Galois groups of co-abelian ball quotient covers

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Abstract

If $X' = (\mathbb{B}/\Gamma)'$ is a torsion free toroidal compactification of a discrete ball quotient $X_\circ = \mathbb{B}/\Gamma$ and $\xi : (X', T = X' \setminus X_\circ) \to (X, D = \xi(T))$ is the blow-down of the $(-1)$-curves to the corresponding minimal model, then $G' = Aut(X', T)$ coincides with the finite group $G = Aut(X, D)$. In particular, for an elliptic curve $E$ with endomorphism ring $R = \text{End}(E)$ and a split abelian surface $X = A = E \times E$, $G$ is a finite subgroup of $Aut(A) = T_A \ltimes GL(2, R)$, where $(T_A, +)$ is the translation group of $A$ and $GL(2, R) = \{g \in R_{2 \times 2} \mid \det(g) \in R^*\}$.

The present work classifies the finite subgroups $H$ of $Aut(A = E \times E)$ for an arbitrary elliptic curve $E$. By the means of the geometric invariants theory, it characterizes the Kodaira-Enriques types of $A/H \simeq (\mathbb{B}/\Gamma)'/H$, in terms of the fixed point sets of $H$ on $A$. The abelian and the K3 surfaces $A/H$ are elaborated in [7]. The first section provides necessary and sufficient conditions for $A/H$ to be a hyper-elliptic, ruled with elliptic base, Enriques or a rational surface. In such a way, it depletes the Kodaira-Enriques classification of the finite Galois quotients $A/H$ of a split abelian surface $A = E \times E$. The second section derives a complete list of the conjugacy classes of the linear automorphisms $g \in GL(2, R)$ of $A$ of finite order, by the means of their eigenvalues. The third section classifies the finite subgroups $H$ of $GL(2, R)$. The last section provides explicit generators and relations for the finite subgroups $H$ of $Aut(A)$ with K3, hyper-elliptic, rules with elliptic base or Enriques quotients $A/H \simeq (\mathbb{B}/\Gamma)'/H$.

Let

$$
\mathbb{B} = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\} \simeq SU_{2,1}/S(U_2 \times U_1).
$$

be the complex 2-ball. In [4] Holzapfel settled the problem of the characterization of the projective surfaces, which are birational to an eventually singular ball quotient $\mathbb{B}/\Gamma$ by a lattice $\Gamma$ of $SU_{2,1}$. Note that if $\gamma \in \Gamma$ is a torsion element with isolated fixed points on $\mathbb{B}$ then $\mathbb{B}/\Gamma$ has isolated cyclic quotient singularity, which ought to be resolved in order to obtain a smooth open surface. The aforementioned resolution creates smooth rational curves of self-intersection $\leq -2$, which alter the local differential geometry of $\mathbb{B}/\Gamma$, modeled by $\mathbb{B}$. That is why we split the problem to the description of the minimal models $X_\circ$ of the smooth toroidal compactifications $X'_\circ = (\mathbb{B}/\Gamma_\circ)'$ of torsion free $\Gamma_\circ$ and to the characterization of the birational equivalence classes of
X_o/H for appropriate finite automorphism groups H. This reduction is based on the
fact that any finitely generated lattice Γ in the simple Lie group SU_{2,1} has a torsion
free normal subgroup Γ_o of finite index [Γ : Γ_o]. Therefore B/Γ = (B/Γ_o) / (Γ/Γ_o)
and the classification of B/Γ is attempted by the classification of B/Γ_o and the finite
automorphism groups H = Γ/Γ_o of B/Γ_o.

According to the next proposition, for any torsion free ball lattice Γ_o and any
Γ < SU_{2,1}, containing Γ_o as a normal subgroup of finite index, the quotient group
Γ/Γ_o acts on the toroidal compactifying divisor T = (B/Γ_o)' \ (B/Γ_o) and provides a
compactification B/Γ = (B/Γ_o)' / (Γ/Γ_o) of B/Γ with at worst isolated cyclic quotient
singularities. Therefore H = Γ/Γ_o is a subgroup of Aut(X_o,T). The birational
equivalence classes of B/Γ are to be described by the numerical invariants of the
minimal resolutions Y of the singularities of B/Γ. These can be computed by the
means of the geometric invariant theory, applied to X_o and a finite subgroup H of the
biholomorphism group Aut(X_o).

**Proposition 1.** Let Γ be a lattice of SU_{2,1} and Γ_o be a normal torsion free subgroup
of Γ with finite index [Γ : Γ_o]. Then the group G = Γ/Γ_o acts on the toroidal
compactifying divisor T = (B/Γ_o)' \ (B/Γ_o) and the quotient (B/Γ_o)' / G = (B/Γ) ∪
(T/G) = B/Γ is a compactification of B/Γ with at worst isolated cyclic quotient
singularities.

**Proof.** Recall that p ∈ ∂_Γ B is a Γ-rational boundary point exactly when the intersection
Γ ∩ Stab_{SU_{2,1}}(p) is a lattice of Stab_{SU_{2,1}}(p). Since [Γ : Γ_o] < ∞, the quotient
Stab_{SU_{2,1}}(p)/[Γ ∩ Stab_{SU_{2,1}}(p)] =

= \{Stab_{SU_{2,1}}(p)/ [Γ_o ∩ Stab_{SU_{2,1}}(p)] \} / \{[Γ ∩ Stab_{SU_{2,1}}(p)]/[Γ_o ∩ Stab_{SU_{2,1}}(p)]\}

has finite invariant volume exactly when Stab_{SU_{2,1}}(p)/[Γ_o ∩ Stab_{SU_{2,1}}(p)] has finite
invariant volume. Therefore the Γ-rational boundary points coincide with the Γ_o-
rational boundary points, ∂_Γ B = ∂_{Γ_o} B. It suffices to establish that the Γ-action on B
admits local extensions on neighborhoods of the liftings of T_i to complex lines through
p_i ∈ ∂_{Γ_o} B with Orb_{Γ_o}(p_i) = κ_i. According to [?], the cusp κ_i, associated with the
smooth elliptic curve T_i has a neighborhood N(κ_i) = T_i × Δ^*(0,ε) ⊂ (B/Γ_o) for a
sufficiently small punctured disc Δ^*(0,ε) = \{ z ∈ C \ | \ |z| < ε \}. The biholomorphisms
γ : B → B from Γ extend to γ : B ∪ ∂_{Γ_o} B → B ∪ ∂_{Γ_o} B, as far as ∂_{Γ_o} B consists of
isolated points. If p_i ∈ ∂_{Γ_o} B, γ(p_i) = p_j ∈ ∂_{Γ_o} B and κ_j = Orb_{Γ_o}(p_j) then there is a
biholomorphism
γ : N(κ_i) ∩ γ^{-1} N(κ_j) → γ N(κ_i) ∩ N(κ_j).

For any q_i ∈ T_i let Δ_{T_i}(q_i,η) be a sufficiently small disc on T_i, centered at q_i,
which is contained in a π_1(T_i)-fundamental domain, centered at q_i. One can view
Δ_{T_i}(q_i,η) = Δ_{\tilde{T_i}}(q_i,η) as a disc on the lifting \tilde{T_i} of T_i to a complex line through p_i.
Then $N(k_i, q_i) := \Delta_{\tilde{T}_i}(q_i, \eta) \times \Delta^*(0, \varepsilon)$ is a bounded neighborhood of $q_i \in T_i$ on $\mathbb{B}/\Gamma_o$ and the holomorphic map

$$\gamma : N(k_i, q_i) \cap \gamma^{-1} N(k_j, q_j) \to \gamma N(k_i, q_i) \cap N(k_j, q_j) \subseteq N(k_j, q_j) = \Delta_{\tilde{T}_j}(q_j, \eta) \times \Delta^*(0, \varepsilon)$$

is bounded. Thus, $\gamma : \mathbb{B} \to \mathbb{B}$ is locally bounded around $\tilde{T} = \sum_{p_i \in \partial \mathbb{B}_o \mathbb{B}} \tilde{T}_i(p_i)$ and admits a holomorphic extension $\gamma : \mathbb{B} \cup \tilde{T} \to \mathbb{B} \cup \tilde{T}$. This induces a biholomorphism $\gamma_{\Gamma_o} : (\mathbb{B}/\Gamma_o)' \to (\mathbb{B}/\Gamma_o)'$.

The next proposition establishes that an arbitrary torsion free toroidal compactification $(\mathbb{B}/\Gamma_o)'$ has finitely many Galois quotients $(\mathbb{B}/\Gamma_o)' / H = \mathbb{B}/\Gamma_H$ with $\Gamma_H / \Gamma_o = H$. For torsion free $(\mathbb{B}/\Gamma_o)'$ with an abelian minimal model $X_o = A$, the result is proved in [6]. Note also that [9] constructs an infinite series $\{(\mathbb{B}/\Gamma_n)\}'$ of mutually non-birationnal torsion free toroidal compactifications with abelian minimal models, which are finite Galois covers of a fixed $(\mathbb{B}/\Gamma_n)’, T(1)/H = ((\mathbb{B}/\Gamma_n), T(n)) / H_n$, $H_n \leq \text{Aut} ((\mathbb{B}/\Gamma_n)’, T(n))$.

**Proposition 2.** Let $X' = (\mathbb{B}/\Gamma)' = (\mathbb{B}/\Gamma) \cup T$ be a torsion free toroidal compactification and $\xi : (X', T) \to (X = \xi(X'), D = \xi(T))$ be the blow-down of the $(-1)$-curves to the minimal model $X$ of $X'$. Then $\text{Aut}(X', T)$ is a finite group, which coincides with $\text{Aut}(X, D)$.

**Proof.** Let us denote $G = \text{Aut}(X, D)$, $G' = \text{Aut}(X', T)$ and observe that $X'$ is the blow-up of $X$ at the singular locus $D^{\text{sing}}$ of $D$. Since $D = \sum_{i=1}^{h} D_i$ has smooth elliptic irreducible components $D_i$, the singular locus $D^{\text{sing}} = \sum_{1 \leq i < j \leq h} D_i \cap D_j$ and its complement $X \setminus D^{\text{sing}}$ are $G$-invariant. We claim that the $G$-action extends to the exceptional divisor $E = \xi^{-1}(D^{\text{sing}})$ of $\xi$, so that $X' = (X \setminus D^{\text{sing}}) \cup E$ is $G$-invariant. Indeed, for any $g \in G$ and $p \in D^{\text{sing}}$ with $q = g(p)$, let us choose local holomorphic coordinates $x = (x_1, x_2)$, respectively, $y = (y_1, y_2)$ on sufficiently small neighborhoods $N(p)$, $N(q)$ of $p$ and $q$ on $X$ with $gN(p) \subseteq N(q)$. Then $g : N(p) \to N(q) \subseteq \mathbb{C}^2$ consists of two local holomorphic functions $g = (g_1, g_2)$ on $N(p)$. By the very definition of a blow-up,

$$\xi^{-1}N(p) = \{(x_1, x_2) \times [x_1 : x_2] \mid (x_1, x_2) \in N(p)\} \quad \text{and} \quad \xi^{-1}N(q) = \{(g_1(x), g_2(x)) \times [g_1(x) : g_2(x)] \mid g(x) = (g_1(x), g_2(x)) \in N(q)\},$$

so that

$$g : \xi^{-1}N(p) \to \xi^{-1}N(q),$$

$$(x_1, x_2) \times [x_1 : x_2] \mapsto (g_1(x), g_2(x)) \times [g_1(x) : g_2(x)]$$

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extends the action of \( g \in G \) to \( \xi^{-1}(D_{\text{sing}}) \) and \( G \subset \text{Aut}(X') \). Towards the \( G \)-invariance of \( T \), note that the birational maps \( \xi : T_i \to \xi(T_i) = D_i \) of the smooth irreducible components \( T_i \) of \( T \) are biregular. Thus, the \( G \)-invariance of \( D = \sum_{i=1}^{h} D_i \) implies the \( G \)-invariance of \( T = \sum_{i=1}^{h} T_i \) and \( G \subseteq G' = \text{Aut}(X', T) \). For the opposite inclusion \( G' = \text{Aut}(X', T) \subseteq G = \text{Aut}(X, D) \) observe that an arbitrary \( g' \in G' \) acts on the union \( E \) of the \((-1)\)-curves on \( X' \) and permutes the finite set \( \xi(E) = D_{\text{sing}} \). In such a way, \( g' \) turns to be a biregular morphism of \( X = (X' \setminus E) \cup D_{\text{sing}} \). The restriction of \( g' \) on \( T_i \) has image \( g'(T_i) = T_j \) for some \( 1 \leq j \leq h \) and induces a biholomorphism \( g' : D_i \to D_j \). As a result, \( g' \in G' \) acts on \( D \) and \( g' \in G = \text{Aut}(X, D) \).

In order to justify that \( G = \text{Aut}(X, D) \) is a finite group, let us consider the natural representation

\[
\varphi : G \to \text{Sym}(D_1, \ldots, D_h)
\]

in the permutation group of the irreducible components \( D_1, \ldots, D_h \) of \( D \). As far as the image \( \varphi(G) \) is a finite group, it suffices to prove that the kernel \( \ker \varphi \) is finite. Fix \( p \in D_{\text{sing}} \) and two local irreducible branches \( U_o \) and \( V_o \) of \( D \) through \( p \). If \( U_o \subset D_i \) and \( V_o \subset D_j \) for \( i \neq j \) then consider the natural representation

\[
\varphi_o : \ker \varphi \to \text{Sym}(D_i \cap D_j).
\]

The group homomorphism \( \varphi_o \) has a finite image, so that the problem reduces to the finiteness of \( G_o := \ker (\varphi_o |_{\ker \varphi}) \). By its very definition, \( G_o \leq \text{Stab}_G(p) \). Let us move the origin of \( D_i \) at \( p \) and realize \( G_o \) as a subgroup of the finite cyclic group \( \text{End}^*(D_i) \). After an eventual shrinking, \( U_o \) is contained in a coordinate chart of \( X \). Then \( U = \cap_{g_o \in G_o} [g_o(U_o)] \) is a \( G_o \)-invariant neighborhood of \( p \) on \( D_i \). Similarly, pass to a \( G_o \)-invariant neighborhood \( V \subset V_o \) of \( p \) on \( D_j \), intersecting transversally \( U \). Through any point \( v \in V \) there is a local complex line \( U(v), \) parallel to \( U \). The union \( W = \cup_{v \in V} U(v) \) is a neighborhood of \( p \) on \( X \), biholomorphic to \( U \times V \). In holomorphic coordinates \((u, v) \in W\), one gets \( G_o \leq \text{End}^*(U) \times \text{End}^*(V) \). Note that \( \text{End}^*(U) \subseteq \text{End}^*(D_i) \) and \( \text{End}^*(D_i) \) is a finite cyclic group of order 1, 2, 3, 4 or 6, so that \( |G_o| < \infty \).

\[ \square \]
1 Kodaira-Enriques classification of the finite Galois quotients of a split abelian surface

Let $A = E \times E$ be the Cartesian square of an elliptic curve $E$. For an arbitrary finite automorphism group $H \leq \text{Aut}(A)$, we characterize the Kodaira-Enriques classification type of $A/H$ in terms of the fixed point set $\text{Fix}_A(H)$ of $H$ on $A$. Partial results for this problem are provided by [7]. Namely, any $A/H$ is a finite cyclic Galois quotient of a smooth abelian surface $A/K$ or a normal model $A/K$ of a K3 surface. The surface $A/K$ is abelian exactly when $K = \mathcal{T}(H)$ is a translation group. The note [7] specifies that a necessary and sufficient condition for $A/\mathcal{T}(H)/h)$ to have irregularity $q(Y) = h^{1,0}(Y) = 1$ is the presence of an entire elliptic curve in the fixed point set $\text{Fix}_A(h)$ of $h$. This result is similar to S. Tokunaga and M. Yoshida’s study [11] of the discrete subgroups $\Lambda \leq \mathbb{C}^n \ltimes U(n)$ with compact quotient $\mathbb{C}^n/\Lambda$. Namely, [11] establishes that if the linear part $\mathcal{L}(\Lambda)$ of such $\Lambda$ does not fix pointwise a complex line on $\mathbb{C}^2$, then $\mathbb{C}^n/\Lambda$ has vanishing irregularity. Further, [7] observes that if some $h \in H$ fixes pointwise an entire elliptic curve on $A$, then the Kodaira dimension $\kappa(A/H) = -\infty$ drops down. Tokunaga and Yoshida prove the same statement for discrete subgroups $\Lambda \leq \mathbb{C}^n \ltimes U(n)$ with compact quotient $\mathbb{C}^n/\Lambda$. The note [7] proves also that if $A/K$ is a K3 double cover of $A/H$ then $A/H$ is birational to an Enriques surface if and only if $A/K \to A/H$ is unramified.

