$. These are the Enriques $A_{-1}/\langle -\operatorname{Id}, \tau_{33}I^2\rangle$, all K3 quotients $A_{-1}/H$ with $\langle \tau_{33}^n\rangle \neq H \leq K = \ker\det\mathcal{L}$, except $A_{-1}/\langle \tau_{33}(-\operatorname{Id})\rangle$ and the rational $A_{-1}/H$ with $\tau_{33}IJ \in H$ for $0 \leq n \leq 1$ or $\langle -\operatorname{Id}, h_1\rangle \leq H$ for

$$h_1 \in \{I^{2m}J^{2-2m},\ \tau_{33}^m I,\ \tau_{33}^m J,\ \tau_{33}^m I^l J^{-l}\theta\ ;\ 0 \leq m \leq 1,\ 0 \leq l \leq 3\}.$$

**Lemma 9.** *The non-trivial subgroups $H \not\ni -\operatorname{Id}$ of $G_{-1}$ are*

*(i) cyclic of order 2:*

$$H_2(m,l) = \langle \tau_{33}I^{2m}J^{2l}\rangle \quad \textit{with} \quad 0 \leq m,l \leq 1;$$
$$H_2^\theta(n,k) = \langle \tau_{33}^n I^k J^{-k}\theta\rangle \quad \textit{with} \quad 0 \leq n \leq 1,\ 0 \leq k \leq 3;$$
$$H_2' = \langle I^2\rangle,\ H_2'' = \langle J^2\rangle;$$

*(ii) cyclic of order 4:*

$$H_4'(n,m) = \langle \tau_{33}^n I J^{2m}\rangle \quad \textit{with} \quad 0 \leq n,m \leq 1;$$
$$H_4''(n,m) = \langle \tau_{33}^n I^{2m} J\rangle \quad \textit{with} \quad 0 \leq n,m \leq 1;$$

*(iii) isomorphic to Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$:*

$$H_{2\times 2}'(m) = \langle \tau_{33}J^{2m}, I^2\rangle \quad \textit{with} \quad 0 \leq m \leq 1;$$
$$H_{2\times 2}''(m) = \langle \tau_{33}I^{2m}, J^2\rangle \quad \textit{with} \quad 0 \leq m \leq 1;$$
$$H_{2\times 2}^\theta(k) = \langle I^k J^{-k}\theta, \tau_{33}\rangle \quad \textit{with} \quad 0 \leq k \leq 1;$$
$$H_{2\times 2}^\theta(n,k) = \langle \tau_{33}^n I^k J^{-k}\theta, \tau_{33}I^2J^2\rangle \quad \textit{with} \quad 0 \leq n,k \leq 1;$$

*(iv) isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$:*

$$H_{4\times 2}'(m,l) = \langle IJ^{2m}, \tau_{33}J^{2l}\rangle \quad \textit{with} \quad 0 \leq m,l \leq 1;$$
$$H_{4\times 2}''(m,l) = \langle I^{2m}J, \tau_{33}I^{2l}\rangle \quad \textit{with} \quad 0 \leq m,l \leq 1.$$

**Proof:** If $H$ is a subgroup of $G_{-1}$, which does not contain $-\mathrm{Id}$, then $H \subseteq S = \{g \in G_{-1} \ ; \ -\mathrm{Id} \not\in \langle g \rangle\}$. Decompose $G_{-1} = G'_{-1} \cup G'_{-1}\theta$ into cosets modulo the abelian subgroup

$$G'_{-1} = \{\tau_{33}^n I^k J^l \ ; \ 0 \leq n \leq 1, 0 \leq k, l \leq 3\} \leq G_{-1}.$$

The cyclic group, generated by $(\tau_{33}^n I^k J^l \theta)^2 = (IJ)^{k+l}$ does not contain $-\mathrm{Id} = (IJ)^2$ if and only if $k + l \equiv 0 (\mathrm{mod}\ 4)$. If $S^{(r)} = \{g \in S \ ; \ g \text{ is of order } r\}$ then

$$S \cap G'_{-1}\theta = \{\tau_{33}^n I^k J^{-k}\theta \ ; \ 0 \leq n \leq 1, \ 0 \leq k \leq 3\} = S^{(2)} \cap G'_{-1}\theta =: S_1^{(2)}$$

and $S \cap G'_{-1}\theta \subseteq S^{(2)}$ consists of elements of order 2. Concerning $S \cap G'_{-1}$, observe that $(\tau_{33}^n I^k J^{k+2m})^2 = (IJ)^{2k} \in S$ for $0 \leq n, m \leq 1$, $0 \leq k \leq 3$ requires $k = 2p$ to be even. Consequently,

$$\{\tau_{33}^n I^k J^l \ ; \ k \equiv l (\mathrm{mod}\ 2)\} \cap S =$$
$$= \{\tau_{33} I^{2m} J^{2l}, \ I^2, \ J^2 \ ; \ 0 \leq m, l \leq 1\} = S^{(2)} \cap G'_{-1} =: S_0^{(2)},$$
$$\{\tau_{33}^n I^k J^l \ ; \ k \equiv l + 1 (\mathrm{mod}\ 2)\} \cap S =$$
$$= \{\tau_{33}^n I^{2m+1} J^{2l}, \tau_{33}^n I^{2m} J^{2l+1} \ ; \ 0 \leq n, m, l \leq 1\} = S^{(4)}.$$

In such a way, one obtains $S = \{\mathrm{Id}\} \cup S_0^{(2)} \cup S_1^{(2)} \cup S^{(4)}$ of cardinality $|S| = 31$. If a subgroup $H$ of $G_{-1}$ is contained in $S$, then $|H| \leq |S| = 31$ divides $|G_{-1}| = 64$, i.e., $|H| = 1, 2, 4, 8$ or 16. The only subgroup $H < G_{-1}$ of $|H| = 1$ is the trivial one $H = \{\mathrm{Id}\}$. The subgroups $-\mathrm{Id} \not\in H < G_{-1}$ of order 2 are the cyclic ones, generated by $h \in S_0^{(2)} \cup S_1^{(2)}$. We denote $H_2(m, l) = \langle \tau_{33} I^{2m} J^{2l} \rangle$ for $0 \leq m, l \leq 1$, $H_2^{\theta}(n, k) = \langle \tau_{33}^n I^k J^{-k} \theta \rangle$ for $0 \leq n \leq 1$, $0 \leq k \leq 3$ and $H'_2 = \langle I^2 \rangle$, $H''_2 = \langle J^2 \rangle$.

For any $h \in S^{(4)}$ one has $\langle h \rangle = \langle h^3 \rangle$, so that the subgroups $-\mathrm{Id} \not\in H \simeq \mathbb{Z}_4$ of $G_{-1}$ are depleted by $H'_4(n, m) = \langle \tau_{33}^n I J^{2m} \rangle$, $H''_4(n, m) = \langle \tau_{33}^n I^{2m} J \rangle$ with $0 \leq n, m \leq 1$.

The subgroups $-\mathrm{Id} \not\in H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ of $G_{-1}$ are generated by commuting $g_1, g_2 \in S^{(2)} = S_0^{(2)} \cup S_1^{(2)}$. If $g_1, g_2 \in S_1^{(2)}$ then $g_1 g_2 \in G'_{-1}$, so that one can always assume that $g_2 \in S_0^{(2)}$. Any $g_1 \neq g_2$ from $S_0^{(2)} \subset G'_{-1}$ generate a Klein group of order 4. Moreover, if

$$S_{0,1}^{(2)} = \{\tau_{33} I^{2m} J^{2l} \ | \ 0 \leq m, l \leq 1\}, \ \ S_{0,0}^{(2)} = \{I^2, J^2\},$$

then for any $g_1, g_2 \in S_{0,1}^{(2)}$ with $g_1 g_2 \in S$ there follows $g_1 g_2 \in S_{0,0}^{(2)}$. Thus, any $S_0^{(2)} \supset H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ has at least one generator $g_2 \in S_{0,0}^{(2)}$. The requirement $I^2 J^2 = -\mathrm{Id} \not\in H$ specifies that $g_1 \in S_{0,1}^{(2)}$. In the case of $g_2 = I^2$ there is no loss of generality to choose $g_1 = \tau_{33} J^{2m}$, in order to form $H'_{2\times 2}(m)$. Similarly, for $g_2 = J^2$ it suffices to take $g_1 = \tau_{33} I^{2m}$, while constructing $H''_{2\times 2}(m)$. In order to determine the subgroups $-\mathrm{Id} \not\in H = \langle g_1 \rangle \times \langle g_2 \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ with $g_1 \in S_1^{(2)}$, $g_2 \in S_0^{(2)}$, note that $g_1 = \tau_{33}^n I^k J^{-k} \theta$ does not commute with $I^2$, $J^2$ and commutes with $g_2 = \tau_{33} I^{2m} J^{2l}$ if and only if $2m \equiv 2l (\mathrm{mod}\ 4)$, i.e., $0 \leq m = l \leq 1$. Bearing in mind that $\langle \tau_{33}^n I^k J^{-k} \theta, \tau_{33} I^{2m} J^{2m} \rangle = \langle \tau_{33}^{n+1} I^{k+2m} J^{-k+2m} \theta, \tau_{33} I^{2m} J^{2m} \rangle$, one restricts the values of $k$ to $0 \leq k \leq 1$. For $m = 0$ denote $H^{\theta}_{2\times 2}(k) = \langle I^k J^{-k} \theta, \tau_{33} \rangle$. For $m = 1$ put $H^{\theta}_{2\times 2}(n,k) = \langle \tau_{33}^n I^k J^{-k} \theta, \tau_{33} I^2 J^2 \rangle$.