The present note establishes that an arbitrary cyclic cover $\zeta^K_H : A/K \to A/H$ of degree $\geq 3$ by a K3 surfaces $A/K$ with isolated cyclic quotient singularities is ramified over a finite set of points and $A/H$ is a rational surface. If a K3 surface $A/K$ is a double cover $\zeta^K_H : A/K \to A/H$ of $A/H$ then $A/H$ is birational to an Enriques surface exactly when $\zeta^K_H$ is unramified. The quotients $A/H$ with ramified K3 double covers $\zeta^K_H : A/K \to A/H$ are rational surfaces. If $H = \mathcal{T}(H)/h)$ and the fixed points of $\mathcal{L}(h)$ on $A$ contain an elliptic curve then $A/H$ is hyper-elliptic (respectively, ruled with an elliptic base) if and only if $H$ has not a fixed point on $A$ (respectively, $H$ has a fixed point on $A$, whereas $H$ has a pointwise fixed elliptic curve on $A$). If $H = \mathcal{T}(H)/h)$ and $\mathcal{L}(h)$ has isolated fixed points on $A$ then $A/H$ is a rational surface.

In order to construct the normal subgroup $K$ of $H$, let us recall that the automorphism group $\text{Aut}(A) = T_A \ltimes \text{Aut}_{\hat{o}_A}(A)$ of $A$ is a semi-direct product of the translation group $T_A \simeq (\mathbb{A}, +)$ and the stabilizer $\text{Aut}_{\hat{o}_A}(A)$ of the origin $\hat{o}_A \in A$. Each $g \in \text{Aut}_{\hat{o}_A}(A)$ is a linear transformation

$$g = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL_2(\mathbb{C}),$$

leaving invariant the fundamental group $\pi_1(A) = \pi_1(E) \times \pi_1(E)$ of $A = E \times E$. Therefore $a_{ij} \pi_1(E) \subseteq \pi_1(E)$ for all $1 \leq i, j \leq 2$ and $a_{ij} \in R$ for the endomorphism ring $R$ of $E$. The same holds for the entries of the inverse matrix

$$g^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \in \text{Aut}_{\hat{o}_A}(A). \quad (1)$$
Now, \( \det(g) \in R \) and \( \det(g^{-1}) = (\det(g))^{-1} \in R \) imply that \( \det(g) \in R^* \) is a unit. Thus, \( \text{Aut}_{\phi_A}(A) \) is contained in

\[
\text{Gl}(2, R) := \{ g \in (R)_{2 \times 2} \mid \det(g) \in R^* \}.
\]

The opposite inclusion \( \text{Gl}(2, R) \subseteq \text{Aut}_{\phi_A}(A) \) is clear from (1) and \( \text{Aut}_{\phi_A}(A) = \text{Gl}(2, R) \).

The map \( \mathcal{L} : \text{Aut}(A) \rightarrow \text{Gl}(2, R) \), associating to \( g \in \text{Aut}(A) \) its linear part \( \mathcal{L}(g) \in \text{Gl}(2, R) \) is a group homomorphism with kernel \( \ker(\mathcal{L}) = T_A \). Denote by \( \mathcal{O}_{-d} \) the integers ring of an imaginary quadratic number field \( \mathbb{Q}(\sqrt{-d}) \). The determinant \( \det : \text{Gl}(2, R) \rightarrow R^* \) is a group homomorphism in the cyclic units group

\[
R^* = \langle \zeta_d \rangle \simeq \begin{cases} 
\mathbb{C}_2 & \text{for } R \neq \mathbb{Z}[i], \mathcal{O}_{-3}, \\
\mathbb{C}_4 & \text{for } R = \mathbb{Z}[i] = \mathcal{O}_{-1}, \\
\mathbb{C}_6 & \text{for } R = \mathcal{O}_{-3}
\end{cases}
\]

of order \( o(R) \). For an arbitrary subgroup \( H \) of \( \text{Aut}(A) \), let us denote by \( K = K_H \) the kernel of the group homomorphism \( \det \mathcal{L} : H \rightarrow R^* \). The image \( \det \mathcal{L}(H) \subseteq (R^*, \cdot) \) is a cyclic group of order \( m \), dividing \( o(R^*) \), i.e., \( \det \mathcal{L}(H) = \langle \zeta_d^k \rangle \) for some natural divisor \( k = \frac{o(R^*)}{m} \) of \( o(R^*) \). For an arbitrary \( h_0 \in H \) with \( \det \mathcal{L}(h_0) = \zeta_d^k \) the first homomorphism theorem reads as

\[
\{K_H, h_0 K_H, \ldots, h_0^{m-1} K_H\} = H/K_H \simeq \langle \zeta_d^k \rangle = \{1, \zeta_d^k, \zeta_d^{2k}, \ldots, \zeta_d^{(m-1)k}\}.
\]

Therefore \( H = K_H \langle h_0 \rangle \) is a product of \( K_H = \ker(\det \mathcal{L}|_H) \) and the cyclic subgroup \( \langle h_0 \rangle \) of \( H \).

Denote by \( E_1(H) \) the set of \( h \in H \), whose linear parts \( \mathcal{L}(h) \in GL_2(R) \) have eigenvalue 1 of multiplicity 1. In other words, \( h \in E_1(H) \) exactly when \( \mathcal{L}(h) \) fixes pointwise an elliptic curve on \( A \) through the origin \( \partial_A \). Put \( E_0(H) \) for the set of \( h \in H \), whose linear parts have no eigenvalue 1. Observe that \( h \in E_0(H) \) if and only if \( \mathcal{L}(h) \in \text{GL}(2, R) \) has isolated fixed points on \( A \).

An automorphism \( h \in H \setminus \{\text{Id}\} \) is called a reflection if fixes pointwise an elliptic curve on \( A \). We claim that \( h \in H \) is a reflection if and only if \( h \in E_1(H) \) and \( h \) has a fixed point on \( A \). Indeed, if \( h \) fixes an elliptic curve \( F \) on \( A \), then one can move the origin \( \partial_A \) of \( A \) on \( F \), in order to represent \( h \) by a linear transformation \( h = \mathcal{L}(h) \in \text{GL}(2, R) \setminus \{\text{Id}\} = E_1(\text{GL}(2, R)) \cup E_0(\text{GL}(2, R)) \). Any \( h = \mathcal{L}(h) \in E_0(\text{GL}(2, R)) \) has isolated fixed points on \( A \), so that \( h = \mathcal{L}(h) \in E_1(H) \) and \( \text{Fix}_A(h) \neq \emptyset \). Conversely, if \( h \in E_1(H) \) and \( \text{Fix}_A(h) \neq \emptyset \), then after moving the origin of \( A \) at \( \partial_A \in \text{Fix}_A(h) \), one attains \( h = \mathcal{L}(h) \). Thus, \( h \) fixes pointwise an elliptic curve on \( A \) or \( h \) is a reflection.

Towards the complete classification of the Kodaira-Enriques type of \( A/H \), we use the following results from [7]:
Proposition 3. (i) (cf. Corollary 5 from [7]) The quotient $A/H$ of $A = E \times E$ by a finite automorphism group $H$ is an abelian surface if and only if $H = \ker(L|_H) = T(H)$ is a translation group.

(ii) (Lemma 7 from [7]) The quotient $A/H$ is birational to a K3 surface if and only if $H = \ker(\det L|_H)$ and $H \supseteq \ker(L|_H) = T(H)$.

Proposition 4. (i) (cf. Lemma 11 from [7]) If a finite automorphism group $H \leq \text{Aut}(A)$ contains a reflection then $A/H$ is of Kodaira dimension $\kappa(A/H) = -\infty$.

(ii) (cf. Proposition 12 from [7]) A smooth model $Y$ of $A/H$ is of irregularity $q(Y) = h^{1,0}(Y) = 1$ if and only if $H = T(H)\langle h \rangle$ is a product of its normal translation subgroup $T(H) = \ker(L|_H)$ and a cyclic group $\langle h \rangle$, generated by $h \in E_1(H)$.

From now on, we consider only subgroups $H \leq \text{Aut}(A,T)$ with $\det L(H) \neq \{1\}$ and distinguish between translation $K = \ker(\det L|_H) = \ker(L|_H) = T(H)$ and non-translation $T = \ker(\det L|_H) \supseteq \ker(L|_H) = T(H)$. Any $h \notin K = \ker(\det L|_H)$ belongs to $E_1(H)$ or to $E_0(H)$.

Proposition 5. Let $H = T(H)\langle h \rangle$ be a product of its (normal) translation subgroup $T(H) = \ker(L|_H)$ and a cyclic group $\langle h \rangle$, generated by $h \in E_1(H)$. Then:

(i) the fixed point set $\text{Fix}_A(H) = \emptyset$ of $H$ on $A$ is empty if and only if $A/H$ is a smooth hyper-elliptic surface;

(ii) the fixed point set $\text{Fix}_A(H) \neq \emptyset$ is non-empty if and only if $A/H$ is a smooth ruled surface with an elliptic base. If so, then $\text{Fix}_A(H)$ is of codimension 1 in $A$.

Proof. According to Proposition 4 (ii), $H = T(H)\langle h \rangle$ with $h \in E_1(H)$ if and only if any smooth model $Y$ of $A/H$ has irregularity $q(Y) = h^{1,0}(Y) = 1$. More precisely, $Y$ is a hyper-elliptic surface or a ruled surface with an elliptic base.

If $\text{Fix}_A(H) = \emptyset$ then $A \to A/H$ is an unramified cover and the Kodaira dimension $\kappa(A/H) = \kappa(A) = 0$. Therefore $A/H$ is hyper-elliptic.

Suppose that there is an $H$-fixed point $p \in \text{Fix}_A(H)$ and move the origin $o_A$ of $A$ at $p$. For any $h_1 \in \text{Stab}_H(o_A) \setminus \{Id_A\}$ one has $o_A = h_1(o_A) = \tau(h_1)L(h_1)(o_A) = \tau(h_1)(o_A)$, so that $h_1$ has trivial translation part $\tau(h_1) = \tau_{o_A}$. As a result, $h_1 = L(h_1) \in E_1(H) \setminus \{Id_A\}$ is a reflection and fixes pointwise an elliptic curve on $A$. In particular, $\text{Fix}_A(H)$ is of complex codimension 1. If

$$L(h) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2(h) \end{pmatrix}$$

with $\lambda_2(h) \neq 1$ then

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2(h)^i \end{pmatrix}$$

with $i \in \mathbb{Z}$, $\lambda_2(h)^i \neq 1$.

By Proposition 4 (i), the quotient $A/\langle h_1 \rangle$ by the cyclic group $\langle h_1 \rangle$, generated by the reflection $h_1 = L(h_1) \in E_1(H)$ is of Kodaira dimension $\kappa(A/\langle h_1 \rangle) = -\infty$. Along the finite (not necessarily Galois) cover $A/\langle h_1 \rangle \to A/H$, one has $\kappa(A/\langle h_1 \rangle) \geq \kappa(A/H)$,
whereas $\kappa(A/H) = -\infty$ and $A/H$ is birational to a ruled surface with an elliptic base. Note that all $h \in H$ with $\text{Fix}_A(h) \neq \emptyset$ are reflections, so that the quotient $A/H$ is a smooth surface by a result of Chevalley [5].

That proves the proposition, as far as the assumption $\text{Fix}_A(H) \neq \emptyset$ for a hyper-elliptic $A/H$ leads to a contradiction, as well as the assumption $\text{Fix}_A(H) = \emptyset$ for a ruled $A/H$ with an elliptic base.

\[ \square \]

**Proposition 6.** Let $H = \mathcal{T}(H)\langle h \rangle$ for some

\[ h \in E_0(H) = \{ h \in H \mid \lambda_j \mathcal{L}(h) \neq 1, \ 1 \leq j \leq 2 \} \]

with $\det \mathcal{L}(h) \neq 1$. Then $A/H$ is a rational surface.