Let $-\mathrm{Id} \not\in H \subset S$ be a subgroup of order 8. The non-abelian such $H$ are isomorphic to quaternionic group $\mathbb{Q}_8 = \langle s, t \ ; \ s^4 = \mathrm{Id}, \ s^2 = t^2, \ sts = t \rangle$ or to dihedral group $\mathbb{D}_4 = \langle s, t \ ; \ s^4 = \mathrm{Id}, \ t^2 = \mathrm{Id}, \ sts = t \rangle$. Note that $s \in S^{(4)}$ and $sts = t$ require $st \neq ts$. As far as $S^{(4)} \cup S_0^{(2)} \subset G'_{-1}$ for the abelian group $G'_{-1} = \langle \tau_{33}, I, J \rangle$, it suffices to consider $t = \tau_{33}^n I^k J^{-k} \theta \in S_1^{(2)}$ and $s = \tau_{33}^m I^p J^{2l+1-p} \in S^{(4)}$ with $0 \leq n, m, l \leq 1$, $0 \leq p, k \leq 3$. However, $sts = \tau_{33}^n I^{k+2l+1} J^{k+2l+1} \theta \neq t$ reveals the non-existence of a non-abelian group $-\mathrm{Id} \not\in H \leq G_{-1}$ of order 8.

The abelian groups $H \subset S = \{\mathrm{Id}\} \cup S^{(2)} \cup S^{(4)}$ of order 8 are isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Any $\mathbb{Z}_4 \times \mathbb{Z}_2 \simeq H \subset S$ is generated by $s = \tau_{33}^m I^p J^{2l+1-p} \in S^{(4)}$ and $t \in S_0^{(2)}$, as far as $t' = \tau_{33}^n I^k J^{-k} \theta \in S_1^{(2)}$ has

$$st' = \tau_{33}^{m+n} I^{p+k} J^{2l+1-(p+k)} \theta \neq \tau_{33}^{m+n} I^{2l+1-(p-k)} J^{p-k} \theta = t's.$$

For $s = \tau_{33}^n I^{2m+1} J^{2l} \in S^{(4)}$ there holds $\langle s, t \rangle = \langle s^3, t \rangle$ and it suffices to consider $s = \tau_{33}^n I J^{2l}$. Further, $t \not\in \langle s^2 \rangle = \langle I^2 \rangle$ and $s^2 t \neq -\mathrm{Id}$ specify that $t = \tau_{33} I^{2p} J^{2q}$ for some $0 \leq p, q \leq 1$. Replacing eventually $t$ by $ts^2 = tI^2$, one attains $t = \tau_{33} J^{2q}$. On the other hand, the generator $s = \tau_{33} I J^{2l} \in S^{(4)}$ of $H = \langle s, t \rangle$ can be restored by $st = I J^{2(l+q)}$, so that $H = H'_{4\times 2}(l, q) = \langle I J^{2l}, \tau_{33} J^{2q} \rangle$ for some $0 \leq l, q \leq 1$. Exchanging $I$ with $J$, one obtains the remaining groups $H''_{4\times 2}(l, q) = \langle I^{2l} J, \tau_{33} I^{2q} \rangle \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$, contained in $S$.

If $-\mathrm{Id} \not\in H \subset S$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ then arbitrary different elements $s, t, r \in H$ of order 2 commute and generate $H$. For any $x \in S$ and $M \subseteq S$, consider the centralizer $C_M(x) = \{y \in M \ ; \ xy = yx\}$ of $x$ in $M$. Looking for $s \in S^{(2)}$, $t \in C_{S^{(2)}}(s)$ and $r \in C_{S^{(2)}}(s) \cap C_{S^{(2)}}(t)$, one computes that

$$C_{S^{(2)}}(\tau_{33}^n I^2) = C_{S^{(2)}}(\tau_{33}^n J^2) = S_0^{(2)},$$

$$C_{S^{(2)}}(\tau_{33} I^{2m} J^{2m}) = S^{(2)} = S_0^{(2)} \cup S_1^{(2)},$$

$$C_{S^{(2)}}(\tau_{33}^n I^{2m} J^{-2m} \theta) = \{\tau_{33}^p I^{2q} J^{-2q} \theta, \ \ \tau_{33} I^{2p} J^{2p} \ ; \ 0 \leq p, q \leq 1\},$$

$$C_{S^{(2)}}(\tau_{33}^n I^{2m+1} J^{-2m-1} \theta) = \{\tau_{33}^p I^{2q+1} J^{-2q-1} \theta, \ \ \tau_{33} I^{2p} J^{2p} \ ; \ 0 \leq p, q \leq 1\}.$$

Any subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \simeq H \subset \{\mathrm{Id}\} \cup S_0^{(2)} \cup S_1^{(2)}$ intersects $S_1^{(2)}$, due to $|S_0^{(2)}| = 6$. That allows to assume that $s \in S_1^{(2)}$ and observe that

$$C_{S^{(2)}}(s) = \{s, \ (-\mathrm{Id})s, \ \tau_{33}s, \ \tau_{33}(-\mathrm{Id})s, \ \tau_{33}, \ \tau_{33}(-\mathrm{Id})\}.$$

If $t = \tau_{33} I^{2p} J^{2p} \in C_{S^{(2)}}(s)$ then $C_{S^{(2)}}(t) = S^{(2)}$, so that

$$H \setminus \{\mathrm{Id}, s, t\} \subseteq [C_{S^{(2)}}(s) \cap C_{S^{(2)}}(t)] \setminus \{s, t\} = C_{S^{(2)}} \setminus \{s, t\} \tag{2}$$

with $5 = |H \setminus \{\mathrm{Id}, s, t\}| \leq |C_{S^{(2)}}(s) \setminus \{s, t\}| = 4$ is an absurd. For $t \in C_{S^{(2)}}(s) \setminus \{\tau_{33} I^{2p} J^{2p} \ ; \ 0 \leq p \leq 1\}$ one has $C_{S^{(2)}}(t) = C_{S^{(2)}}(s)$, which again leads to (2). Therefore, there is no subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \simeq H \not\ni -\mathrm{Id}$ of $G_{-1}$.

Concerning the non-existence of subgroups $-\mathrm{Id} \notin H \subset S$ of order 16, the abelian $-\mathrm{Id} \notin H \subset S$ of order 16 may be isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Any $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_4$ is generated by $s, t \in S^{(4)}$ with $s^2 \neq t^2$. Replacing, eventually, $s$ by $s^3$ and $t$ by $t^3$, one has $s = \tau_{33}^n I J^{2m}$, $t = \tau_{33}^p I^{2q} J$ with $0 \leq n, m, p, q \leq 1$. Then $s^2 t^2 = I^2 J^2 = -\mathrm{Id} \in H$ is an absurd. The groups $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are generated by $s \in S^{(4)}$ and $t, r \in C_{S^{(2)}}(s)$ with $r \in C_{S^{(2)}}(t)$. In the case of $s = \tau_{33}^n I J^{2m}$, the centralizer $C_{S^{(2)}}(s) = S_0^{(2)}$. Bearing in mind that $s^2 = I^2$, one observes that $\langle t, r \rangle \cap \{I^2, J^2\} = \emptyset$. Therefore $t, r \in \{\tau_{33} I^{2p} J^{2q} \ ; \ 0 \leq p, q \leq 1\}$, whereas $tr \in \{\mathrm{Id}, I^2, J^2, -\mathrm{Id}\}$. That reveals the non-existence of $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \simeq H \not\ni -\mathrm{Id}$. The groups $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ contain 15 elements of order 2, while $|S^{(2)}| = 14$. Therefore there is no abelian group $-\mathrm{Id} \notin H \leq G_{-1}$ of order 16.