**Proof.** We claim that $A/H$ with $A = E \times \tilde{E}$ is simply connected. To this end, let us denote by $R$ the endomorphism ring of $E$ and lift $H$ to a subgroup $\tilde{H}$ of the affine-linear group $\text{Aff}(\mathbb{C}^2, R) = (\mathbb{C}^2, +) \ltimes \text{GL}(2, R)$, containing $(\pi_1(A), +)$ as a normal subgroup with quotient $\tilde{H}/\pi_1(A) = H$. Then

\[ A/H = [\mathbb{C}^2/\pi_1(A)] / [\tilde{H}/\pi_1(A)] \simeq \mathbb{C}^2/\tilde{H}. \]

The universal cover $\tilde{A} = \mathbb{C}^2$ of $A$ is a path connected, simply connected locally compact metric space and $\tilde{H}$ is a discontinuous group of homeomorphisms of $\mathbb{C}^2$. That allows to apply Armstrong’s result [1] and conclude that

\[ \pi_1(A/H) = \pi_1 \left( \mathbb{C}^2/\tilde{H} \right) \simeq \hat{H}/\hat{N}, \]

where $\hat{N}$ is the normal subgroup of $\hat{H}$ generated by $\hat{h} \in \hat{H}$ with $\text{Fix}_{\mathbb{C}^2}(\hat{h}) \neq \emptyset$. There remains to be shown the coincidence $\hat{H} = \hat{N}$. In the case under consideration, let us choose generators $\tau_{(p_i, q_i)}$ of $\mathcal{T}(H)$, $1 \leq i \leq m$ and fix liftings $(p_i, q_i) \in \mathbb{C}^2 = \tilde{A}$ of $(p_i + \pi_1(E), q_i + \pi_1(E)) = (P_i, Q_i)$. If $\pi_1(E) = \lambda_1 \mathbb{Z} + \lambda_2 \mathbb{Z}$ for some $\lambda_1, \lambda_2 \in \mathbb{C}^*$ with $\lambda_2 \notin \mathbb{C} \setminus \mathbb{R}$, then $\pi_1(A) = \pi_1(E) \times \pi_1(E)$ is generated by

\[ \Lambda_{11} = (\lambda_1, 0), \quad \Lambda_{12} = (\lambda_2, 0), \quad \Lambda_{21} = (0, \lambda_1) \quad \text{and} \quad \Lambda_{22} = (0, \lambda_2). \]

Let $\tilde{h} = \tau_{(u, v)} \mathcal{L}(h) \in \tilde{H}$ be a lifting of $h = \tau_{(U, V)} \mathcal{L}(h) \in H$, i.e., $(u + \pi_1(E), v + \pi_1(E)) = (U, V)$. Then $\tilde{H}$ is generated by its subset

\[ S = \left\{ \Lambda_{ij}, \quad \tau_{(p_k, q_k)} \tilde{h}, \quad 1 \leq i, j \leq 2, \ 1 \leq k \leq m \right\}. \]

Since $\mathcal{L}(h)$ has eigenvalues $\lambda_1 \mathcal{L}(h) \neq 1, \lambda_2 \mathcal{L}(h) \neq 1$, for any $(a, b) \in \mathbb{C}^2$ the automorphism $\tau_{(a,b)} \mathcal{L}(h) \in \text{Aut}(\mathbb{C}^2)$ has a fixed point on $\mathbb{C}^2$. One can replace the generators $\Lambda_{ij}$ and $\tau_{(p_k, q_k)}$ of $\tilde{H}$ by $\Lambda_{ij}\tilde{h}$, respectively, $\tau_{(p_k, q_k)}\tilde{h}$, since

\[ \langle S \rangle \supseteq \left\{ \Lambda_{ij}\tilde{h}, \quad \tau_{(p_k, q_k)}\tilde{h}, \quad \tilde{h}, \quad 1 \leq i, j \leq 2, \ 1 \leq k \leq m \right\}. \]
and $A_{ij}, \tau_{(p_k,q_k)} \in \langle \{ A_{ij}, \tau_{(p_k,q_k)}, \tilde{h} \mid 1 \leq i, j \leq 2, 1 \leq k \leq m \} \rangle$. Thus
\[
\tilde{H} = \langle A_{ij}, \tau_{(p_k,q_k)}, \tilde{h} \mid 1 \leq i, j \leq 2, 1 \leq k \leq m \rangle
\]
coincides with $\tilde{N}$, because $\tilde{H}$ is generated by elements with fixed points. As a result, \( \pi_1(A/H) = \{1\} \).

Note that the simply connected surfaces $A/H$ are either rational or K3. According to $\det L(h) \neq 1$, the quotient $A/H$ is not birational to a K3 surface, so that $A/H$ is a rational surface with isolated cyclic quotient singularities.

\[ \square \]

**Proposition 7.** Let $H < Aut(A)$ be a finite subgroup of the form $H = K\langle h \rangle$ with $L(K) < SL(2, R)$ and $\det L(H) = \langle \det(h) \rangle \neq \{1\}$.

(i) The complement $H \setminus K$ has fixed points on $A$, $Fix_A(H \setminus K) \neq \emptyset$ if and only if $A/H$ is a rational surface;

(ii) The complement $H \setminus K$ has no fixed points on $A$, $Fix_A(H \setminus K) = \emptyset$ if and only if $A/H$ is birational to an Enriques surface $Y$. If so, then the K3 universal cover $\tilde{Y}$ of $Y$ is birational to $A/K$ and the index $[H : K] = 2$.

**Proof.** First of all, the $H/K$-Galois cover $\zeta : A/K \to A/H$ is ramified if and only if the complement $H \setminus K$ has a fixed point on $A$. More precisely, a point $\text{Orb}_K(p) \in A/K$, $p \in A$ is fixed by $hK \in H/K \setminus \{K\}$ exactly when $h\text{Orb}_K(p) = \text{Orb}_K(p)$ or
\[
\{hk(p) \mid k \in K\} = \{h(p) \mid k \in K\}. \tag{2}
\]
The condition (2) implies the existence of $k_0 \in K$ with $h(p) = k_0(p)$. Therefore $h_1 = k_0^{-1} h \in \text{Stab}_H(p) \setminus K$ has a fixed point and
\[
h_1 K = (k_0^{-1} h) K = k_0^{-1} (hK) = k_0^{-1} K h = Kh = hK,
\]
as far as $K$ is a normal subgroup of $H$. Conversely, if $h_1(p) = p$ for some $h_1 \in H \setminus K$ then $K_p = Kh_1(p) = h_1 K(p)$ and the point $\text{Orb}_K(p) \in A/K$ is fixed by $h_1 K \in H/K$.

Note that the presence of a covering $\zeta : A/K \to A/H$ by a (singular) K3 model $A/K$ implies the vanishing $q(X) = h^{1,0}(X)$ of the irregularity of any smooth model $X$ of $A/H$, as far as $q(X) \leq q(Y) = 0$ for any smooth $H/K$-Galois cover $Y$ of $X$, birational to $A/K$. The smooth projective surfaces $S$ with irregularity $q(S) = 0$ and Kodaira dimension $\kappa(S) \leq 0$ are the rational, K3 and Enriques $S$. Due to $L(h) \neq 1$, the smooth model $X$ of $A/H$ is not a K3 surface. Thus, $X$ is either an Enriques or a rational surface.

If $\text{Fix}_A(H \setminus K) = \emptyset$ and $\zeta : A/K \to A/H$ in unramified, then $\kappa(X) = \kappa(Y) = 0$ by [10] and $X$ is an Enriques surface.

Let us assume that $\text{Fix}_A(H \setminus K) \neq \emptyset$ and the minimal resolution $Y$ of the singularities of $A/H$ is an Enriques surface. Consider the minimal resolution $\rho_1 : Y \to A/K$
of the singularities of $A/K$ and the resolution $\nu_2 : X_2 \to A/H$ of $\zeta(A/H)^{\text{sing}}$. Then there is a commutative diagram

$$
\begin{array}{ccc}
A/K & \xrightarrow{\rho_1} & Y \\
\downarrow{\zeta} & & \downarrow{\zeta_1} \\
A/H & \xleftarrow{\nu_2} & X_2
\end{array}
$$

(3)

with $H/K$-Galois cover $\zeta_1$, ramified over the pull-back $\nu_2^{-1}B(\zeta)$ of the branch locus $B(\zeta) \subseteq A/H$ of $\zeta$. The minimal resolution $\mu_2 : X \to X_2$ of the singularities $X_2^{\text{sing}} = (A/H)^{\text{sing}} \setminus \zeta(A/K)^{\text{sing}}$ of $X_2$ and $\zeta_1 : Y \to X_2$ give rise to the fibered product commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\text{pr}_1} & Z = Y \times_{X_2} X \\
\downarrow{\zeta_1} & & \downarrow{\zeta_2} \\
X_2 & \xleftarrow{\mu_2} & X
\end{array}
$$

(4)

with ramified $H/K$-Galois cover $\zeta_2$ and birational $\text{pr}_1$. Note that $Z$ is a smooth surface, since otherwise $\emptyset \neq \text{pr}_1(Z^{\text{sing}}) \subseteq X^{\text{sing}} = \emptyset$. Moreover, $Z$ is of type K3. Let us consider the universal double covering $U_X : \tilde{X} \to X$ of $X$ by a K3 surface $\tilde{X}$. Since $Z$ is simply connected and $U_X : \tilde{X} \to X$ is unramified, the finite cover $\zeta_2 : Z \to X$ lifts to a morphism $\tilde{\zeta} : Z \tilde{X}$, closing the commutative diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{U_X} & X \\
\downarrow{\tilde{\zeta}} & & \downarrow{\zeta_2} \\
Z & & X
\end{array}
$$

(5)

The finite ramified morphism $\zeta_2 = U_X \tilde{\zeta}$ has finite ramified factor $\tilde{\zeta}$, as far as the universal covering $U_X : \tilde{X} \to X$ is unramified. If $B(\tilde{\zeta}) \subseteq Z$ is the branch locus of $\tilde{\zeta}$ then the canonical divisor

$$
O_Z = K_Z = \tilde{\zeta}^*K_{\tilde{X}} + B(\tilde{\zeta}) = \tilde{\zeta}^*O_{\tilde{X}} + B(\tilde{\zeta}),
$$

which is an absurd. Therefore, $\text{Fix}_A(H \setminus K) \neq \emptyset$ implies that $A/H$ is a rational surface.

If $\zeta : A/K \to A/H$ is unramified and $A/H$ is an Enriques surface then $\zeta_1 : Y \to X_2$ from diagram (3) and $\zeta_2 : Z \to X$ from (4) are unramified. As a result, $\tilde{\zeta} : Z \to \tilde{X}$ from diagram (5) is a finite ramified cover of smooth simply connected surfaces,
whereas \( \deg(\tilde{\zeta}) = 1 \) and \( Z \) coincides with the universal cover \( \tilde{X} \) of \( X \). Thus, \( \tilde{X} \) is birational to \( A/K \) and

\[
\deg(\zeta) = \deg(\zeta_1) = \deg(\zeta_2) = \deg(U_X) = 2,
\]

so that \([H : K] = |H/K| = \deg(\zeta) = 2\). \( \square \)

By the very construction, the surfaces \( A/H \) and \( \overline{B/\Gamma_H} = (\overline{B/\Gamma})' / H \) are simultaneously singular. The classical work [5] of Chevalley establishes that \( A/H \) is singular if and only if there is \( h \in H \), whose linear part \( L(h) \in GL(2, R) \) has eigenvalues \( \{\lambda_1 L(h), \lambda_2 L(h)\} \not\in 1 \). Thus, \( A/H \) and \( \overline{B/\Gamma_H} \) are smooth exactly when birational to a hyper-elliptic or a ruled surface with an elliptic base.

Let \( T_i \) be an irreducible component of \( T = (\overline{B/\Gamma})' \setminus (\overline{B/\Gamma}) \) of \( \overline{B/\Gamma} \). Then the irreducible component \( Orb_H(T_i)/H \) of \( T/H = \left( \overline{B/\Gamma_H} \right) \setminus (\overline{B/\Gamma_H}) \) is elliptic (respectively, rational) if and only if \( Fix_A(H) \cap D_i = \emptyset \) (respectively, \( Fix_A(H) \cap D_i \neq \emptyset \)) for the image \( D_i = \xi(T_i) \) of \( T_i \) under the blow-down \( \xi : (\overline{B/\Gamma})' \to A \) of the \((-1)\)-curves.
2 Linear automorphisms of finite order

Throughout this section, let $R$ be the endomorphism ring of an elliptic curve $E$. It is well known that $R = \mathbb{Z} + f\mathcal{O}_{-d}$ for a natural number $f \in \mathbb{N}$, called the conductor of $E$ and integers ring $\mathcal{O}_{-d}$ of an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$. More precisely, $\mathcal{O}_{-d} = \mathbb{Z} + \omega_{-d}\mathbb{Z}$ with

$$\omega_{-d} = \begin{cases} \sqrt{-d} & \text{for } -d \not\equiv 1 \pmod{4}, \\ \frac{1 + \sqrt{-d}}{2} & \text{for } -d \equiv 1 \pmod{4}. \end{cases}$$

and $R = \mathbb{Z} + f\omega_{-d}\mathbb{Z}$ for $R \not\equiv \mathbb{Z}$. In particular, $R$ is a subring of $\mathbb{Q}(\sqrt{-d})$. We write $R \subset \mathbb{Q}(\sqrt{-d})$ both, for the case of $R = \mathbb{Z} + f\omega_{-d}\mathbb{Z}$ or $R = \mathbb{Z}$, without specifying the presence of a complex multiplication on $E$. (For $R = \mathbb{Z}$ one hat $R \subset \mathbb{Q}(\sqrt{-d})$ for $\forall d \in \mathbb{N}$.)

The automorphism group of the abelian surface $A = E \times E$ is a semi-direct product

$$\text{Aut}(A) = (A, +) \rtimes \text{GL}(2, R)$$

of its translation subgroup $(A, +)$ and the isotropy group

$$\text{Aut}_{\partial A}(A) = \text{GL}(2, R) = \{ g \in R_{2x2} \mid \det(g) \in R^* \}$$

of the origin $\partial A \in A$.

**Lemma 8.** Let $R$ be the endomorphism ring of an elliptic curve $E$. If $R$ is different from $\mathcal{O}_{-1} = \mathbb{Z}[i]$ and $\mathcal{O}_{-3}$ then

$$R^* = \langle -1 \rangle = \{ \pm 1 \} = \mathbb{C}_2$$

is the cyclic group of the square roots of the unity.

If $R = \mathbb{Z}[i]$ is the ring of the Gaussian integers then

$$R^* = \langle i \rangle = \{ \pm 1, \pm i \} = \mathbb{C}_4$$

is the cyclic group of the roots of unity of order 4.

The units group of Eisenstein integers $R = \mathcal{O}_{-3}$ is the cyclic group

$$R^* = \langle e^{\frac{2\pi i}{3}} \rangle = \{ \pm 1, \ e^{\frac{2\pi i}{3}}, \ e^{\frac{4\pi i}{3}} \} = \mathbb{C}_6$$

of the sixth roots of unity.

**Proof.** Recall that the units group $\mathcal{O}^*_{-d}$ of the integers ring $\mathcal{O}_{-d}$ of an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ is

$$\mathcal{O}^*_{-d} = \langle -1 \rangle \simeq \mathbb{C}_2 \quad \text{for } d \not\equiv 1, 3 \quad \text{and}$$

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\[O_{-1}^* = \mathbb{Z}[i]^* = \langle i \rangle = \mathbb{C}_4,\]
\[O_{-3}^* = \langle e^{\frac{2\pi i}{6}} \rangle = \mathbb{C}_6.\]

The units group \(R^*\) of the subring \(R = \mathbb{Z} + fO_{-d}\) of \(O_{-d}\) is a subgroup of \(O_{-d}^*\), so that \(R^* = \langle -1 \rangle \simeq \mathbb{C}_2\) for \(R = \mathbb{Z}\) or \(R = \mathbb{Z} + fO_{-d}\) with \(d \in \mathbb{N} \setminus \{1, 3\}, f \in \mathbb{N}\). In the case of \(R = \mathbb{Z} + fO_{-1}\), the assumption \(i \in R^*\) implies \(R = O_{-1}\) and happens only for the conductor \(f = 1\). Similarly, the existence of \(e^{\frac{2\pi i}{3}} \in R^* \setminus \{\pm 1\}\) for \(R = \mathbb{Z} + fO_{-3}\) forces
\[e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}i}{2} = -1 + \frac{1 + \sqrt{-3}}{2} = -1 + \omega_{-3} \in R^*,\]
whereas \(\omega_{-3} \in R\) and \(R = O_{-3}\).