There are three non-abelian groups of order 16, which do not contain a non-abelian subgroup of order 8 and consist of elements of order 1, 2 or 4. If

$$\langle s, t \ | \ s^4 = e, \ t^4 = e, \ st = ts^3 \rangle \simeq H \subset S$$

then $s, t \in S^{(4)} \subset G'_{-1} = \langle \tau_{33}, I, J \rangle$ commute and imply that $s$ is of order 2. The assumption

$$\langle a, b, c \ | \ a^4 = e, \ b^2 = e, \ c^2 = e, \ cbca^2b = e, \ ba = ab, \ ca = ac \rangle \simeq H \subset S$$

requires $b, c \in C_{S^{(2)}}(a) = S_0^{(2)} = \{\tau_{33} I^{2m} J^{2l}, \ I^2, \ J^2 \ ; \ 0 \leq m, l \leq 1\}$. Then $b$ and $c$ commute and imply that $cbca^2b = e = a^2 = e$. Finally, for

$$G_{4,4} = \langle s, t \ | \ s^4 = e, \ t^4 = e, \ stst = e, \ ts^3 = st^3 \rangle$$

there follows $s, t \in S^{(4)} \subset G'_{-1}$, whereas $st = ts$. Consequently, $s^2 = t^2$ and $G_{4,4} = \{s^i t^j \ ; \ 0 \leq i \leq 3, \ 0 \leq j \leq 1\}$ is of order $\leq 8$, contrary to $|G_{4,4}| = 16$. Thus, there is no subgroup $-\mathrm{Id} \notin H \leq G_{-1}$ of order 16.

□

Throughout, we use the notations $H_\alpha^\beta(\gamma)$ from lemma 9 and denote by $\Gamma_\alpha^\beta(\gamma)$ the corresponding lattices with $\Gamma_\alpha^\beta(\gamma)/\Gamma_{-1}^{(6,8)} = H_\alpha^\beta(\gamma)$.

**Theorem 5.** *For* $H = H'_{4\times 2}(p,q) = \langle IJ^{2p}, \tau_{33}J^{2q}\rangle$, $H''_{4\times 2}(p,q) = \langle I^{2p}J, \tau_{33}I^{2q}\rangle$, $H'_4(1-m,m) = \langle \tau_{33}^{1-m} IJ^{2m}\rangle$, $H''_4(1-m,m) = \langle \tau_{33}^{1-m} I^{2m}J\rangle$, $H'_{2\times 2}(1) = \langle \tau_{33}J^2, I^2\rangle$, $H''_{2\times 2}(1) = \langle \tau_{33}I^2, J^2\rangle$, $H^{\theta}_{2\times 2}(n,m) = \langle \tau_{33}^n I^m J^{-m}\theta, \tau_{33}I^2J^2\rangle$ *with* $0 \leq p,q \leq 1$, $(p,q) \neq (1,1)$ *and* $0 \leq n,m \leq 1$ *the logarithmic-canonical map*

$$\Phi^H : \widehat{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, 1]) = \mathbb{P}^1$$

*is dominant and not globally defined. The Baily-Borel compactifications* $\widehat{\mathbb{B}/\Gamma_H}$ *are birational to ruled surfaces with elliptic bases whenever* $H = H'_{4\times 2}(0,0)$, $H''_{4\times 2}(0,0)$, $H'_4(1,0)$ *or* $H''_4(1,0)$. *All the other* $\widehat{\mathbb{B}/\Gamma_H}$ *are rational surfaces.*

**Proof:** According to Lemma 4(v), it suffices to prove the theorem for $H'_{4\times 2}(p,q)$ with $(p,q) \neq (1,1)$, $H'_4(1-m,m)$ $H'_{2\times 2}(1)$ and $H^{\theta}_{2\times 2}(n,m)$.
If $H = H'_4(1,0) = \langle \tau_{33}I\rangle$, then $\mathcal{L}^H$ is generated by $1 \in \mathbb{C}$ and Reynolds operators

$$R_H(f_{56}) = 0, \;\; R_H(f_{78}) = 0, \;\; R_H(f_{157}) = f_{157} - \mathrm{e}^{\frac{\pi}{2}} f_{258} + \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{357} + \mathrm{i} f_{458},$$

$$R_H(f_{168}) = f_{168} - \mathrm{i} f_{267} + \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}} f_{368} + \mathrm{e}^{-\frac{\pi}{2}} f_{467} = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}} R_H(f_{368}).$$

There are four $\Gamma'_4(1,0)$-cusps : $\overline{\kappa_1} = \overline{\kappa_2} = \overline{\kappa_3} = \overline{\kappa_4}$, $\overline{\kappa_5}$, $\overline{\kappa_6}$, $\overline{\kappa_7} = \overline{\kappa_8}$. Applying Lemma 4(ii) to $T_1 \subset (R_H(f_{157}))_{\infty}, R_H(f_{168})_{\infty} \subseteq \sum_{i=1}^{8} T_i$, one concludes that $R_H(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{157})$. Therefore $\mathcal{L}^H \simeq \mathbb{C}^2$ and $\Phi^{H'_4(1,0)}$ is a dominant map to $\mathbb{P}(\mathcal{L}^H) \simeq \mathbb{P}^1$. Since $R_H(f_{157})|_{T_6} \neq \infty$, the entire $[\Gamma'_4(1,0), 1]$ vanishes at $\overline{\kappa_6}$ and $\Phi^{H'_4(1,0)}$ is not defined at $\overline{\kappa_6}$.
The group $H = H'_{4\times 2}(0,0) = \langle I, \tau_{33}\rangle$ contains $F = H'_4(1,0)$ as a subgroup of index 2 with non-trivial coset representative $I$. Therefore $R_H(f_{56}) = R_F(f_{56}) + IR_F(f_{56}) = 0$, $R_H(f_{78}) = 0$ and $\mathrm{rk}\Phi^{H'_{4\times 2}(0,0)} \leq 1$. Due to

$$R_H(f_{157}) = f_{157} - \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{168} - \mathrm{e}^{\frac{\pi}{2}} f_{258} - \mathrm{e}^{\frac{\pi}{2}} f_{267} + f_{368} + \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{357} + \mathrm{i} f_{458} - \mathrm{i} f_{467},$$

$\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{157}))$. Lemma 6 provides $\left.\frac{f_{157} - \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\right|_{T_1} = -2\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}} \neq 0$, whereas $R_H(f_{157})|_{T_1} = \infty$. Therefore $\dim_{\mathbb{C}} \mathcal{L}^H = 2$ and $\Phi^{H'_{4\times 2}(0,0)}$ is a dominant map to $\mathbb{P}^1$. The $\Gamma_{4\times 2}(0,0)$-cusps are $\overline{\kappa_1} = \overline{\kappa_2} = \overline{\kappa_3} = \overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_6}$ and $\overline{\kappa_7} = \overline{\kappa_8}$. Again from Lemma 6, $\left.\frac{f_{157} - \mathrm{e}^{\frac{\pi}{2}} f_{258} + \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{357} + \mathrm{i} f_{458}}{\Sigma_5}\right|_{T_5} = 0$, so that $R_H(f_{157})$ is regular over $T_5 + T_6$. As a result, $\Phi^{H'_{4\times 2}(0,0)}$ is not defined at $\overline{\kappa_5} = \overline{\kappa_6}$.
For $H = H'_4(0,1) = \langle IJ^2\rangle$, the space $\mathcal{L}^H$ is spanned by 1 and Reynolds operators

$$R_H(f_{56}) = 0, \;\; R_H(f_{78}) = 0, \;\; R_H(f_{157}) = f_{157} + \mathrm{e}^{\frac{\pi}{2}} f_{267} + \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{357} + \mathrm{i} f_{467},$$

$$R_H(f_{168}) = f_{168} + \mathrm{i} f_{258} + \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}} f_{368} + \mathrm{e}^{-\frac{\pi}{2}} f_{458} = \mathrm{i} R_H(f_{258}).$$

The $\Gamma'_4(0,1)$-cusps are $\overline{\kappa_1} = \overline{\kappa_2} = \overline{\kappa_3} = \overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_6}$, $\overline{\kappa_7}$ and $\overline{\kappa_8}$. Note that $T_1 \subset (R_H(f_{157}))_\infty, (R_H(f_{168}))_\infty \subseteq \sum_{i=1}^{8} T_i$, in order to conclude that $R_H(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{157}))$ by Lemma 4 (ii). Therefore $\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{157})) \simeq \mathbb{C}^2$ and $\Phi^{H'_4(0,1)}$ is a dominant map to $\mathbb{P}^1$. Lemma 6 supplies $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}}f_{357}}{\Sigma_5}\Big|_{T_5} = 0$ and justifies that $\Phi^{H'_4(0,1)}$ is not defined at $\overline{\kappa_5}$.