Towards the description of \(g \in GL(2, R)\) of finite order, let us recall that the polynomials
\[f(x) = x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n \in \mathbb{Z}[x]\]
with leading coefficient 1 are called monic.

**Definition 9.** If \(A\) is a subring with unity of a ring \(B\) then \(b \in B\) is integral over \(A\) if \(b\) annihilates a monic polynomial
\[f(x) = x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n \in A[x]\]
with coefficients from \(A\).

It is well known (cf. [2]) that \(b \in B\) is integral over \(A\) if and only if the polynomial ring \(A[b]\) is a finitely generated \(A\)-module.

**Definition 10.** The complex numbers \(c \in \mathbb{C}\), which are integral over \(\mathbb{Z}\) are called algebraic integers.

Any algebraic integer \(c\) is algebraic over \(\mathbb{Q}\). If \(g(x) \in \mathbb{Q}[x] \setminus \mathbb{Q}\) is a polynomial of minimal degree \(k\) with a root \(c\) then \(g(x)\) divides any \(h(x) \in \mathbb{Q}[x] \setminus \mathbb{Q}\) with \(h(c) = 0\). An arbitrary \(g'(x) \in \mathbb{Q}[x]\) of degree \(k\) with a root \(c\) is of the form \(g'(x) = qg(x)\) for some \(\mathbb{Q}^*\). The polynomials \(qg(x)\) with arbitrary \(q \in \mathbb{Q}^*\) are referred to as minimal polynomials of \(c\) over \(\mathbb{Q}\). If \(c\) is algebraic over \(\mathbb{Q}\) then the ring of the polynomials \(\mathbb{Q}[c]\) of \(c\) with rational coefficients coincides with the field \(\mathbb{Q}(c)\) of the rational functions of \(c\), \(\mathbb{Q}[c] = \mathbb{Q}(c)\) and the degree \([\mathbb{Q}(c) : \mathbb{Q}]\) equals the degree of a minimal polynomial of \(c\) over \(\mathbb{Q}\).

**Definition 11.** If \(c \in \mathbb{C}\) is algebraic over \(\mathbb{Q}\), then \([\mathbb{Q}(c) : \mathbb{Q}] = \dim_{\mathbb{Q}} \mathbb{Q}(c)\) is called the degree of \(c\) over \(\mathbb{Q}\).
Let \( c \) be an algebraic integer and \( f(x) \in \mathbb{Z}[x] \setminus \mathbb{Z} \) be a monic polynomial of minimal degree with a root \( c \). Then any \( h(x) \in \mathbb{Z}[x] \) with \( h(c) = 0 \) is divisible by \( f(x) \). Thus, \( f(x) \) is unique and referred to as the minimal integral relation of \( c \). If \( f(x) \in \mathbb{Z}[x] \setminus \mathbb{Z} \) is the minimal integral relation of \( c \in \mathbb{C} \) and \( g(x) \in \mathbb{Q}[x] \setminus \mathbb{Q} \) is a minimal polynomial of \( c \) over \( \mathbb{Q} \), then \( g(x) = a f(x) \) for the leading coefficient \( a = \text{LC}(g) \in \mathbb{Q}^* \) of \( g(x) \). More precisely, \( g(x) \) divides \( f(x) \) and \( f(x) \) is indecomposable over \( \mathbb{Q} \), as far as it is indecomposable over \( \mathbb{Z} \). In such a way, one obtains the following

**Lemma 12.** If \( c \in \mathbb{C} \) is an algebraic integer, then the degree \( \deg_{\mathbb{Q}}(c) = [\mathbb{Q}(c) : \mathbb{Q}] \) of \( c \) over \( \mathbb{Q} \) equals the degree of the minimal integral relation

\[
f(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n \in \mathbb{Z}[x] \quad \text{of} \quad c.
\]

**Lemma 13.** Let \( E \) be an elliptic curve, \( R = \text{End}(E) \) and \( g \in \text{GL}(2, R) \). Then any eigenvalue \( \lambda_1 \) of \( g \) is an algebraic integer of degree 1, 2 or 4 over \( \mathbb{Q} \).

**Proof.** It suffices to observe that if \( A \subset B \) are subrings with unity of a ring \( C \), \( A \) is a Noetherian ring, \( B \) is a finitely generated \( A \)-module and \( c \in C \) is integral over \( B \), then \( c \) is integral over \( A \). Indeed, let \( f \in \mathbb{N} \) be the conductor of \( E \) and

\[
\omega_{-d} = \begin{cases} \sqrt{-d} & \text{for } -d \not\equiv 1(\text{mod}4), \\ \frac{1+\sqrt{-d}}{2} & \text{for } -d \equiv 1(\text{mod}4). \end{cases}
\]

Then the integers ring \( \mathbb{Z} \) is Noetherian and the endomorphism ring

\[
R = \mathbb{Z} + f\mathcal{O}_{-d} = \mathbb{Z} + f\omega_{-d}\mathbb{Z}
\]

of \( E \) is a free \( \mathbb{Z} \)-module of rank 2. The eigenvalue \( \lambda_1 \in \mathbb{C} \) of \( g \in \text{GL}(2, R) \) is a root of the characteristic polynomial

\[
\mathcal{X}_g(\lambda) = \lambda^2 - \text{tr}(g)\lambda + \text{det}(g) \in R[\lambda]
\]

of \( g \), so that \( \lambda_1 \) is integral over \( R \). According to the claim, \( \lambda_1 \) is integral over \( \mathbb{Z} \) or \( \lambda_1 \in \mathbb{C} \) is an algebraic integer. On one hand, the degree of \( \lambda_1 \) over \( \mathbb{Q}(\sqrt{-d}) \) is

\[
\deg_{\mathbb{Q}(\sqrt{-d})}(\lambda_1) = [\mathbb{Q}(\sqrt{-d}, \lambda_1) : \mathbb{Q}(\sqrt{-d})] = 1 \quad \text{or} \quad 2,
\]

so that

\[
[\mathbb{Q}(\sqrt{-d}, \lambda_1) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{-d}, \lambda_1) : \mathbb{Q}(\sqrt{-d})][\mathbb{Q}(\sqrt{-d}) : \mathbb{Q}] = 2 \quad \text{or} \quad 4.
\]

On the other hand, the inclusions

\[
\mathbb{Q} \subseteq \mathbb{Q}(\lambda_1) \subseteq \mathbb{Q}(\sqrt{-d}, \lambda_1)
\]

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of subfields imply that

\[ [\mathbb{Q}(\lambda_1) : \mathbb{Q}] = \frac{[\mathbb{Q}(\sqrt{-d}, \lambda_1) : \mathbb{Q}]}{[\mathbb{Q}(\sqrt{-d}, \lambda_1) : \mathbb{Q}(\lambda_1)]}. \]

Therefore, the degree \( \deg_{\mathbb{Q}}(\lambda_1) = [\mathbb{Q}(\lambda_1) : \mathbb{Q}] \) of \( \lambda_1 \) over \( \mathbb{Q} \) is a divisor of the degree \( [\mathbb{Q}(\sqrt{-d}, \lambda_1) : \mathbb{Q}] \) or \( \deg_{\mathbb{Q}}(\lambda_1) \in \{1, 2, 4\} \).

In order to justify the claim, recall that \( c \in \mathbb{C} \) is integral over \( \mathbb{B} \) if and only if the polynomial ring \( \mathbb{B}[c] = \mathbb{B} + \mathbb{B}c + \ldots + \mathbb{B}c^{n-1} \) is a finitely generated \( \mathbb{B} \)-module. If \( \mathbb{B} = \mathbb{A}\beta_1 + \ldots + \mathbb{A}\beta_s \) is a finitely generated \( \mathbb{A} \)-module, then

\[ B[c] = \sum_{i=1}^{s} \sum_{j=0}^{n-1} \mathbb{A}\beta_ic^j \]

is a finitely generated \( \mathbb{A} \)-module. Since \( \mathbb{A} \) is a Noetherian ring, the \( \mathbb{A} \)-submodule \( \mathbb{A}[c] \) of \( B[c] \) is a finitely generated \( \mathbb{A} \)-module.

\[ \blacksquare \]

Note that if \( h = \tau_{(U,V)} \mathcal{L}(h) \in H \leq \text{Aut}(\mathbb{A}) \) is an automorphism of \( \mathbb{A} = E \times E \) of finite order \( r \) then

\[ h^r = \tau_{r-1} \sum_{s=0}^{r-1} \mathcal{L}(h)^s (\frac{U}{V}) \mathcal{L}(h)^r = I_d \]

implies that \( \sum_{s=0}^{r-1} \mathcal{L}(h)^s (\frac{U}{V}) = \delta_A \) and \( \mathcal{L}(h)^r = I_2 \). In other words, the automorphisms \( h \in \text{Aut}(\mathbb{A}) \) of finite order have linear parts \( \mathcal{L}(h) \in \text{GL}(2, R) \) of finite order.

From now on, we concentrate on \( g \in \text{GL}(2, R) \) of finite order.

**Proposition 14.** If \( R \) is the endomorphism ring of an elliptic curve \( E \) and \( g \in \text{GL}(2, R) \) is of finite order \( r \), then \( g \) is diagonalizable and the eigenvalues \( \lambda_j \) of \( g \) are primitive roots of unity of degree \( r_j = 1, 2, 3, 4, 6, 8 \) or 12.

**Proof.** Let us assume that \( g \in \text{GL}(2, R) \) of finite order \( r \) is not diagonalizable. Then there exists \( S \in \text{GL}(2, \mathbb{C}) \), reducing \( g \) to its Jordan normal form

\[ J = S^{-1}gS = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}. \]

By an induction on \( n \), one verifies that

\[ J^n = \begin{pmatrix} \lambda_1^n & (n-1)\lambda_1^{n-1} \\ 0 & \lambda_1^n \end{pmatrix} \quad \text{for} \quad \forall n \in \mathbb{N}. \]

In particular,

\[ I_2 = S^{-1}I_2S = S^{-1}g^rS = (S^{-1}gS)^r = J^r = \begin{pmatrix} \lambda_1^r & (r-1)\lambda_1^{r-1} \\ 0 & \lambda_1^r \end{pmatrix} \]

\[ J = S^{-1}gS = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}. \]

By an induction on \( n \), one verifies that

\[ J^n = \begin{pmatrix} \lambda_1^n & (n-1)\lambda_1^{n-1} \\ 0 & \lambda_1^n \end{pmatrix} \quad \text{for} \quad \forall n \in \mathbb{N}. \]

In particular,

\[ I_2 = S^{-1}I_2S = S^{-1}g^rS = (S^{-1}gS)^r = J^r = \begin{pmatrix} \lambda_1^r & (r-1)\lambda_1^{r-1} \\ 0 & \lambda_1^r \end{pmatrix} \]

\[ J = S^{-1}gS = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}. \]

By an induction on \( n \), one verifies that

\[ J^n = \begin{pmatrix} \lambda_1^n & (n-1)\lambda_1^{n-1} \\ 0 & \lambda_1^n \end{pmatrix} \quad \text{for} \quad \forall n \in \mathbb{N}. \]

In particular,

\[ I_2 = S^{-1}I_2S = S^{-1}g^rS = (S^{-1}gS)^r = J^r = \begin{pmatrix} \lambda_1^r & (r-1)\lambda_1^{r-1} \\ 0 & \lambda_1^r \end{pmatrix} \]

\[ J = S^{-1}gS = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}. \]
is an absurd, justifying the diagonalizability of \( g \).

If

\[
D = S^{-1}gS = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}
\]

is a diagonal form of \( g \) then

\[
I_2 = S^{-1}I_2S = S^{-1}g^rS = (S^{-1}gS)^r = \begin{pmatrix} \lambda_1^r & 0 \\ 0 & \lambda_2^r \end{pmatrix}
\]

reveals that \( \lambda_1 \) and \( \lambda_2 \) are \( r \)-th roots of unity.

Thus, \( \lambda_j \) are of finite order \( r_j \), dividing \( r \) and the least common multiple \( m = LCM(r_1, r_2) \in \mathbb{N} \) divides \( r \). Conversely,

\[
I_2 = \begin{pmatrix} \lambda_1^m & 0 \\ 0 & \lambda_2^m \end{pmatrix} = (S^{-1}gS)^m = S^{-1}g^mS
\]

implies that \( g^m = SI_2S^{-1} = I_2 \), so that \( r \in \mathbb{N} \) divides \( m \in \mathbb{N} \) and \( r = m \).

Let \( \lambda_j \in \mathbb{C}^* \) be a primitive \( r_j \)-th root of unity. Then the cyclotomic polynomials \( \Phi_{r_j}(x) \in \mathbb{Z}[x] \) are the minimal integral relations of \( \lambda_j \). More precisely, the minimal integral relations \( f_j(x) \in \mathbb{Z}[x] \setminus \mathbb{Z} \) of \( \lambda_j \) are monic polynomials of degree \( \deg_{\mathbb{Q}}(\lambda_j) \).

On the other hand, \( \Phi_{r_j}(x) \in \mathbb{Z}[x] \setminus \mathbb{Z} \) are irreducible over \( \mathbb{Z} \) and \( \mathbb{Q} \). Therefore \( \Psi_{r_j}(x) \) are minimal polynomials of \( \lambda_j \) over \( \mathbb{Q} \) and \( \Psi_{r_j}(x) = qf_j(x) \) for some \( q \in \mathbb{Q}^* \). As far as \( \Phi_{r_j}(x) \) and \( f_j(x) \) are monic, there follows \( q = 1 \) and \( \Phi_{r_j}(x) \equiv f_j(x) \in \mathbb{Z}[x] \).

Recall Euler’s function

\[
\varphi : \mathbb{N} \rightarrow \mathbb{N},
\]

associating to each \( n \in \mathbb{N} \) the number of the residues \( 0 \leq r \leq n - 1 \) modulo \( n \), which are relatively prime to \( n \). The degree of \( \Phi_{r_j}(x) \) is \( \varphi(r_j) \). If \( r_j = p_1^{a_1} \cdots p_m^{a_m} \) is the unique factorization of \( r_j \in \mathbb{N} \) into a product of different prime numbers \( p_s \), then

\[
\varphi(p_1^{a_1} \cdots p_m^{a_m}) = \varphi(p_1^{a_1}) \cdots \varphi(p_m^{a_m}) = p_1^{a_1-1}(p_1-1) \cdots p_m^{a_m-1}(p_m-1).
\]

According to Lemma 13, the algebraic integers \( \lambda_j \) are of degree

\[
\deg_{\mathbb{Q}}(\lambda_j) = \deg \Phi_{r_j}(x) = \varphi(r_j) = 1, 2, \text{ or } 4.
\]

If \( r_j \) has a prime divisor \( p \geq 7 \) then \( \varphi(r_j) \) has a factor \( p-1 \geq 6 \), so that \( \varphi(r_j) > 4 \).