For $H = H'_{4\times 2}(1,0) = \langle IJ^2, \tau_{33}\rangle$ note that $R_H(f_{56}) = 0$, $R_H(f_{78}) = 0$, as far as $H'_4(1,0)$ is a subgroup of $H'_{4\times 2}(1,0)$. Further,

$$R_H(f_{157}) = f_{157} - \mathrm{i}e^{\frac{\pi}{2}}f_{168} + e^{\frac{\pi}{2}}f_{267} + e^{\frac{\pi}{2}}f_{258} + \mathrm{i}e^{\frac{\pi}{2}}f_{357} + f_{368} + \mathrm{i}f_{467} - \mathrm{i}f_{458}$$

has a pole over $\sum_{i=1}^{4} T_i$, according to $\frac{f_{157}-\mathrm{i}e^{\frac{\pi}{2}}f_{168}}{\Sigma_1}\Big|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$ by Lemma 6 and the transitiveness of the $H'_4(1,0)$-action on $\{\kappa_i \ ; \ 1 \leq i \leq 4\}$. Therefore $\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{157})) \simeq \mathbb{C}^2$ and $\Phi^{H'_{4\times 2}(1,0)}$ is a dominant map to $\mathbb{P}^1$. One computes immediately that the $\Gamma'_{4\times 2}(1,0)$-cusps are $\overline{\kappa_1} = \overline{\kappa_2} = \overline{\kappa_3} = \overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_6}$ and $\overline{\kappa_7} = \overline{\kappa_8}$. Again from Lemma 6, $\frac{f_{157}+e^{\frac{\pi}{2}}f_{258}+\mathrm{i}e^{\frac{\pi}{2}}f_{357}-\mathrm{i}f_{458}}{\Sigma_5}\Big|_{T_5} = 0$, $R_H(f_{157})$ has no pole at $T_5 + T_6$ and $\Phi^{H'_{4\times 2}(1,0)}$ is not defined at $\overline{\kappa_5} = \overline{\kappa_6}$.

If $H = H'_{2\times 2}(1) = \langle I^2, \tau_{33}J^2\rangle$ then

$$R_H(f_{56}) = 0, \ \ R_H(f_{78}) = 4f_{78}, \ \ R_H(f_{157}) = f_{157} + \mathrm{i}e^{\frac{\pi}{2}}f_{168} + \mathrm{i}e^{\frac{\pi}{2}}f_{357} - f_{368},$$

$$R_H(f_{258}) = f_{258} - f_{267} - \mathrm{i}e^{-\frac{\pi}{2}}f_{467} - \mathrm{i}e^{-\frac{\pi}{2}}f_{458} \quad \text{and} \quad 1 \in \mathbb{C}$$

span $\mathcal{L}^H$. The $\Gamma'_{2\times 2}(1)$-cusps are $\overline{\kappa_1} = \overline{\kappa_3}$, $\overline{\kappa_2} = \overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_6}$ and $\overline{\kappa_7} = \overline{\kappa_8}$. Lemma 6 reveals that $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}}f_{168}}{\Sigma_1}\Big|_{T_1} = \frac{\mathrm{i}e^{\frac{\pi}{2}}f_{357}-f_{368}}{\Sigma_3}\Big|_{T_3} = \frac{f_{258}-f_{267}}{\Sigma_2}\Big|_{T_2} = \frac{f_{467}+f_{458}}{\Sigma_4}\Big|_{T_4} = 0$, so that $R_H(f_{157})$, $R_H(f_{258}) \in \mathrm{Span}_{\mathbb{C}}(1, f_{78})$ and $\mathcal{L}^H \simeq \mathbb{C}^2$. As a result, $\Phi^{H'_{2\times 2}(1)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\overline{\kappa_1}$ and $\overline{\kappa_2}$.

For the group $H = H'_{4\times 2}(0,1) = \langle I, \tau_{33}J^2\rangle$, containing $H'_{2\times 2}(1) = \langle I^2, \tau_{33}J^2\rangle$ there follows $R_H(f_{56}) = 0$ and $\mathrm{rk}\Phi^{H'_{4\times 2}(0,1)} \leq 1$. Therefore $R_H(f_{78}) = 8f_{78}$,

$$R_H(f_{157}) = f_{157} + \mathrm{i}e^{\frac{\pi}{2}}f_{168} + e^{\frac{\pi}{2}}f_{258} - e^{\frac{\pi}{2}}f_{267} + \mathrm{i}e^{\frac{\pi}{2}}f_{357} - f_{368} - \mathrm{i}f_{458} - \mathrm{i}f_{467}$$

and $1 \in \mathbb{C}$ span $\mathcal{L}^H$. The $\Gamma'_{4\times 2}(0,1)$-cusps are $\overline{\kappa_1} = \overline{\kappa_2} = \overline{\kappa_3} = \overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_6}$ and $\overline{\kappa_7} = \overline{\kappa_8}$. By Lemma 6, $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}}f_{168}}{\Sigma_1}\Big|_{T_1} = 0$, so that $R_H(f_{157}) \in \mathrm{Span}_{\mathbb{C}}(1, f_{78}) \simeq \mathbb{C}^2$. Thus, $\Phi^{H'_{4\times 2}(0,1)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\overline{\kappa_1}$.

If $H = H^{\theta}_{2\times 2}(0,0) = \langle \theta, \tau_{33}I^2J^2\rangle$ then $\mathcal{L}^H$ is spanned by $1 \in \mathbb{C}$,

$$R_H(f_{56}) = 2(f_{56} + f_{78}), \ \ R_H(f_{157}) = f_{157} + \mathrm{i}e^{\frac{\pi}{2}}f_{168} - e^{\frac{\pi}{2}}f_{357} - \mathrm{i}f_{368}$$

and $R_H(f_{467}) = 2(f_{467} + f_{458})$, due to $R_H(f_{258}) = 0$. The $\Gamma_2^\theta(0,0)$-cusps are $\overline{\kappa_1} = \overline{\kappa_3}$, $\overline{\kappa_2}$, $\overline{\kappa_4}$ and $\overline{\kappa_5} = \overline{\kappa_6} = \overline{\kappa_7} = \overline{\kappa_8}$. Lemma 6 provides $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}}f_{168}}{\Sigma_1}\Big|_{T_1} = 0$, $\frac{f_{467}+f_{458}}{\Sigma_4}\Big|_{T_4} = 0$, whereas $R_H(f_{157}), R_H(f_{467}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56})) \simeq \mathbb{C}^2$. Therefore $\Phi^{H_2^\theta(0,0)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\overline{\kappa_1}$, $\overline{\kappa_2}$ and $\overline{\kappa_4}$.
For $H = H_{2\times 2}^\theta(0,1) = \langle IJ^{-1}\theta, \tau_{33}I^2J^2\rangle$ one has

$$R_H(f_{56}) = 2(f_{56} + \mathrm{i}f_{78}), \quad R_H(f_{157}) = 0, \quad R_H(f_{168}) = 0,$$

$$R_H(f_{368}) = 2(f_{368} - \mathrm{i}e^{\frac{\pi}{2}}f_{357}), \quad R_H(f_{258}) = f_{258} - f_{267} - e^{-\frac{\pi}{2}}f_{458} - e^{-\frac{\pi}{2}}f_{467}.$$

The $\Gamma_{2\times 2}^\theta(0,1)$-cusps are $\overline{\kappa_1}$, $\overline{\kappa_3}$, $\overline{\kappa_2} = \overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_6} = \overline{\kappa_7} = \overline{\kappa_8}$. Lemma 6 implies that $\frac{f_{368}-\mathrm{i}e^{\frac{\pi}{2}}f_{357}}{\Sigma_3}\Big|_{T_3} = 0$, $\frac{f_{258}-f_{267}}{\Sigma_2}\Big|_{T_2} = 0$, $\frac{f_{458}+f_{467}}{\Sigma_4}\Big|_{T_4} = 0$, whereas $R_H(f_{368}), R_H(f_{258}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56})) \simeq \mathbb{C}$. Consequently, $\Phi^{H_{2\times 2}^\theta(0,1)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\overline{\kappa_1}$, $\overline{\kappa_2}$ and $\overline{\kappa_4}$.
In the case of $H = H_{2\times 2}^\theta(1,0) = \langle \tau_{33}\theta, \tau_{33}I^2J^2\rangle$, Reynolds operators

$$R_H(f_{56}) = 2(f_{56} - f_{78}), \quad R_H(f_{157}) = f_{157} + \mathrm{i}e^{\frac{\pi}{2}}f_{168} + \mathrm{i}f_{368} + e^{\frac{\pi}{2}}f_{357},$$

$$R_H(f_{258}) = 2(f_{258} - f_{267}), \quad R_H(f_{458}) = 0, \quad R_H(f_{467}) = 0.$$