Therefore \( r_j = 2^a3^b5^c \) for some non-negative integers \( a, b, c \). If \( c \geq 1 \) then

\[
\varphi(r_j) = \varphi(2^a3^b)\varphi(5^c) = \varphi(2^a3^b)5^{c-1} \cdot 4 \in \{1, 2, 4\}
\]

exactly when \( \varphi(r_j) = 4 \), \( c = 1 \) and \( \varphi(2^a3^b) = 1 \). For \( b \geq 1 \) one has

\[
\varphi(2^a3^b) = \varphi(2^a)3^{b-1} \cdot 2 > 1,
\]
so that \( \varphi(2^a3^b) = 1 \) requires \( b = 0 \) and \( \varphi(2^a) = 1 \). As a result, \( a = 0 \) or 1 and \( r_j = 5 \) or 10, if 5 divides \( r_j \). From now on, let us assume that \( r_j = 2^a3^b \) with \( a, b \in \mathbb{N} \cup \{0\} \).

If \( b \geq 2 \) then \( \varphi(r_j) = \varphi(2^a).3^{b-1}.2 \) with \( b - 1 \geq 1 \) is divisible by 3 and cannot equal 1, 2 or 4. Therefore \( r_j = 2^a.3 \) or \( r_j = 2^a \) with \( a \geq 0 \). Straightforwardly,

\[
\varphi(2^a.3) = 2\varphi(2^a) \in \{1, 2, 4\}
\]

exactly when \( \varphi(2^a) = 1 \) or \( \varphi(2^a) = 2 \). These amount to \( a \in \{0, 1, 2\} \) and reveal that 3, 6, 12 are possible values for \( r_j \). Finally, \( \varphi(r_j) = \varphi(2^a) \in \{1, 2, 4\} \) for \( r_j = 1, 2, 4 \) or 8. Thus, \( \varphi(r_j) \in \{1, 2, 4\} \) if and only if

\[
r_j \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}.
\]

In order to exclude \( r_j = 5 \) and \( r_j = 10 \) with \( \varphi(5) = \varphi(10) = 4 \), recall that \( \lambda_j \) is of degree \( \deg_{\mathbb{Q}(\sqrt{-d})}(\lambda_j) = \lceil \mathbb{Q}(\sqrt{-d}, \lambda_j) : \mathbb{Q}(\sqrt{-d}) \rceil \leq 2 \) over \( \mathbb{Q}(\sqrt{-d}) \), so that

\[
[\mathbb{Q}(\sqrt{-d}, \lambda_j) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{-d}, \lambda_j) : \mathbb{Q}(\lambda_j)] [\mathbb{Q}(\lambda_j) : \mathbb{Q}] = 4 [\mathbb{Q}(\sqrt{-d}, \lambda_j) : \mathbb{Q}(\lambda_j)] \geq 4,
\]

whereas \( [\mathbb{Q}(\sqrt{-d}, \lambda_j) : \mathbb{Q}] = [\mathbb{Q}(\lambda_j) : \mathbb{Q}] = 4 \) and \( [\mathbb{Q}(\sqrt{-d}, \lambda_j) : \mathbb{Q}(\lambda_j)] = 1 \). Therefore \( \mathbb{Q}(\sqrt{-d}, \lambda_j) = \mathbb{Q}(\lambda_j) \), so that \( \sqrt{-d} \in \mathbb{Q}(\lambda_j) \) and \( \mathbb{Q}(\sqrt{-d}) \subset \mathbb{Q}(\lambda_j) \) with

\[
[\mathbb{Q}(\lambda_j) : \mathbb{Q}(\sqrt{-d})] = \frac{[\mathbb{Q}(\lambda_j) : \mathbb{Q}]}{[\mathbb{Q}(\sqrt{-d}) : \mathbb{Q}]} = \frac{4}{2} = 2.
\]

As far as \( \mathbb{Q}(\sqrt{-d}) \) and \( \mathbb{Q}(\lambda_j) \) are finite Galois extensions of \( \mathbb{Q} \) (i.e., normal and separable), the subfield \( \mathbb{Q}(\sqrt{-d}) \) of \( \mathbb{Q}(\lambda_j) \) of index \( [\mathbb{Q}(\lambda_j) : \mathbb{Q}(\sqrt{-d})] = 2 \) is the fixed point set of a subgroup \( H \) of the Galois group \( \text{Gal}(\mathbb{Q}(\lambda_j)/\mathbb{Q}) \) with \( |H| = 2 \). The minimal polynomial of \( \lambda_j \) over \( \mathbb{Q} \) is the cyclotomic polynomial \( \Phi_{r_j}(x) \in \mathbb{Z}[x] \) of degree \( \deg(\Phi_{r_j}) = \varphi(r_j) = 4 \) for \( r_j \in \{5, 10\} \) and the Galois group

\[
\text{Gal}(\mathbb{Q}(\lambda_j)/\mathbb{Q}) \simeq \mathbb{Z}_{r_j}^*
\]

coincides with the multiplicative group \( \mathbb{Z}_{r_j}^* \) of the congruence ring \( \mathbb{Z}_{r_j} \) modulo \( r_j \). More precisely, the roots of \( \Phi_{r_j}(x) \) are \( \{\lambda_j^s \mid s \in \mathbb{Z}_{r_j}^*\} \) and for any \( s \in \mathbb{Z}_{r_j}^* \) the correspondence \( \lambda_j \mapsto \lambda_j^s \) extends to an automorphism of \( \mathbb{Q}(\lambda_j) \), fixing \( \mathbb{Q} \). The groups

\[
\mathbb{Z}_5^* = \{\pm 1(\text{mod}5), \pm 3(\text{mod}5)\} = \langle 3(\text{mod}5) \rangle = \langle -3(\text{mod}5) \rangle \simeq \mathbb{C}_4
\]

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and
\[ \mathbb{Z}_{10}^* = \{ \pm 1 (\text{mod} 10), \pm 3 (\text{mod} 10) \} = \langle 3 (\text{mod} 10) \rangle = \langle -3 (\text{mod} 10) \rangle \simeq \mathbb{C}_4 \]
are cyclic and contain unique subgroups \( H_5 = \langle -1 (\text{mod} 5) \rangle \), respectively, \( H_{10} = \langle -1 (\text{mod} 10) \rangle \) or order 2. Denote by \( h \) the generator of \( H_5 \) or \( H_{10} \) with \( h(\lambda_j) = \lambda_j^{-1} \), \( h | Q = Id_Q \). In both cases, the degree
\[ \deg_{\mathbb{Q}(\sqrt{-d})}(\lambda_j) = [\mathbb{Q}(\lambda_j, \sqrt{-d}) : \mathbb{Q}(\sqrt{-d})] = [\mathbb{Q}(\lambda_j) : \mathbb{Q}(\sqrt{-d})] = 2, \]
so that the characteristic polynomial
\[ \chi_g(\lambda) = \lambda^2 - \text{tr}(g) \lambda + \det(g) \in R[\lambda] \subset \mathbb{Q}(\sqrt{-d})[\lambda] \]
of \( g \) is irreducible over \( \mathbb{Q}(\sqrt{-d}) \). In fact, \( \chi_g(\lambda) \) is a minimal polynomial of \( \lambda_j \) over \( \mathbb{Q}(\sqrt{-d}) \) and divides the cyclotomic polynomial \( \Phi_{r_j}(\lambda) \in \mathbb{Z}[\lambda] \subset \mathbb{Q}(\sqrt{-d})[\lambda] \) with \( \Phi_{r_j}(\lambda_j) = 0 \). In particular, the other eigenvalue \( \lambda_{3 - j} \) of \( g \) is a root of \( \Phi_{r_j}(\lambda) \) or a primitive \( r_j \)-th root of unity. That allows to express \( \lambda_{3 - j} = \lambda_j^t \) by some \( t \in \mathbb{Z}_{r_j}^* \).

According to
\[ \lambda_j^{t+1} = \lambda_j \lambda_j^t = \lambda_j \lambda_{3 - j} = \det(g) \in R^* \subset \mathbb{Q}(\sqrt{-d}) = \mathbb{Q}(\lambda_j)^{(h)}, \]
one has
\[ \lambda_j^{t+1} = h(\lambda_j^{t+1}) = \lambda_j^{t-1} \quad \text{or} \quad \lambda_j^{2(t+1)} = 1. \]
If \( \lambda_j \) is a primitive fifth root of unity then \( \lambda_j^{2(t+1)} = 1 \) requires that \( 2(t + 1) \) to be divisible by 5. Since \( GCD(2, 5) = 1 \), 5 is to divide \( t + 1 \) or \( t \equiv -1 (\text{mod} 5) \). Similarly, if \( \lambda_j \) is a primitive tenth root of unity then 10 divides \( 2(t + 1) \), i.e., \( 2(t + 1) = 10z \) for some \( z \in \mathbb{Z} \). As a result, 5 divides \( t + 1 \) and \( t \equiv -1 (\text{mod} 10) \). Thus, for any \( r_1 \in \{ 5, 10 \} \) there follows \( \lambda_{3 - j} = \lambda_j^t = \lambda_j^{-1} \).

Expressing \( \lambda_j = e^{2\pi i s / r_j} \) for some natural number \( 1 \leq s \leq r_j - 1 \), relatively prime to \( r_j \), one observes that
\[ \text{tr}(g) = \lambda_j + \lambda_{3 - j} = \lambda_j + \lambda_j^{-1} = e^{2\pi i s / r_j} + e^{-2\pi i s / r_j} = 2 \cos \left( \frac{2\pi s}{r_j} \right) \in R \cap \mathbb{R}. \]

We claim that \( R \cap \mathbb{R} = \mathbb{Z} \). The inclusion \( \mathbb{Z} \subseteq R \cap \mathbb{R} \) is clear. Conversely, let
\[ r \in \mathbb{R} \cap R = \mathbb{R} \cap (\mathbb{Z} + f \omega_{-d} \mathbb{Z}) \]
for the conductor \( f \in \mathbb{N} \) of \( E \) and \( \omega_{-d} \) from (6). In the case of \( -d \not\equiv 1 (\text{mod} 4) \) there exist \( a, b \in \mathbb{Z} \) with \( r = a + f \sqrt{-db} \). The complex number \( a - r + f \sqrt{-db} = 0 \) vanishes exactly when its real part \( a - r \) and its imaginary part \( f \sqrt{db} = 0 \) are zero. Therefore \( b = 0 \) and \( r = a \in \mathbb{Z} \), i.e., \( \mathbb{R} \cap R \subseteq \mathbb{Z} \) for \( -d \not\equiv 1 (\text{mod} 4) \).

If \( -d \equiv 1 (\text{mod} 4) \) then
\[ r = a + fb \left( \frac{1 + \sqrt{-d}}{2} \right) \quad \text{for some} \quad a, b \in \mathbb{Z} \]
yields
\[ r = a + \frac{\sqrt{3}}{2}b \]
\[ \frac{\sqrt{3}}{2}b = 0 \]
by comparison of the real and imaginary parts. As a result, again \( b = 0 \) and \( r = a \in \mathbb{Z} \), i.e., \( \mathbb{R} \cap R \subseteq \mathbb{Z} \) for \(-d \equiv 1 \text{ (mod } 4\)). That justifies \( \mathbb{R} \cap R = \mathbb{Z} \) and implies that \( \text{tr}(g) = 2 \cos \left( \frac{2 \pi s}{r_j} \right) \in \mathbb{Z} \). Bearing in mind the \( \cos \left( \frac{2 \pi s}{r_j} \right) \in [-1, 1] \), one concludes
\[ \text{tr}(g) = 2 \cos \left( \frac{2 \pi s}{r_j} \right) \in [-2, 2] \cap \mathbb{Z} = \{0, \pm 1, \pm 2\} \quad \text{or} \quad (7) \]
\[ \cos \left( \frac{2 \pi s}{r_j} \right) \in \left\{0, \pm \frac{1}{2}, \pm 1\right\}. \]
For a natural number \( 1 \leq s \leq r_j-1 \), one has \( \frac{2 \pi s}{r_j} \in [0, 2\pi) \). The solutions of \( \cos(x) = 0 \) in \( [0, 2\pi) \) are \( \frac{\pi}{2} \) and \( \frac{3\pi}{2} \), while \( \cos(x) = \pm 1 \) holds for \( x \in \{0, \pi\} \). Finally, \( \cos(x) = \pm \frac{1}{2} \) is satisfied by \( x \in \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\} \), so that (7) implies
\[ \frac{2 \pi s}{r_j} \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}. \quad (8) \]
For \( r_j = 5 \) or 10 this is an absurd, so that
\[ r_j \in \{1, 2, 3, 4, 6, 8, 12\}. \]
\[ \square \]

Now we are ready to describe the elements of \( GL(2, R) \) of finite order, by specifying their eigenvalues \( \lambda_1, \lambda_2 \). The roots \( \lambda_1, \lambda_2 \) of the characteristic polynomial
\[ \chi_g(\lambda) = \lambda^2 - \text{tr}(g)\lambda + \text{det}(g) \in R[\lambda] \]
of \( g \) are in a bijective correspondence with the trace \( \text{tr}(g) = \lambda_1 + \lambda_2 \in R \) and the determinant \( \text{det}(g) = \lambda_1\lambda_2 \in R^* \) of \( g \). Making use of Lemma 8, we subdivide the problem to the description of finite order \( g \in GL(2, R) \) with a fixed determinant \( \text{det}(g) \in R^* \). The traces of such \( g \) take finitely many values and allow to list explicitly the eigenvalues of all \( g \in GL(2, R) \) of finite order. The classification of the unordered pairs of eigenvalues \( \lambda_1, \lambda_2 \) of \( g \in GL(2, R) \) of finite order is a more specific result than Proposition 14. Note that the next classification of \( \lambda_1, \lambda_2 \) is derived independently of Proposition 14.

Let us start with the case of \( \text{det}(g) = 1 \). The next proposition puts in a bijective correspondence the traces \( \text{tr}(g) \) of \( g \in SL(2, R) \) with the orders \( r \) of \( g \).
Proposition 15. If \( g \in SL(2, R) \) is of finite order \( r \) then the trace

\[
\text{tr}(g) \in \{ \pm 2, \pm 1, 0 \}. \tag{9}
\]

The eigenvalues \( \lambda_1, \lambda_2 \) of \( g \) are of order

\[
r_1 = r_2 = r \in \{ 1, 2, 3, 4, 6 \}. \tag{10}
\]

More precisely,

(i) \( \text{tr}(g) = 2 \) or \( \lambda_1 = \lambda_2 = 1, g = I_2 \) if and only if \( g \) is of order 1;

(ii) \( \text{tr}(g) = -2 \) or \( \lambda_1 = \lambda_2 = -1, g = -I_2 \) if and only if \( g \) is of order 2;

(iii) \( \text{tr}(g) = 1 \) or \( \lambda_1 = e^{\frac{2\pi i}{r_1}}, \lambda_2 = e^{-\frac{2\pi i}{r_1}} \) if and only if \( g \) is of order 6;

(iv) \( \text{tr}(g) = -1 \) or \( \lambda_1 = e^{\frac{2\pi i}{r}}, \lambda_2 = e^{-\frac{2\pi i}{r}} \) if and only if \( g \) is of order 3;

(v) \( \text{tr}(g) = 0 \) or \( \lambda_1 = i, \lambda_2 = -i \) if and only if \( g \) is of order 4.

Proof. If \( g \in SL(2, R) \) is of order \( r \) then the eigenvalues \( \lambda_j \) of \( g \) are of finite order \( r_j \), dividing \( r = \text{LCM}(r_1, r_2) \). According to

\[
1 = \det(g) = \lambda_1 \lambda_2,
\]

one has \( \lambda_1 = e^{\frac{2\pi i}{r_1}}, \lambda_2 = e^{-\frac{2\pi i}{r_1}} \) for some natural number \( 1 \leq s \leq r_1 - 1 \), relatively prime to \( r_1 \). Thus, \( \lambda_2 \) is a primitive \( r_1 \)-th root and \( r_1 = r_2 = \text{LCM}(r_1, r_2) = r \). As in the proof of Proposition 14,

\[
\text{tr}(g) = \lambda_1 + \lambda_2 = e^{\frac{2\pi i}{r_1}} + e^{-\frac{2\pi i}{r_1}} = 2 \cos \left( \frac{2\pi s}{r_1} \right) \in \mathbb{R} \cap R = \mathbb{Z}
\]

and \( \cos \left( \frac{2\pi s}{r_1} \right) \in [-1, 1] \) specify (9). Consequently,

\[
\cos \left( \frac{2\pi s}{r_1} \right) \in \left\{ 0, \pm \frac{1}{2}, \pm 1 \right\} \quad \text{and}
\]

\[
\frac{2\pi s}{r_1} \in \left\{ 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \right\},
\]

as in (8). Straightforwardly, \( \lambda_1 = e^0 = 1 \) is of order 1, \( \lambda_1 = e^{\pi i} = -1 \) is of order 2, \( \lambda_1 \in \left\{ e^{\frac{2\pi i}{r_1}}, e^{\frac{4\pi i}{r_1}} \right\} \) are of order 4, \( \lambda_1 \in \left\{ e^{\frac{2\pi i}{r}}, e^{\frac{4\pi i}{r}} \right\} \) are of order 3 and \( \lambda_1 \in \left\{ e^{\frac{2\pi i}{r}}, e^{\frac{4\pi i}{r}} \right\} \) are of order 6. That justifies (10).

If \( g \) is of order \( r = 1 \) then \( \lambda_1 \in \mathbb{C}^* \) is of order \( r_1 = 1 \), so that \( \lambda_1 = 1 \). Consequently, \( \lambda_2 = 1 \) and \( g = I_2 \), as far as \( I_2 \) is the only conjugate of the scalar matrix \( I_2 \). The trace \( \text{tr}(g) = \text{tr}(I_2) = 2 \). Conversely, if \( \lambda_1 = \lambda_2 = 1 \), then \( g = I_2 \) is of order 1.

An automorphism \( g \in SL(2, R) \) of order \( r = 2 \) has eigenvalues \( \lambda_1, \lambda_2 \in \mathbb{C}^* \) of order 2, or \( \lambda_1 = \lambda_2 = -1 \). Consequently, \( g = -I_2 \) and \( \text{tr}(g) = -2 \). Conversely, for \( \lambda_1 = \lambda_2 = -1 \) the matrix \( g = -I_2 \) is of order 2.
Let us suppose that \( g \in SL(2, R) \) is of order 3. Then the eigenvalues \( \lambda_1, \lambda_2 \) of \( g \) are of order 3 or \( \lambda_1 = e^{\frac{2\pi i}{3}}, \lambda_2 = e^{-\frac{2\pi i}{3}} \), up to a transposition. The trace \( \text{tr}(g) = \lambda_1 + \lambda_2 = -1 \). Conversely, if \( \lambda_1 = e^{\frac{2\pi i}{3}}, \lambda_2 = e^{-\frac{2\pi i}{3}} \) then \( r = r_1 = r_2 = 3 \).

For \( g \in SL(2, R) \) of order 4 one has \( \lambda_1, \lambda_2 \in \mathbb{C}^* \) of order 4 or \( \lambda_1 = i, \lambda_2 = -i \), up to a transposition. The trace \( \text{tr}(g) = \lambda_1 + \lambda_2 = 0 \). Conversely, for \( \lambda_1 = i, \lambda_2 = -i \) there follows \( r = r_1 = r_2 = 4 \).

Suppose that \( g \in SL(2, R) \) is of order 6. Then \( \lambda_1, \lambda_2 \in \mathbb{C}^* \) are of order 6 or \( \lambda_1 = e^{\frac{2\pi i}{3}}, \lambda_2 = e^{-\frac{2\pi i}{3}} \), up to a transposition. The trace \( \text{tr}(g) = \lambda_1 + \lambda_2 = 1 \). Conversely, the assumption \( \lambda_1 = e^{\frac{2\pi i}{3}}, \lambda_2 = e^{-\frac{2\pi i}{3}} \) implies \( r = r_1 = r_2 = 6 \).

Note that

\[
g_1 = \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix} \in SL(2, \mathbb{Z}) \subseteq SL(2, R)
\]

with \( \text{tr}(g_1) = -1, \text{tr}(g_2) = 0, \text{tr}(g_3) = 1 \) realize all the possibilities, listed in the statement of the proposition.

\[\square\]

If \( E \) is an elliptic curve with complex multiplication by an imaginary quadratic number field \( \mathbb{Q}(\sqrt{-d}) \) and conductor \( f \in \mathbb{N} \) then we denote the endomorphism ring of \( E \) by

\[
R_{-d,f} = \mathbb{Z} + f\mathcal{O}_{-d} = \mathbb{Z} + f\omega_{-d}\mathbb{Z},
\]

where \( \omega_{-d} \) is the non-trivial generator of \( \mathcal{O}_{-d} \) as a \( \mathbb{Z} \)-module, given in (6). If \( E \) has no complex multiplication, we put

\[
R_{0,1} := \mathbb{Z}.
\]

**Proposition 16.** Let \( g \in GL(2, R_{-d,f}) \) be a linear automorphism of \( A = E \times E \) of order \( r \), with \( \det(g) = -1 \) and eigenvalues \( \lambda_1(g), \lambda_2(g) \in \mathbb{C}^* \).

(i) The automorphism \( g \) is of order 2 if and only if its trace is \( \text{tr}(g) = 0 \) or, equivalently, \( \lambda_1(g) = -1, \lambda_2(g) = 1 \).

(ii) If \( R_{-d,f} \neq \mathbb{Z}[i], \mathcal{O}_{-2}, \mathcal{O}_{-3}, R_{-3,2} \) then any \( g \in GL(2, R_{-d,f}) \setminus SL(2, R) \) is of order 2.

(iii) If \( g \in GL(2, \mathcal{O}_{-2}) \) is of order \( r > 2 \) and \( \det(g) = -1 \) then \( r = 8 \) and the trace \( \text{tr}(g) \in \{ \pm \sqrt{-2} \} \).

More precisely,

(a) \( \text{tr}(g) = \sqrt{-2} \) if and only if \( \lambda_1(g) = e^{\frac{2\pi i}{8}}, \lambda_2(g) = e^{-\frac{2\pi i}{8}} \);

(b) \( \text{tr}(g) = -\sqrt{-2} \) if and only if \( \lambda_1(g) = e^{\frac{3\pi i}{8}}, \lambda_2(g) = e^{-\frac{3\pi i}{8}} \).

(iv) If \( g \in GL(2, \mathbb{Z}[i]) \) is of order \( r > 2 \) and \( \det(g) = -1 \), then \( r \in \{4, 12\} \) and the trace \( \text{tr}(g) \in \{ \pm i, \pm 2i \} \).

More precisely,

(a) \( \text{tr}(g) = 2i \) exactly when \( g = iI_2 \);

(b) \( \text{tr}(g) = -2i \) exactly when \( g = -iI_2 \);

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(c) \( \text{tr}(g) = i \) exactly when \( \lambda_1(g) = e^{\frac{2\pi i}{3}}, \lambda_2(g) = e^{\frac{4\pi i}{3}} \);

(d) \( \text{tr}(g) = -i \) exactly when \( \lambda_1(g) = e^{\frac{2\pi i}{3}}, \lambda_2(g) = e^{-\frac{4\pi i}{3}} \).

(v) If \( g \in GL(2, R_{-3,f}) \) with \( R_{-3,f} \in \{ R_{-3,1} = O_{-3}, R_{-3,2} = Z + \sqrt{-3}Z \} \) is of order \( r > 2 \) and \( \text{det}(g) = -1 \) then \( r = 6 \) and the trace \( \text{tr}(g) \in \{ \pm \sqrt{-3} \} \).

More precisely,

(a) \( \text{tr}(g) = \sqrt{-3} \) if and only if \( \lambda_1(g) = e^{\frac{2\pi i}{3}}, \lambda_2(g) = e^{\frac{4\pi i}{3}} \);

(b) \( \text{tr}(g) = -\sqrt{-3} \) if and only if \( \lambda_1(g) = e^{-\frac{2\pi i}{3}}, \lambda_2(g) = e^{-\frac{4\pi i}{3}} \).

Proof. The eigenvalues \( \lambda_1(g), \lambda_2(g) \in C^* \) of \( g \in GL(2, R_{-d,f}) \) with \( \text{det}(g) = -1 \) are subject to \( \lambda_2(g) = -\lambda_1(g)^{-1} \). More precisely, if \( \lambda_1(g) = e^{\frac{2\pi i}{r_1}} \) is a primitive \( r_1 \)-th root of unity then \( \lambda_2(g) = e^{-\frac{2\pi i}{r_1}} \). The trace

\[
\text{tr}(g) = \lambda_1(g) + \lambda_2(g) = e^{\frac{2\pi i}{r_1}} - e^{-\frac{2\pi i}{r_1}} = 2i \sin \left( \frac{2\pi s}{r_1} \right) \in R_{-d,f} \cap i\mathbb{R}.
\] (11)

We claim that

\[
R_{-d,f} \cap i\mathbb{R} = \begin{cases} 
 f\sqrt{-d}\mathbb{Z} & \text{for } -d \not\equiv 1(\text{mod}4) \text{ or } -d \equiv 1(\text{mod}4), f \equiv 1(\text{mod}2), \\
 \frac{1}{2}f\sqrt{-d}\mathbb{Z} & \text{for } -d \equiv 1(\text{mod}4), f \equiv 0(\text{mod}2).
\end{cases}
\]

Indeed, if \( -d \not\equiv 1(\text{mod}4) \) then \( O_{-d} = Z + \sqrt{-d}Z \) and \( R_{-d,f} = Z + f\sqrt{-d}Z \) contains \( f\sqrt{-d} \), i.e., \( f\sqrt{-d}Z \subseteq R_{-d,f} \cap i\mathbb{R} \). Any \( ir = a + bf\sqrt{-d} \in i\mathbb{R} \cap R_{-d,f} \) with \( r \in \mathbb{R}, a, b \in \mathbb{Z} \) has imaginary part \( r = bf\sqrt{-d} \), so that \( i\mathbb{R} \cap R_{-d,f} \subseteq f\sqrt{-d}Z \) and \( i\mathbb{R} \cap R_{-d,f} = f\sqrt{-d}Z \).

Suppose that \( -d \equiv 1(\text{mod}4) \) and the conductor \( f = 2k + 1 \in \mathbb{N} \) is odd. Then \( R_{-d,2k+1} = Z + f(1+\sqrt{-d})Z \) contains \( f\sqrt{-d} = -f + (2f)^{\frac{1}{2}(1+\sqrt{-d})} \), so that \( f\sqrt{-d}Z \subseteq R_{-d,2k+1} \cap i\mathbb{R} \). Any \( ir = a + \frac{bf}{2}(1+\sqrt{-d}) \) with \( r \in \mathbb{R}, a, b \in \mathbb{Z} \) has real part \( a + \frac{bf}{2} = 0 \) and imaginary part \( r = \frac{bf}{2}\sqrt{d} \). Note that \( \frac{bf}{2} = \frac{b(2k+1)}{2} = -a \in \mathbb{Z} \) is an integer only for an even \( b = 2b_1, b_1 \in \mathbb{Z} \), so that \( r = b_1f\sqrt{d} \) and \( i\mathbb{R} \cap R_{-d,2k+1} \subseteq f\sqrt{-d}Z \). That justifies \( i\mathbb{R} \cap R_{-d,2k+1} = f\sqrt{-d}Z \) for \( -d \equiv 1(\text{mod}4), f \equiv 1(\text{mod}2) \).

Finally, for \( -d \equiv 1(\text{mod}4) \) and an even conductor \( f = 2k \in \mathbb{N} \) the endomorphism ring \( R_{-d,2k} = Z + k(1+\sqrt{-d})Z \) contains \( k\sqrt{-d} \), so that \( k\sqrt{-d}Z \subseteq i\mathbb{R} \cap R_{-d,2k} \). Note that \( ir = a + bk(1+\sqrt{-d}) \) with \( r \in \mathbb{R}, a, b \in \mathbb{Z} \) has real part \( a + bk = 0 \) and imaginary part \( r = bk\sqrt{d} \), so that \( i\mathbb{R} \cap R_{-d,2k} \subseteq k\sqrt{-d}Z \) and \( i\mathbb{R} \cap R_{-d,2k} = k\sqrt{-d}Z \).

Now, (11) implies that

\[
2 \sin \left( \frac{2\pi s}{r_1} \right) \in [-2, 2] \cap i(R_{-d,f} \cap i\mathbb{R}) = \begin{cases} 
 [-2, 2] \cap f\sqrt{-d}Z & \text{for } -d \not\equiv 1(\text{mod}4) \text{ or } -d \equiv 1(\text{mod}4), f \equiv 1(\text{mod}2), \\
 [-2, 2] \cap \frac{1}{2}f\sqrt{-d}Z & \text{for } -d \equiv 1(\text{mod}4), f \equiv 0(\text{mod}2).
\end{cases}
\]
If $d \geq 5$ then $\sqrt{d} \geq \sqrt{5} > 2$ and $[-2, \sqrt{d}] \cap f \sqrt{d} \mathbb{Z} = \{0\}$ for all $f \in \mathbb{N}$ and $[-2, \sqrt{d}] \cap \frac{1}{2} \sqrt{d} \mathbb{Z} = \{0\}$ for all $f \in 2\mathbb{N}$. Note that $\sin \left( \frac{2\pi s}{r_1} \right) = 0$ for some natural number $1 \leq s \leq r_1 - 1$ with $\text{GCD}(s, r_1) = 1$ has unique solution $\frac{2\pi s}{r_1} = \pi$, since $\frac{2\pi s}{r_1} \in (0, 2\pi)$. That implies $2s = r_1$, whereas $s$ divides $r_1$ and $s = \text{GCD}(s, r_1) = 1$, $r_1 = 2$. Thus, $\lambda_1 = e^{\frac{2\pi i}{2}} = e^{\pi i} = -1$, $\lambda_2 = -(-1) = 1$ and $g$ is conjugate to

$$D_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

In particular, $g$ is of order 2. Note that the case of $g \in \text{GL}(2, R)$ with $\lambda_1 = -1$, $\lambda_2 = 1$ is realized by the diagonal matrix $D_2 \in \text{GL}(2, \mathbb{Z}) \leq \text{GL}(2, R_{-d,f})$.