The $\Gamma_{2\times 2}^\theta(1,0)$-cusps are $\overline{\kappa_1}$, $\overline{\kappa_3}$, $\overline{\kappa_2} = \overline{\kappa_4}$ and $\overline{\kappa_5} = \overline{\kappa_6} = \overline{\kappa_7} = \overline{\kappa_8}$. Lemma 6 yields $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}}f_{168}}{\Sigma_1}\Big|_{T_1} = \frac{\mathrm{i}f_{368}+e^{\frac{\pi}{2}}f_{357}}{\Sigma_3}|_{T_3} = \frac{f_{258}-f_{267}}{\Sigma_2}\Big|_{T_2} = 0$. Consequently, $R_H(f_{157}), R_H(f_{258}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}))$. Bearing in mind that $R_H(f_{56})|_{T_5} = \infty$, one concludes that $\Phi^{H_{2\times 2}^\theta(1,0)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\overline{\kappa_1}$, $\overline{\kappa_2}$ and $\overline{\kappa_3}$
Finally, for $H = H_{2\times 2}^\theta(1,1) = \langle \tau_{33}IJ^{-1}\theta, \tau_{33}I^2J^2\rangle$ one has

$$R_H(f_{56}) = 2(f_{56} - \mathrm{i}f_{78}), \quad R_H(f_{157}) = 2(f_{157} + \mathrm{i}e^{\frac{\pi}{2}}f_{168}), \quad R_H(f_{357}) = 0,$$

$$R_H(f_{368}) = 0 \quad \text{and} \quad R_H(f_{258}) = f_{258} - f_{267} + e^{-\frac{\pi}{2}}f_{467} + e^{-\frac{\pi}{2}}f_{458}.$$

The $\Gamma_{2\times 2}^\theta(1,1)$-cusps are $\overline{\kappa_1}$, $\overline{\kappa_3}$, $\overline{\kappa_2} = \overline{\kappa_4}$ and $\overline{\kappa_5} = \overline{\kappa_6} = \overline{\kappa_7} = \overline{\kappa_8}$. Lemma 6 implies that $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}}f_{168}}{\Sigma_1}\Big|_{T_1} = \frac{f_{258}-f_{267}}{\Sigma_2}\Big|_{T_2} = 0$, so that $R_H(f_{157}), R_H(f_{258}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56})) \simeq \mathbb{C}^2$. As a result, $\Phi^{H_{2\times 2}^\theta(1,1)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\overline{\kappa_1}$, $\overline{\kappa_3}$ and $\overline{\kappa_2}$.

□

**Theorem 6.** *If* $H = H'_{2\times 2}(0) = \langle \tau_{33}, I^2\rangle$, $H''_{2\times 2}(0) = \langle \tau_{33}, J^2\rangle$, $H_{2\times 2}^\theta(n) = \langle I^nJ^{-n}\theta, \tau_{33}\rangle$ *with* $0 \leq n \leq 1$, $H'_4(n,n) = \langle \tau_{33}^n IJ^{2n}\rangle$, $H''_4(n,n) = \langle \tau_{33}^n I^{2n}J\rangle$ *with* $0 \leq n \leq 1$ *or* $H_2(1,1) = \langle \tau_{33}I^2J^2\rangle$ *then the logarithmic-canonical map*

$$\Phi^H : \widehat{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, 1]) = \mathbb{P}^2$$

*is dominant and not globally defined. The surface $\widehat{\mathbb{B}/\Gamma_H}$ is K3 for $H = H_2(1,1)$, rational for $H = H_4'(1,1)$, $H_4''(1,1)$ and ruled with an elliptic base for all the other aforementioned $H$.*

**Proof:** By Lemma 4 (v), it suffices to consider $H'_{2\times 2}(0)$, $H^{\theta}_{2\times 2}(n)$, $H'_4(n,n)$ and $H_2(1,1)$.
In the case of $H = H'_{2\times 2}(0) = \langle \tau_{33}, I^2 \rangle$, $\mathcal{L}^H$ is spanned by

$$R_H(f_{56}) = 0,\ \ R_H(f_{78}) = 0,\ \ R_H(f_{157}) = f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} + f_{368},$$

$$R_H(f_{258}) = f_{258} + f_{267} - \mathrm{i}e^{-\frac{\pi}{2}} f_{458} + \mathrm{i}e^{-\frac{\pi}{2}} f_{467} \quad \text{and} \quad 1 \in \mathbb{C}.$$

The $\Gamma'_{2\times 2}(0)$-cusps are $\overline{\kappa_1} = \overline{\kappa_3}$, $\overline{\kappa_2} = \overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_6}$ and $\overline{\kappa_7} = \overline{\kappa_8}$. Lemma 6 provides $\left.\frac{f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\right|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$, whereas $R_H(f_{157})|_{T_1} = \infty$. Similarly, $\left.\frac{f_{258}+f_{267}}{\Sigma_2}\right|_{T_2} = 2e^{-\pi} \neq 0$ suffices for $R_H(f_{258})|_{T_2} = \infty$. Therefore 1, $R_H(f_{157})$, $R_H(f_{258})$ are linearly independent, according to Lemma 4(iii) and constitute a $\mathbb{C}$-basis for $\mathcal{L}^H$. In order to assert that $\mathrm{rk}\Phi^{H'_{2\times 2}(0)} = 2$, we use that $R_H(f_{258})|_{T_2} = \infty$ and $R_H(f_{157})|_{T_2} \not\equiv \mathrm{const}$ by Lemma 7 with $c = \mathrm{i}e^{\frac{\pi}{2}}$. Lemma 6 provides $\left.\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}} f_{357}}{\Sigma_5}\right|_{T_5} = 0$, in order to conclude that $R_H(f_{157})|_{T_5} \neq \infty$ and the entire $[\Gamma'_{2\times 2}(0), 1]$ vanishes at $\overline{\kappa_5}$. Therefore $\Phi^{H'_{2\times 2}(0)}$ is a dominant map to $\mathbb{P}([\Gamma'_{2\times 2}(0), 1]) = \mathbb{P}^2$, which is not defined at $\overline{\kappa_5}$.
For $H = H^{\theta}_{2\times 2}(0) = \langle \theta, \tau_{33} \rangle$, Reynolds operators

$$R_H(f_{56}) = 0,\ \ R_H(f_{78}) = 0,\ \ R_H(f_{157}) = f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168} - e^{\frac{\pi}{2}} f_{357} + \mathrm{i}f_{368},$$

$$R_H(f_{258}) = 2(f_{258} + f_{267}),\ \ R_H(f_{467}) = 0.$$

generate $\mathcal{L}^H$. The $\Gamma^{\theta}_{2\times 2}(0)$-cusps are $\overline{\kappa_1} = \overline{\kappa_3}$, $\overline{\kappa_2}$, $\overline{\kappa_4}$ and $\overline{\kappa_5} = \overline{\kappa_6} = \overline{\kappa_7} = \overline{\kappa_8}$. According to Lemma 6, $\left.\frac{f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\right|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$, so that $R_H(f_{157})|_{T_1} = \infty$. Further, $\left.\frac{f_{258}+f_{267}}{\Sigma_2}\right|_{T_2} = 2e^{-\pi} \neq 0$ by the same lemma provides $R_H(f_{258})|_{T_2} = \infty$. Therefore 1, $R_H(f_{157})$, $R_H(f_{258})$ are linearly independent and $\mathcal{L}^H \simeq \mathbb{C}^3$ by Lemma 4(iii). We claim that

$$R_H(f_{258})|_{T_1} = -2e^{-\pi \mathrm{i} v} \frac{\sigma((1+i)v + \omega_3)}{\sigma((1+i)v)} \left[ \frac{\sigma(v-\omega_1)^2}{\sigma(v-\omega_2)^2} + e^{2\pi(1+\mathrm{i})v} \frac{\sigma(v-\omega_2)^2}{\sigma(v-\omega_1)^2} \right]$$

is non-constant. On one hand, $R_H(f_{258})|_{T_1}$ has no poles on $\mathbb{C} \setminus \mathbb{Q}(\mathrm{i})$. On the other hand, $\left. \left[ \left.\frac{1}{2} R_H(f_{258})\right|_{T_1} \right] \sigma((1+\mathrm{i})v) \right|_{v=0} = -\sigma(\omega_3) \left[ \frac{1}{\mathrm{i}^2} + \mathrm{i}^2 \right] \neq 0$, so that $\lim_{v\to 0} [R_H(f_{258})|_{T_1}] = \infty$. According to Lemma 4(iv), $R_H(f_{157})|_{T_1} = \infty$ and

$R_H(f_{258})|_{T_1} \not\equiv \mathrm{const}$ suffice for $\Phi^{H^{\theta}_{2\times 2}(0)}$ to be a dominant map to $\mathbb{P}^2$. The entire $\mathcal{L}^H$ takes finite values on $T_4$, so that $\Phi^{H^{\theta}_{2\times 2}(0)}$ is not defined at $\overline{\kappa_4}$.

Concerning $H = H^{\theta}_{2\times 2}(1) = \langle IJ^{-1}\theta, \tau_{33}\rangle$, one computes that

$$R_H(f_{56}) = 0,\ \ R_H(f_{78}) = 0,\ \ R_H(f_{157}) = 2(f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}),$$

$$R_H(f_{368}) = 0,\ \ R_H(f_{258}) = f_{258} + f_{267} - e^{-\frac{\pi}{2}} f_{458} + e^{-\frac{\pi}{2}} f_{467}.$$

The $\Gamma^{\theta}_{2\times 2}(1)$-cusps are $\overline{\kappa_1}$, $\overline{\kappa_3}$, $\overline{\kappa_2} = \overline{\kappa_4}$ and $\overline{\kappa_5} = \overline{\kappa_6} = \overline{\kappa_7} = \overline{\kappa_8}$. By Lemma 6 we have $\frac{f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\Big|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$ and $\frac{f_{258}+f_{267}}{\Sigma_2}\Big|_{T_2} = 2e^{-\pi} \neq 0$. Therefore $R_H(f_{157})|_{T_1} = \infty$, $R_H(f_{258})|_{T_2} = \infty$ and $1$, $R_H(f_{157})$, $R_H(f_{258})$ constitute a $\mathbb{C}$-basis of $\mathcal{L}^H$, according to Lemma 4(iii). Applying Lemma 7 with $c = 0$, one concludes that $R_H(f_{157})|_{T_2} \not\equiv \mathrm{const}$. Then Lemma 4(iv) implies that $\Phi^{H^{\theta}_{2\times 2}(1)}$ is a dominant map to $\mathbb{P}^2$. The lack of $f \in \mathcal{L}^H$ with $f|_{T_3} = \infty$ reveals that $\Phi^{H^{\theta}_{2\times 2}(1)}$ is not defined at $\overline{\kappa_3}$.

If $H = H'_4(0,0) = \langle I\rangle$ then Reynolds operators

$$R_H(f_{56}) = 0,\ \ R_H(f_{78}) = 4f_{78},\ \ R_H(f_{157}) = f_{157} - e^{\frac{\pi}{2}} f_{267} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} - \mathrm{i}f_{467},$$

$$R_H(f_{168}) = f_{168} - \mathrm{i}f_{258} + \mathrm{i}e^{-\frac{\pi}{2}} f_{368} - e^{-\frac{\pi}{2}} f_{458} \quad \text{and} \quad R_H(1) = 1 \in \mathbb{C}$$

span $\mathcal{L}^H$. The $\Gamma'_4(0,0)$-cusps are $\overline{\kappa_1} = \overline{\kappa_2} = \overline{\kappa_3} = \overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_6}$, $\overline{\kappa_7}$ and $\overline{\kappa_8}$. According to Lemma 4(ii), the inclusions $T_1 \subset (R_H(f_{157}))_{\infty}, (R_H(f_{168}))_{\infty} \subseteq \sum_{i=1}^{8} T_i$ suffice for $R_H(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{78}), R_H(f_{157})$. Therefore $\mathcal{L}^H \simeq \mathbb{C}^3$. Observe that $R_H(f_{78})|_{T_1} = 4\Sigma_{12}(v) \not\equiv \mathrm{const}$, in order to apply Lemma 4(iv) and assert that $\Phi^{H'_4(0,0)}$ is a dominant map to $\mathbb{P}^2$. As far as $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}} f_{357}}{\Sigma_5}\Big|_{T_5} = 0$ by Lemma 6, the abelian function $R_H(f_{157})$ has no pole on $T_5$. Therefore $\Phi^{H'_4(0,0)}$ is not defined at $\overline{\kappa_5}$.

For $H'_4(1,1) = \langle \tau_{33} IJ^2\rangle$ Reynolds operators

$$R_h(f_{56}) = 0,\ \ R_H(f_{78}) = 4f_{78},\ \ R_H(f_{157}) = f_{157} + e^{\frac{\pi}{2}} f_{258} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} - \mathrm{i}f_{458},$$

$$R_H(f_{168}) = f_{168} + \mathrm{i}f_{267} + \mathrm{i}e^{-\frac{\pi}{2}} f_{368} - e^{-\frac{\pi}{2}} f_{467}.$$

The $\Gamma'_4(1,1)$-cusps are $\overline{\kappa_1} = \overline{\kappa_2} = \overline{\kappa_3} = \overline{\kappa_4}$, $\overline{\kappa_5}$, $\overline{\kappa_6}$ and $\overline{\kappa_7} = \overline{\kappa_8}$. Due to $T_1 \subset (R_H(f_{157}))_{\infty}, (R_H(f_{168}))_{\infty} \subseteq \sum_{i=1}^{8} T_i$, Lemma 4(ii) applies to provide $R_H(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{78}), R_H(f_{157}))$. Thus, $\mathcal{L}^H \simeq \mathbb{C}^3$. According to Lemma 4(iv), $R_H(f_{78})|_{T_1} = 4\Sigma_{12}(v) \not\equiv \mathrm{const}$ suffices for $\Phi^{H'_4(1,1)}$ to be a dominant rational map to $\mathbb{P}^2$. Further, $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}} f_{357}}{\Sigma_5}\Big|_{T_5} = 0$ by Lemma 6 implies that $R_H(f_{157})$ has no pole over $T_5$ and $\Phi^{H'_4(1,1)}$ is not defined at $\overline{\kappa_5}$.

If $H = H_2(1,1) = \langle \tau_{33} I^2 J^2 \rangle$ then $\mathcal{L}^H$ is generated by

$$1 \in \mathbb{C}, \ \ R_H(f_{56}) = 2f_{56}, \ \ R_H(f_{78}) = 2f_{78}, \ \ R_H(f_{157}) = f_{157} + \mathrm{i}e^{\frac{\pi}{2}} f_{168},$$

$R_H(f_{368}) = f_{368} - \mathrm{i}e^{\frac{\pi}{2}} f_{357}, \ \ R_H(f_{258}) = f_{258} - f_{267}, \ \ R_H(f_{467}) = f_{467} + f_{458}$. The $\Gamma_2(1,1)$-cusps are $\overline{\kappa_1}$, $\overline{\kappa_2}$, $\overline{\kappa_3}$, $\overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_6}$ and $\overline{\kappa_7} = \overline{\kappa_8}$. By Lemma 6 one has $\frac{f_{157} + \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\Big|_{T_1} = \frac{f_{368} - \mathrm{i}e^{\frac{\pi}{2}} f_{357}}{\Sigma_3}\Big|_{T_3} = \frac{f_{258} - f_{267}}{\Sigma_2}\Big|_{T_2} = \frac{f_{467} + f_{458}}{\Sigma_4}\Big|_{T_4} = 0$. Thus, $R_H(f_{157}), R_H(f_{368}), R_H(f_{258}), R_H(f_{467}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{78}))$ and $\mathcal{L}^H \simeq \mathbb{C}^3$. Bearing in mind that $R_H(f_{56})|_{T_5} = \infty$, $R_H(f_{78})|_{T_5} \not\equiv \mathrm{const}$, one applies Lemma 4(iv) and concludes that $\Phi^{H_2(1,1)}$ is a dominant map to $\mathbb{P}^2$. Since $\mathcal{L}^H$ has no pole over $\sum_{i=1}^{4} T_i$, the map $\Phi^{H_2(1,1)}$ is not defined at $\overline{\kappa_1}, \overline{\kappa_2}, \overline{\kappa_3}, \overline{\kappa_4}$.

□

Let us recall from Hacon and Pardini's [1] that the geometric genus $p_g(X) = \dim_{\mathbb{C}} H^0(X, \Omega^2_X)$ of a smooth minimal surface $X$ of general type is at most 4. The next theorem provides a smooth toroidal compactification $Y = \left(\mathbb{B}/\Gamma_{\langle \tau_{33} \rangle}\right)'$ with abelian minimal model $A_{-1}/\langle \tau_{33} \rangle$ and $\dim_{\mathbb{C}} H^0(Y, \Omega^2_Y(T')) = 5$.