If $d = 1$ and $f \geq 3$ then $2\sin \left( \frac{2\pi s}{r_1} \right) \in [-2, 2] \cap f \mathbb{Z} = \{0\}$ and $D_2$ is the only diagonal form for $g$. For $d = 2$ and $f \geq 2$ the intersection $[-2, 2] \cap f \sqrt{2} \mathbb{Z} = \{0\}$, so that any $g \in \text{GL}(2, R_{-d,f})$ with $f \geq 2$ and $\det(g) = -1$ is conjugate to $D_2$. If $d = 3$ and $f = 2k + 1 \geq 3$ then $[-2, 2] \cap f \sqrt{3} \mathbb{Z} = \{0\}$. Similarly, for $d = 3$ and $f = 2k \geq 4$ one has $[-2, 2] \cap k \sqrt{3} \mathbb{Z} = \{0\}$. In such a way, the existence of $g \in \text{GL}(2, R_{-d,f})$ with $\det(g) = -1$, $\text{tr}(g) \neq 0$ requires $R_{-d,f}$ to be among

$$R_{-1,1} = O_{-1} = \mathbb{Z}[i], \quad R_{-1,2} = \mathbb{Z} + 2i \mathbb{Z}, \quad R_{-2,1} = O_{-2} = \mathbb{Z} + \sqrt{-2} \mathbb{Z},$$

$$R_{-3,1} = O_{-3} = \mathbb{Z} + \frac{1 + \sqrt{-3}}{2} \mathbb{Z} \quad \text{or} \quad R_{-3,2} = \mathbb{Z} + 2 \left( \frac{1 + \sqrt{-3}}{2} \right) \mathbb{Z} = \mathbb{Z} + \sqrt{-3} \mathbb{Z}.$$  

The next considerations exploit the following simple observation: If $a, b$ are relatively prime natural numbers and $s, r_1$ are relatively prime natural numbers then $as = br_1$ if and only if $s = b$ and $r_1 = a$. Namely, $b$ divides $as$ and $\text{GCD}(a, b) = 1$ requires $b$ to divide $s$. Thus, $s = bs_1$ for some $s_1 \in \mathbb{N}$ and $as_1 = r_1$. Now $s_1$ is a natural common divisor of the relatively prime $s, r_1$, so that $s_1 = 1$, $s = b$ and $r_1 = a$.

For $d = 1$ and $f = 2$ one has $2 \sin \left( \frac{2\pi s}{r_1} \right) \in [-2, 2] \cap f \mathbb{Z} = \{0, \pm 2\}$. Let $\text{tr}(g) = 2i$ or $\sin \left( \frac{2\pi s}{r_1} \right) = 1$ for $r_1 \in \mathbb{N}$ and some natural number $1 \leq s \leq r_1 - 1$, $\text{GCD}(s, r_1) = 1$. Then $\frac{2\pi s}{r_1} = \frac{\pi}{2}$ or $4s = r_1$. As a result, $s = 1$, $r_1 = 4$ and $\lambda_1 = e^{\frac{\pi i}{2}} = i$, $\lambda_2 = -e^{-\frac{\pi i}{2}} = i$. Now $g = iI_2$ as the unique matrix, conjugate to the scalar matrix $iI_2$. However, $iI_2 \notin \text{GL}(2, R_{-1,2}) = \text{GL}(2, \mathbb{Z} + 2i \mathbb{Z})$, so that $g = iI_2$ is not a solution of the problem. For $\text{tr}(g) = -2i$ one has $\sin \left( \frac{2\pi s}{r_1} \right) = -1$, whereas $\frac{2\pi s}{r_1} = \frac{3\pi}{2}$ and $4s = 3r_1$. Thus, $s = 3$, $r_1 = 4$ and $\lambda_1 = e^{\frac{3\pi i}{4}} = -i$, $\lambda_2 = -e^{-\frac{3\pi i}{4}} = -i$. That determines a unique $g = -iI_2$. But $-iI_2 \notin \text{GL}(2, R_{-1,2}) = \text{GL}(2, \mathbb{Z} + 2i \mathbb{Z})$, so that $\lambda_1 = 1$, $\lambda_2 = -1$ are the only possible eigenvalues for $g \in \text{GL}(2, R_{-1,2})$ of finite order with $\det(g) = -1$.

In the case of $d = 1$ and $f = 1$, note that $2 \sin \left( \frac{2\pi s}{r_1} \right) \in [-2, 2] \cap Z = \{0, \pm 1, \pm 2\}$. Besides $g \in \text{GL}(2, \mathbb{Z}[i])$ with $\det(g) = -1$, $\text{tr}(g) = 0$, one has $g = iI_2 \in \text{GL}(2, \mathbb{Z}[i])$ and $g = -iI_2 \in \text{GL}(2, \mathbb{Z}[i])$. The case of $\text{tr}(g) = i$ corresponds to $\sin \left( \frac{2\pi s}{r_1} \right) = \frac{1}{2}$.
and holds for $\frac{2\pi s}{r_1} = \frac{\pi}{6}$ or $\frac{2\pi s}{r_1} = \frac{5\pi}{6}$. Note that $12s = r_1$ implies $s = 1$, $r_1 = 12$ and
\[ \lambda_1 = e^{\frac{s\pi i}{6}} = \frac{\sqrt{3}}{2} + \frac{i}{2}, \quad \lambda_2 = -e^{-\frac{s\pi i}{6}} = -\frac{\sqrt{3}}{2} + \frac{i}{2} = e^{\frac{5s\pi i}{6}}. \]
Thus, $g$ is of order $r = LCM(12, 12) = 12$. This possibility is realized, for instance, by
\[
g(i) = \begin{pmatrix} 1 & 1 \\ i & -1 + i \end{pmatrix} \in GL(2, \mathbb{Z}[i]) \quad \text{with} \quad \det(g(i)) = -1, \quad \text{tr}(g(i)) = i.
\]
If $12s = 5r_1$ then $s = 5$, $r_1 = 12$ and
\[ \lambda_1 = e^{\frac{5s\pi i}{6}} = \frac{\sqrt{3}}{2} + \frac{i}{2}, \quad \lambda_2 = e^{-\frac{5s\pi i}{6}} = \frac{\sqrt{3}}{2} - \frac{i}{2}, \]
which was already obtained. Note that $\text{tr}(g) = -i$ amounts to
\[ \sin \left( \frac{2\pi s}{r_1} \right) = -\frac{1}{2} \quad \text{and holds for} \quad \frac{2\pi s}{r_1} = \frac{7\pi}{6} \quad \text{or} \quad \frac{2\pi s}{r_1} = \frac{11\pi}{6}. \]
If $12s = 7r_1$ then $s = 7$, $r_1 = 12$ and
\[ \lambda_1 = e^{\frac{7s\pi i}{6}} = -\frac{\sqrt{3}}{2} - \frac{i}{2}, \quad \lambda_2 = e^{-\frac{7s\pi i}{6}} = \frac{\sqrt{3}}{2} - \frac{i}{2} = e^{-\frac{7s\pi i}{6}}. \]
and $g$ is of order $r = LCM(12, 12) = 12$. Note that
\[
g(-i) = \begin{pmatrix} 1 & 1 \\ -i & -1 - i \end{pmatrix} \in GL(2, \mathbb{Z}[i]) \quad \text{with} \quad \det(g(-i)) = -1, \quad \text{tr}(g(-i)) = -i
\]
realizes the aforementioned possibility.

In the case of $12s = 11r_1$ one has $s = 11$, $r_1 = 12$ and
\[ \lambda_1 = e^{\frac{11s\pi i}{6}} = \frac{\sqrt{3}}{2} - \frac{i}{2}, \quad \lambda_2 = -e^{-\frac{11s\pi i}{6}} = -\frac{\sqrt{3}}{2} - \frac{i}{2}, \]
which is already listed as a solution. That concludes the considerations for $g \in GL(2, \mathbb{Z}[i])$ with $\det(g) = -1$.

If $d = 2$ and $f = 1$ then $2\sin \left( \frac{2\pi s}{r_1} \right) \in [-2, 2] \cap \sqrt{2}\mathbb{Z} = \{0, \pm \sqrt{2}\}$. Note that
\[ \sin \left( \frac{2\pi s}{r_1} \right) = \frac{\sqrt{7}}{2} \quad \text{holds for} \quad \frac{2\pi s}{r_1} = \frac{\pi}{4} \quad \text{or} \quad \frac{2\pi s}{r_1} = \frac{3\pi}{4}. \]
The equality $r_1 = 8s$ implies $s = 1$ and $r_1 = 8$. As a result, $\lambda_1 = e^{\frac{\pi i}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $\lambda_2 = -e^{-\frac{\pi i}{4}} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i = e^{\frac{3\pi i}{4}}$. Observe that
\[
g(\sqrt{-2}) = \begin{pmatrix} 1 & 1 \\ \sqrt{-2} & \sqrt{-2} \end{pmatrix} \in GL(2, \mathbb{O}_2), \mathbb{O}_2 = \mathbb{Z} + \sqrt{-2}\mathbb{Z}
\]
with $\det(g(\sqrt{-2})) = -1, \quad \text{tr}(g(\sqrt{-2})) = \sqrt{-2}$ realizes the aforementioned possibility. If $8s = 3r_1$ then $s = 3$, $r_1 = 8$ and
\[ \lambda_1 = e^{\frac{3s\pi i}{4}} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{7}}{2}i, \quad \lambda_2 = -e^{-\frac{3s\pi i}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{7}}{2}i = e^{\frac{3s\pi i}{4}}. \]
The corresponding automorphism $g$ is of order $r = LCM(8, 8) = 8$. Note that
\[
g(-\sqrt{-2}) = \begin{pmatrix} 1 & 1 \\ -\sqrt{-2} & -\sqrt{-2} \end{pmatrix} \in GL(2, \mathbb{O}_2)
\]
with $\det(g(-\sqrt{-2})) = -1, \quad \text{tr}(g(-\sqrt{-2})) = -\sqrt{-2}$ realizes this possibility. In the case of $8s = 7r_1$, one has $s = 7$, $r_1 = 8$. The eigenvalues
\[ \lambda_1 = e^{\frac{7s\pi i}{4}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{7}}{2}i, \]
and
\[ \lambda_2 = -e^{-\frac{7s\pi i}{4}} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{7}}{2}i = e^{-\frac{7s\pi i}{4}}. \]
Proposition 17. If $g \in GL(2, \mathbb{Z}[i])$ is of finite order $r$ and $\det(g) = i$ then
\[ \text{tr}(g) \in \{0, \pm(1 + i)\}, \quad r \in \{4, 8\}. \]

More precisely,
(i) $\text{tr}(g) = 0$ or $\lambda_1 = e^{\frac{4\pi i}{8}}$, $\lambda_2 = e^{-\frac{\pi i}{8}}$ if and only if $g$ is of order 8;
(ii) if $\text{tr}(g) = 1 + i$ or $\lambda_1 = i$, $\lambda_2 = 1$ then $g$ is of order 4;
(iii) if $\text{tr}(g) = -1 - i$ or $\lambda_1 = -i$, $\lambda_2 = -1$ then $g$ is of order 4.

Proof. If $\lambda_1 = e^{\frac{4\pi i}{8}}$ for the order $r_1 \in \mathbb{N}$ of $\lambda_1 \in \mathbb{C}^*$ and some natural number $1 \leq s < r_1$, $\text{GCD}(s, r_1) = 1$, then $\lambda_2 = \det(g)\lambda_1^{-1} = ie^{-\frac{2\pi is}{r_1}}$. Therefore, the trace
\[ \text{tr}(g) = \lambda_1 + \lambda_2 = \left[ \cos \left( \frac{2\pi s}{r_1} \right) + \sin \left( \frac{2\pi s}{r_1} \right) \right] (1 + i) = \]
\[ \]
if and only if the real part
\[
    \sqrt{2} \sin \left( \frac{2\pi s}{r_1} + \frac{\pi}{4} \right) \in \mathbb{Z} \cap [-\sqrt{2}, \sqrt{2}] = \{0, \pm 1\}.
\]
As a result, \( \text{tr}(g) \in \{0, \pm (1 + i)\} \). If \( \text{tr}(g) = 0 \) or, equivalently, \( \sin \left( \frac{2\pi s}{r_1} + \frac{\pi}{4} \right) = 0 \) for \( \frac{2\pi s}{r_1} + \frac{\pi}{4} \in \left( \frac{\pi}{4}, \frac{9\pi}{4} \right) \) then \( \frac{2\pi s}{r_1} + \frac{\pi}{4} = \pi \) or \( \frac{2\pi s}{r_1} + \frac{\pi}{4} = 2\pi \). For \( \frac{2\pi s}{r_1} = \frac{3}{4} \) there follows \( 8s = 3r_1 \) and \( s = 3, r_1 = 8 \). As a result, \( \lambda_1 = e^{\frac{3\pi i}{4}} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i, \lambda_2 = ie^{-\frac{3\pi i}{4}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i = e^{-\frac{\pi i}{4}} \) and \( g \) is of order \( r = \text{LCM}(8, 8) = 8 \). For instance,
\[
    g_i(0) = \begin{pmatrix} i & i \\ -1 - i & -i \end{pmatrix} \in \text{GL}(2, \mathbb{Z}[i])
\]
with \( \det(g_i(0)) = i, \text{tr}(g_i(0)) = 0 \) attains this possibility.

If \( \frac{2\pi s}{r_1} = \frac{7}{4} \) then \( 8s = 7r_1 \) and \( s = 7, r_1 = 8 \). The eigenvalues \( \lambda_1 = e^{\frac{7\pi i}{4}} = e^{-\frac{\pi i}{4}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i, \lambda_2 = ie^{\frac{7\pi i}{4}} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i = e^{\frac{3\pi i}{4}} \) are already obtained.