**Theorem 7.** *(i) For $H = H_2' = \langle I^2 \rangle$, $H_2'' = \langle J^2 \rangle$, $H_2(n, 1-n) = \langle \tau_{33} I^{2n} J^{2-2n} \rangle$ or $H_2^{\theta}(n,k) = \langle \tau_{33}^n I^k J^{-k} \theta \rangle$ with $0 \leq n \leq 1$, $0 \leq k \leq 3$ the logarithmic-canonical map*

$$\Phi^H : \widehat{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, 1]) = \mathbb{P}^3$$

*has maximal $\mathrm{rk}\Phi^H = 2$. For $H \neq H_2(n, 1-n)$ the rational map $\Phi^H$ is not globally defined and $\widehat{\mathbb{B}/\Gamma_H}$ are ruled surfaces with elliptic bases. In the case of $H = H_2(n, 1-n)$ the surface $\widehat{\mathbb{B}/\Gamma_H}$ is hyperelliptic.*
*(ii) For $H = H_2(0,0) = \langle \tau_{33} \rangle$ the smooth surface $\left(\mathbb{B}/\Gamma_{\langle \tau_{33} \rangle}\right)'$ has abelian minimal model $A_{-1}/\langle \tau_{33} \rangle$ and the logarithmic-canonical map*

$$\Phi^{\langle \tau_{33} \rangle} : \widehat{\mathbb{B}/\Gamma_{\langle \tau_{33} \rangle}} \dashrightarrow \mathbb{P}([\Gamma_{\langle \tau_{33} \rangle}, 1]) = \mathbb{P}^4$$

*is of maximal $\mathrm{rk}\Phi^{\langle \tau_{33} \rangle} = 2$.*

**Proof:** (i) By Lemma 4(v), it suffices to prove the statement for $H_2'$, $H_2(1,0)$ and $H_2^{\theta}(n,k) = \langle \tau_{33}^n I^k J^{-k} \theta \rangle$ with $0 \leq n \leq 1$, $0 \leq k \leq 2$.
Note that $H_2'$, $H_2(1,0)$ are subgroups of $H'_{2\times 2}(0) = \langle \tau_{33}, I^2 \rangle$ and $\mathrm{rk}\Phi^{H'_{2\times 2}(0)} = 2$. By Lemma 4(iv) that suffices for $\mathrm{rk}\Phi^{H_2'} = \mathrm{rk}\Phi^{H_2(1,0)} = 2$.
In the case of $H = H_2' = \langle I^2 \rangle$, Reynolds operators

$$R_H(f_{56}) = 0, \ \ R_H(f_{78}) = 2f_{78},$$

$$R_H(f_{157}) = f_{157} + \mathrm{i}e\frac{\pi}{2} f_{357}, \ \ R_H(f_{168}) = f_{168} + \mathrm{i}e^{-\frac{\pi}{2}} f_{368},$$

$$R_H(f_{258}) = f_{258} - \mathrm{ie}^{-\frac{\pi}{2}} f_{458}, \ \ R_H(f_{267}) = f_{267} + \mathrm{ie}^{-\frac{\pi}{2}} f_{467}.$$

The $\Gamma_2'$-cusps are $\overline{\kappa_1} = \overline{\kappa_3}$, $\overline{\kappa_2} = \overline{\kappa_4}$, $\overline{\kappa_5}$, $\overline{\kappa_6}$, $\overline{\kappa_7}$ and $\overline{\kappa_8}$. According to Lemma 4(ii), the inclusions $T_1 \subset (R_H(f_{157}))_\infty, (R_H(f_{168}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$ suffice for $R_H(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{78}), R_H(f_{157}))$. Similarly, from $T_2 \subset (R_H(f_{258}))_\infty, (R_H(f_{267}))_\infty \subseteq T_2 + T_4 + \sum_{\alpha=5}^{8} T_\alpha$ there follows $R_H(f_{267}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{78}), R_H(f_{258}))$. As a result, one concludes that the space of the invariants $\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{78}), R_H(f_{157}), R_H(f_{258})) \simeq \mathbb{C}^4$. Since $\mathcal{L}^H$ has no pole over $T_6$, the rational map $\Phi^{H_2'}$ is not defined at $\overline{\kappa_6}$.

If $H = H_2(1,0) = \langle \tau_{33} I^2 \rangle$, then $\mathcal{L}^H$ is spanned by

$$1 \in \mathbb{C}, \ \ R_H(f_{56}) = 2f_{56}, \ \ R_H(f_{78}) = 0,$$

$$R_H(f_{157}) = f_{157} + f_{368}, \ \ R_H(f_{258}) = f_{258} + \mathrm{ie}^{-\frac{\pi}{2}} f_{467}.$$

The $\Gamma_2(1,0)$-cusps are $\overline{\kappa_1} = \overline{\kappa_3}$, $\overline{\kappa_2} = \overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_6}$, $\overline{\kappa_7} = \overline{\kappa_8}$. According to Lemma 4(iii), the inclusions $T_1 + T_3 \subset (R_H(f_{157}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$ and $T_2 + T_4 \subset (R_H(f_{258}))_\infty \subseteq T_2 + T_4 + \sum_{\alpha=5}^{8} T_\alpha$ suffice for the linear independence of 1, $R_H(f_{56})$, $R_H(f_{157})$, $R_H(f_{258})$.

Further, observe that $H_2^\theta(n,0) = \langle \tau_{33}^n \theta \rangle$ are subgroups of $H_{2\times 2}^\theta(0) = \langle \tau_{33}, \theta \rangle$ with $\mathrm{rk}\Phi^{H_{2\times 2}^\theta(0)} = 2$. Therefore $\mathrm{rk}\Phi^{H_2^\theta(n,0)} = 2$ by Lemma 4(iv).

If $H = H_2^\theta(0,0) = \langle \theta \rangle$ then

$$R_H(f_{56}) = f_{56} + f_{78}, \ \ R_H(f_{157}) = f_{157} - \mathrm{e}^{\frac{\pi}{2}} f_{357}, \ \ R_H(f_{368}) = f_{368} - \mathrm{e}^{\frac{\pi}{2}} f_{168},$$

$$R_H(f_{258}) = f_{258} + f_{267}, \ \ R_H(f_{467}) = f_{467} + f_{458}.$$

The $\Gamma_2^\theta(0,0)$-cusps are $\overline{\kappa_1} = \overline{\kappa_3}$, $\overline{\kappa_2}$, $\overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_7}$ and $\overline{\kappa_6} = \overline{\kappa_8}$. According to Lemma 4(ii), $T_1 \subset (R_H(f_{157}))_\infty, (R_H(f_{168}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$ implies $R(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R(f_{157}))$. Lemma 6 supplies $\left.\frac{f_{258}+f_{267}}{\Sigma_2}\right|_{T_2} = 2\mathrm{e}^{-\pi} \neq 0$ and $\left.\frac{f_{467}+f_{458}}{\Sigma_4}\right|_{T_4} = 0$. Therefore $R_H(f_{258})|_{T_2} = \infty$ and $R_H(f_{467}) \subset \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}))$. Thus, $\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{157}), R_H(f_{258})) \simeq \mathbb{C}^4$. The entire $[\Gamma_2^\theta(0,0), 1]$ vanishes at $\overline{\kappa_4}$ and $\Phi^{H_2^\theta(0,0)}$ is not globally defined.

For $H = H_2^\theta(1,0) = \langle \tau_{33}\theta \rangle$ the space $\mathcal{L}^H$ is generated by

$$1 \in \mathbb{C}, \ \ R_H(f_{56}) = f_{56} - f_{78},$$

$$R_H(f_{157}) = f_{157} + \mathrm{i} f_{368}, \ \ R_H(f_{258}) = 2f_{258}, \ \ R_H(f_{467}) = 0.$$

The $\Gamma_2^\theta(1,0)$-cusps are $\overline{\kappa_1} = \overline{\kappa_3}$, $\overline{\kappa_2}$, $\overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_8}$ and $\overline{\kappa_6} = \overline{\kappa_7}$. Making use of $T_1 \subset (R_H(f_{157}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$ and $T_2 \subset (R_H(f_{258}))_\infty \subset T_2 + \sum_{\alpha=5}^{8} T_\alpha$, one applies Lemma 4(iii), in order to conclude that

$$\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{157}), R_H(f_{258})) \simeq \mathbb{C}^4.$$

The abelian functions from $\mathcal{L}^H$ have no poles along $T_4$, so that $\Phi^{H_2^\theta(1,0)}$ is not defined at $\overline{\kappa_4}$.

Observe that $H_2^\theta(n,1) = \langle \tau_{33}^n IJ^{-1}\theta\rangle$ are subgroups of $H_{2\times 2}^\theta(1) = \langle \tau_{33}, IJ^{-1}\theta\rangle$ with $\mathrm{rk}\Phi^{H_{2\times 2}^\theta(1)} = 2$, so that $\mathrm{rk}\Phi^{H_2^\theta(n,1)} = 2$ as well.

More precisely, Reynolds operators for $H = H_2^\theta(0,1) = \langle IJ^{-1}\theta\rangle$ are

$$R_H(f_{56}) = f_{56} + \mathrm{i} f_{78},\ \ R_H(f_{157}) = f_{157} - \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{168},\ \ R_H(f_{368}) = f_{368} - \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{357},$$

$$R_H(f_{258}) = f_{258} - \mathrm{e}^{-\frac{\pi}{2}} f_{458},\ \ R_H(f_{267}) = f_{267} + \mathrm{e}^{-\frac{\pi}{2}} f_{467}.$$

The $\Gamma_2^\theta$-cusps are $\overline{\kappa_1}$, $\overline{\kappa_3}$, $\overline{\kappa_2} = \overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_8}$, $\overline{\kappa_6} = \overline{\kappa_7}$. By Lemma 6 one has $\frac{f_{157} - \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\Big|_{T_1} = -2\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}} \neq 0$, $\frac{f_{368} - \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{357}}{\Sigma_3}\Big|_{T_3} = 0$, whereas $R_H(f_{157})|_{T_1} = \infty$, $R_H(f_{368}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}))$. Applying Lemma 4(ii) to the inclusions $T_2 \subset (R_H(f_{258}))_\infty, (R_H(f_{267}))_\infty \subseteq T_2 + T_4 + \sum_{\alpha=5}^{8} T_\alpha$, one concludes that $R_H(f_{267}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{258}))$. Altogether,

$$\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{157}), R_H(f_{258})) \simeq \mathbb{C}^4.$$

Since $\mathcal{L}^H$ has no pole over $T_3$, the rational map $\Phi^{H_2^\theta(0,1)}$ is not defined at $\overline{\kappa_3}$.

If $H = H_2^\theta(1,1) = \langle \tau_{33} IJ^{-1}\theta\rangle$ then

$$R_H(f_{56}) = f_{56} - \mathrm{i} f_{78},\ \ R_H(f_{157}) = 2 f_{157},$$

$$R_H(f_{368}) = 0,\ \ R_H(f_{258}) = f_{258} + \mathrm{e}^{-\frac{\pi}{2}} f_{467}.$$

The $\Gamma_2^\theta(1,1)$-cusps are $\overline{\kappa_1}$, $\overline{\kappa_3}$, $\overline{\kappa_2} = \overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_7}$ and $\overline{\kappa_6} = \overline{\kappa_8}$. Making use of $R_H(f_{157})|_{T_1} = \infty$, $T_H(f_{258})|_{T_2} = \infty$, one applies Lemma 4(iii), in order to conclude that $\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{157}), R_H(f_{258})) \simeq \mathbb{C}^4$. Since $\mathcal{L}^H$ has no pole over $T_3$, the rational map $\Phi^{H_2^\theta(1,1)}$ is not defined at $\overline{\kappa_3}$.

Reynolds operators for $H = H_2^\theta(0,2) = \langle I^2 J^2 \theta\rangle$ are

$$R_H(f_{56}) = f_{56} - f_{78},\ \ R_H(f_{157}) = f_{157} + \mathrm{e}^{\frac{\pi}{2}} f_{357},\ \ R_H(f_{168}) = f_{168} + \mathrm{e}^{-\frac{\pi}{2}} f_{368},$$

$$R_H(f_{258}) = f_{258} - f_{267},\ \ R_H(f_{467}) = f_{467} - f_{458}.$$

The $\Gamma_2^\theta(0,2)$-cusps are $\overline{\kappa_1} = \overline{\kappa_3}$, $\overline{\kappa_2}$, $\overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_7}$, $\overline{\kappa_6} = \overline{\kappa_8}$. Lemma 4(ii) applies to $T_1 \subset (R_H(f_{157}))_\infty, (R_H(f_{168}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$ to provide

$R_H(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{157}))$. By Lemma 6 one has $\frac{f_{258}-f_{267}}{\Sigma_2}\Big|_{T_2} = 0$ and $\frac{f_{467}-f_{458}}{\Sigma_4}\Big|_{T_4} = 2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$. As a result, $R_H(f_{258}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}))$ and $R_H(f_{467})|_{T_4} = \infty$. Lemma 4(iii) reveals that $1 \in \mathbb{C}$, $R_H(f_{56})$, $R_H(f_{157})$, $R_H(f_{467})$ form a $\mathbb{C}$-basis of $\mathcal{L}^H$. Since $\mathcal{L}^H$ has no pole over $T_2$, the rational map $\Phi^{H_2^\theta(0,2)}$ is not defined over $\overline{\kappa_2}$.

In the case of $H = H_2^\theta(1,2) = \langle \tau_{33} I^2 J^2 \theta \rangle$ one has

$$R_H(f_{56}) = f_{56} + f_{78}, \ \ R_H(f_{157}) = f_{157} - \mathrm{i} f_{368},$$

$$R_H(f_{258}) = 0, \ \ R_H(f_{467}) = 2 f_{467}.$$

The $\Gamma_2^\theta(1,2)$-cusps are $\overline{\kappa_1} = \overline{\kappa_3}$, $\overline{\kappa_2}$, $\overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_8}$ and $\overline{\kappa_6} = \overline{\kappa_7}$. Lemma 4(iii) applies to $T_1 \subset (R_H(f_{157}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$, $T_4 \subset (R_H(f_{467}))_\infty \subseteq T_4 + T_6 + T_7$, in order to justify the linear independence of $1$, $R_H(f_{56})$, $R_H(f_{157})$, $R_H(f_{467})$. Since $\mathcal{L}^H \simeq \mathbb{C}^4$ has no pole over $T_2$, the rational map $\Phi^{H_2^\theta(1,2)}$ is not defined at $\overline{\kappa_2}$.

(ii) For $H = H_2(0,0) = \langle \tau_{33} \rangle$ one has Reynolds operators

$$R_H(f_{56}) = 0, \ \ R_H(f_{78}) = 0, \ \ R_H(f_{157}) = f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168},$$

$$R_H(f_{258}) = f_{258} + f_{267}, \ \ R_H(f_{368}) = f_{368} + \mathrm{i}e^{\frac{\pi}{2}} f_{357}, \ \ R_H(f_{467}) = f_{467} - f_{458}.$$

There are six $\Gamma_{\langle \tau_{33} \rangle}$-cusps: $\overline{\kappa_1}$, $\overline{\kappa_2}$, $\overline{\kappa_3}$, $\overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_6}$ and $\overline{\kappa_7} = \overline{\kappa_8}$. By the means of Lemma 6 one observes that $\frac{f_{157}-\mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\Big|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$, $\frac{f_{258}+f_{267}}{\Sigma_2}\Big|_{T_2} = 2e^{-\pi} \neq 0$, $\frac{f_{368}+\mathrm{i}e^{\frac{\pi}{2}} f_{357}}{\Sigma_3}\Big|_{T_3} = 2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$, $\frac{f_{467}-f_{458}}{\Sigma_4}\Big|_{T_4} = 2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$. Therefore $T_i \subset (R_H(f_{i,\alpha_i,\beta_i}))_\infty \subseteq T_i + \sum_{\delta=5}^{8} T_\delta$ for $1 \leq i \leq 4$, $(\alpha_1, \beta_1) = (5,7)$, $(\alpha_2, \beta_2) = (5,8)$, $(\alpha_3, \beta_3) = (6,8)$, $(\alpha_4, \beta_4) = (6,7)$. According to Lemma 4(iii), that suffices for $1$, $R_H(f_{157})$, $R_H(f_{258})$, $R_H(f_{368})$, $R_H(f_{467})$ to be a $\mathbb{C}$-basis of $\mathcal{L}^H$. Bearing in mind that $H_2(0,0) = \langle \tau_{33} \rangle$ is a subgroup of $H'_{2\times 2}(0) = \langle \tau_{33}, I^2 \rangle$ with $\mathrm{rk}\Phi^{H'_{2\times 2}(0)} = 2$, one concludes that $\mathrm{rk}\Phi^{\langle \tau_{33} \rangle} = 2$.

□

## References


[1] Hacon Ch. and Pardini R., *Surfaces with $p_g = q = 3$*, Trans. Amer. Math. Soc. **354** (2002) 2631–1638.

[2] Hemperly J. *The Parabolic Contribution to the Number of Independent Automorphic Forms on a Certain Bounded Domain*, Amer. J. Math. **94** (1972) 1078–1100.

[3] Holzapfel R.-P., *Jacobi Theta Embedding of a Hyperbolic 4-space with Cusps*, In: Geometry, Integrability and Quantization, I. Mladenov and G. Naber (Eds), Coral Press, Sofia 2002, pp 11–63.

[4] Holzapfel R.-P., *Complex Hyperbolic Surfaces of Abelian Type*, Serdica Math. Jour. **30** (2004) 207–238.

[5] Kasparian A. and Kotzev B., *Normally Generated Subspaces of Logarithmic Canonical Sections*, to appear in Ann. Univ. Sofia **101**.

[6] Kasparian A. and Kotzev B., *Weak Form of Holzapfel's Conjecture*, In: Geometry, Integrability and Quantization, I. Mladenov, G. Villasi and A. Yoshioka (Eds), Avangard Prima, Sofia 2010, pp 134–145.

[7] Kasparian A., Nikolova L., *Ball Quotients of Non-Positive Kodaira Dimension*, submitted to Comp. rend. Acad. bulg. Sci.

[8] Lang S., *Elliptic Functions*, Addison-Wesley, London 1973, pp 233–237.

[9] Momot A., *Irregular Ball-Quotient Surfaces with Non-Positive Kodaira Dimension*, Math. Res. Lett. **15** (2008) 1187–1195.