In the case of \( \text{tr}(g) = 1 + i \), one has \( \sin \left( \frac{2\pi s}{r_1} + \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2} \), which is equivalent to \( \frac{2\pi s}{r_1} + \frac{\pi}{4} = \frac{3\pi}{4} \) for \( \frac{2\pi s}{r_1} + \frac{\pi}{4} \in \left( \frac{\pi}{4}, \frac{9\pi}{4} \right) \). Now, \( \frac{2\pi s}{r_1} = \frac{1}{2} \), whereas \( 4s = r_1 \) and \( s = 1, r_1 = 4 \). The eigenvalues are \( \lambda_1 = e^{\frac{\pi i}{4}} = i, \lambda_2 = ie^{-\frac{\pi i}{4}} = 1 \) and \( g \) is of order \( r = \text{LCM}(4, 1) = 4 \). Note that
\[
    g_i(1 + i) = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}(2, \mathbb{Z}[i])
\]
with \( \det(g_i(1 + i)) = i, \text{tr}(g_i(1 + i)) = 1 + i \) realizes this case.

Finally, for \( \text{tr}(g) = -1 - i \) there follows \( \sin \left( \frac{2\pi s}{r_1} + \frac{\pi}{4} \right) = -\frac{\sqrt{2}}{2} \). Consequently, \( \frac{2\pi s}{r_1} + \frac{\pi}{4} = \frac{5\pi}{4} \) or \( \frac{2\pi s}{r_1} + \frac{\pi}{4} = \frac{7\pi}{4} \) for \( \frac{2\pi s}{r_1} + \frac{\pi}{4} \in \left( \frac{\pi}{4}, \frac{9\pi}{4} \right) \). In the case of \( \frac{2\pi s}{r_1} = 1 \) one has \( s = 1, r_1 = 2 \). The eigenvalues of \( g \) are \( \lambda_1 = e^{\pi i} = -1, \lambda_2 = ie^{-\pi i} = -i \), so that \( g \) is of order \( r = \text{LCM}(2, 4) = 4 \). This possibility is realized by
\[
    g_i(-1 - i) = \begin{pmatrix} -i & 0 \\ 0 & -1 \end{pmatrix} \in \text{GL}(2, \mathbb{Z}[i])
\]
with \( \det(g_i(-1 - i)) = i, \text{tr}(g_i(-1 - i)) = -1 - i \).

If \( \frac{2\pi s}{r_1} = \frac{3}{2} \) then \( 4s = 3r_1 \) and \( s = 3, r_1 = 4 \). The eigenvalues \( \lambda_1 = e^{\frac{3\pi i}{2}} = -i, \lambda_2 = ie^{\frac{3\pi i}{2}} = -1 \) are already obtained. That concludes the description of the eigenvalues of all \( g \in \text{GL}(2, \mathbb{Z}[i]) \) of finite order with \( \det(g) = i \).

\[\square\]

**Proposition 18.** If \( g \in \text{GL}(2, \mathbb{Z}[i]) \) is of finite order \( r \) and \( \det(g) = -i \) then
\[
\text{tr}(g) \in \{0, \pm (1 - i)\}, \quad r \in \{4, 8\}.
\]
More precisely,

(i) \( \text{tr}(g) = 0 \) or \( \lambda_1 = e^{\frac{\pi i}{r_1}}, \lambda_2 = e^{\frac{5\pi i}{r_1}} \) if and only if \( g \) is of order 8;

(ii) if \( \text{tr}(g) = 1 - i \) or \( \lambda_1 = -i, \lambda_2 = 1 \) then \( g \) is of order 4;

(iii) if \( \text{tr}(g) = -1 + i \) or \( \lambda_1 = i, \lambda_2 = -1 \) then \( g \) is of order 4.

**Proof.** If one of the eigenvalues of \( g \) is \( \lambda_1 = e^{\frac{2\pi s}{r_1}} \) then the other one is \( \lambda_2 = -ie^{-\frac{2\pi s}{r_1}} \). Thus, the trace

\[
\text{tr}(g) = \lambda + \lambda_2 = \left[ \cos \left( \frac{2\pi s}{r_1} \right) - \sin \left( \frac{2\pi s}{r_1} \right) \right] (1 - i) = \sqrt{2} \cos \left( \frac{2\pi s}{r_1} + \frac{\pi}{4} \right) (1 - i)
\]

belongs to \( \mathbb{Z}[i] = \mathbb{Z} + \mathbb{Z}i \) if and only if \( \sqrt{2} \cos \left( \frac{2\pi s}{r_1} + \frac{\pi}{4} \right) \in \mathbb{Z} \). As a result,

\[
\sqrt{2} \cos \left( \frac{2\pi s}{r_1} + \frac{\pi}{4} \right) \in \mathbb{Z} \cap [-\sqrt{2}, \sqrt{2}] = \{0, \pm1\}
\]

or \( \text{tr}(g) \in \{0, \pm(1 - i)\} \). Note that \( \text{tr}(g) = 0 \) reduces to \( \cos \left( \frac{2\pi s}{r_1} + \frac{\pi}{4} \right) = 0 \) with solutions \( \frac{2\pi s}{r_1} + \frac{\pi}{4} = \frac{\pi}{2} \) or \( \frac{2\pi s}{r_1} + \frac{\pi}{4} = \frac{3\pi}{2} \). If \( \frac{2\pi s}{r_1} = \frac{1}{4} \) then \( 8s = r_1 \) and \( s = 1, r_1 = 8 \). The eigenvalues of \( g \) are \( \lambda_1 = e^{\frac{\pi i}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \lambda_2 = -ie^{-\frac{\pi i}{4}} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \) and \( g \) is of order \( r = LCM(8, 8) = 8 \). Note that

\[
g_{-i}(0) = \left( \begin{array}{cc} -i & -1 \\ -1 & i \end{array} \right) \in GL(2, \mathbb{Z}[i])
\]

with \( \text{det}(g_{-i}(0)) = -i, \text{tr}(g_{-i}(0)) = 0 \) realizes the aforementioned possibility. In the case of \( 2\pi s = \frac{5\pi}{4} \) there holds \( 8s = 5r_1 \), whereas \( s = 5, r_1 = 8 \) and \( \lambda_1 = e^{\frac{5\pi i}{4}} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \lambda_2 = -ie^{-\frac{5\pi i}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \) and \( g \) is of order \( r = LCM(4, 1) = 4 \). This case has been already discussed.

For \( \text{tr}(g) = 1 - i \) one has \( \cos \left( \frac{2\pi s}{r_1} + \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2} \), which reduces to \( \frac{2\pi s}{r_1} + \frac{\pi}{4} = \frac{7\pi}{4} \) for \( \frac{2\pi s}{r_1} + \frac{\pi}{4} \in \left( \frac{\pi}{4}, \frac{9\pi}{4} \right) \). Now \( \frac{2\pi s}{r_1} = \frac{3}{2} \) reads as \( 4s = 3r_1 \) and determines \( s = 3, r_1 = 4 \). The eigenvalues of \( g \) are \( \lambda_1 = e^{\frac{3\pi i}{4}} = -i, \lambda_2 = -ie^{-\frac{3\pi i}{4}} = 1 \) and \( g \) is of order \( r = LCM(4, 1) = 4 \). This possibility is realized by

\[
g_{-i}(1 - i) = \left( \begin{array}{cc} -i & 0 \\ 0 & 1 \end{array} \right) \in GL(2, \mathbb{Z}[i])
\]

with \( \text{det}(g_{-i}(1 - i)) = -i, \text{tr}(g_{-i}(1 - i)) = 1 - i \).

Finally, \( \text{tr}(g) = -1 + i \) is equivalent to \( \cos \left( \frac{2\pi s}{r_1} + \frac{\pi}{4} \right) = -\frac{\sqrt{2}}{2} \) and holds for \( \frac{2\pi s}{r_1} + \frac{\pi}{4} = \frac{3\pi}{4} \) or \( \frac{2\pi s}{r_1} + \frac{\pi}{4} = \frac{5\pi}{4} \). In the case of \( \frac{2\pi s}{r_1} = \frac{1}{2} \), one has \( 4s = r_1 \) and \( s = 1, r_1 = 4 \). The eigenvalues of \( g \) are \( \lambda_1 = e^{\frac{\pi i}{4}} = i, \lambda_2 = -ie^{-\frac{\pi i}{4}} = -1 \) and \( g \) is of order \( r = LCM(4, 2) = 4 \). The automorphism

\[
g_{-i}(-1 + i) = \left( \begin{array}{cc} i & 0 \\ 0 & -1 \end{array} \right) \in GL(2, \mathbb{Z}[i])
\]
realizes the case under discussion. For \( \frac{2s}{r_1} = 1 \) there follow \( s = 1, r_1 = 2 \) and \( \lambda_1 = e^{\pi i} = -1, \lambda_2 = -ie^{-\pi i} = i \), which was already discussed. That concludes the description of the automorphisms \( g \in GL(2, \mathbb{Z}[i]) \) with \( \det(g) = -i \).

\[ \square \]

**Proposition 19.** If \( g \in GL(2, \mathcal{O}_{-3}) \) is of finite order \( r \) and \( \det(g) = e^{\pi i} \) then

\[
r = 6 \quad \text{and} \quad \text{tr}(g) \in \left\{ 0, \pm \left( \frac{3}{2} + \frac{\sqrt{3}}{2} \right) \right\}.
\]

More precisely,

(i) \( \text{tr}(g) = 0 \) exactly when \( \lambda_1 = e^{\pi i} \), \( \lambda_2 = e^{-\pi i} \);

(ii) \( \text{tr}(g) = \frac{3}{2} + \frac{\sqrt{3}}{2} \) exactly when \( \lambda_1 = e^{\pi i} \), \( \lambda_2 = 1 \);

(iii) \( \text{tr}(g) = -\frac{3}{2} - \frac{\sqrt{3}}{2} \) exactly when \( \lambda_1 = e^{-\pi i} \), \( \lambda_2 = -1 \).

**Proof.** If \( \lambda_1 = e^{\frac{2\pi si}{r_1}} \) then \( \lambda_2 = e^{\frac{\pi i}{3}} e^{-\frac{2\pi si}{r_1}} \) and the trace

\[
\text{tr}(g) = \lambda_1 + \lambda_2 = (\sqrt{3} + i) \sin \left( \frac{2\pi s}{r_1} + \frac{\pi}{3} \right)
\]

belongs to \( \mathcal{O}_{-3} = \mathbb{Z} + \frac{1+\sqrt{-3}}{2} \mathbb{Z} \) if and only if \( \sin \left( \frac{2\pi s}{r_1} + \frac{\pi}{3} \right) \in \frac{\sqrt{3}}{2} \mathbb{Z} \). Combining with \( \sin \left( \frac{2\pi s}{r_1} + \frac{\pi}{3} \right) \in [-1, 1] \), one gets \( \sin \left( \frac{2\pi s}{r_1} + \frac{\pi}{3} \right) \in \frac{\sqrt{3}}{2} \mathbb{Z} \cap [-1, 1] = \left\{ 0, \pm \frac{\sqrt{3}}{2} \right\} \) and, respectively, \( \text{tr}(g) \in \left\{ 0, \pm \left( \frac{3}{2} + \frac{\sqrt{3}}{2} \right) \right\} \).

If \( \sin \left( \frac{2\pi s}{r_1} + \frac{\pi}{3} \right) = 0 \) then \( \frac{2\pi s}{r_1} + \frac{\pi}{3} = \pi \) or \( \frac{2\pi s}{r_1} + \frac{\pi}{3} = 2\pi \). For \( \frac{2s}{r_1} = \frac{2}{3} \) there follows \( s = 1, r_1 = 3 \) and \( \lambda_1 = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}, \lambda_2 = e^{\frac{\pi i}{3}} e^{\frac{2\pi i}{3}} = e^{\frac{\pi i}{3}} = \frac{1}{2} - \frac{\sqrt{3}}{2} \). The automorphisms \( g \in GL(2, \mathcal{O}_{-3}) \) with such eigenvalues are of order \( r = \text{LCM}(3, 6) = 6 \). For instance,

\[
\begin{pmatrix}
 e^{\frac{2\pi i}{3}} & 0 \\
 0 & e^{-\frac{\pi i}{3}}
\end{pmatrix} \in GL(2, \mathcal{O}_{-3})
\]

attains the aforementioned possibility.

In the case of \( \frac{2s}{r_1} = \frac{5}{3} \) one has \( s = 5, r_1 = 6 \) and \( \lambda_1 = e^{-\frac{\pi i}{3}}, \lambda_2 = e^{\frac{\pi i}{3}} e^{\frac{2\pi i}{3}} = e^{\frac{2\pi i}{3}}, \)

which was already obtained.

Note that \( \sin \left( \frac{2\pi s}{r_1} + \frac{\pi}{3} \right) = \frac{\sqrt{3}}{2} \) for \( \frac{2\pi s}{r_1} + \frac{\pi}{3} \in \left( \frac{\pi}{3}, \frac{7\pi}{3} \right) \) implies \( \frac{2\pi s}{r_1} + \frac{\pi}{3} = \frac{2\pi}{3} \), whereas

\( 6s = r_1 \) and \( s = 1, r_1 = 6 \). The corresponding eigenvalues are \( \lambda_1 = e^{\frac{\pi i}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2} i, \lambda_2 = e^{\frac{\pi i}{3}} e^{-\frac{\pi i}{3}} = 1 \) and \( g \) is of order \( r = \text{LCM}(6, 1) = 6 \). Note that

\[
\begin{pmatrix}
 e^{\frac{\pi i}{3}} & 0 \\
 0 & 1
\end{pmatrix} \in GL(2, \mathcal{O}_{-3})
\]

realizes this possibility.

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The equality \( \sin \left( \frac{2\pi s}{r_1} + \frac{\pi}{3} \right) = -\frac{\sqrt{3}}{2} \) holds for \( \frac{2\pi s}{r_1} + \frac{\pi}{3} = \frac{4\pi}{3} \) or \( \frac{2\pi s}{r_1} + \frac{\pi}{3} = \frac{5\pi}{3} \). If \( 2s = r_1 \) then \( s = 1, \ r_1 = 2 \) and \( \lambda_1 = e^{\pi i} = -1, \ \lambda_2 = e^{\frac{\pi i}{3}} e^{-\pi i} = e^{-\frac{2\pi i}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i. \) The automorphism \( g \) is of order \( r = \text{LCM}(2, 3) = 6. \) Note that

\[
\begin{pmatrix}
  e^{-\frac{2\pi i}{3}} & 0 \\
  0 & -1
\end{pmatrix} \in GL(2, \mathcal{O}_3)
\]

attains this possibility and concludes the proof of the proposition.

**Proposition 20.** If \( g \in GL(2, \mathcal{O}_3) \) is of finite order \( r \) and \( \det(g) = e^{-\frac{\pi i}{r}} \) then

\[
r = 6 \quad \text{and} \quad \text{tr}(g) \in \left\{ 0, \pm \left( \frac{3}{2} - \frac{\sqrt{3}}{2} \right) \right\}.
\]

More precisely,

(i) \( \text{tr}(g) = 0 \) exactly when \( \lambda_1 = e^{\frac{2\pi i}{3}}, \ \lambda_2 = e^{-\frac{2\pi i}{3}} \);

(ii) \( \text{tr}(g) = \frac{3}{2} - \frac{\sqrt{3}}{2} i \) exactly when \( \lambda_1 = e^{-\frac{\pi i}{3}}, \ \lambda_2 = 1 \);

(iii) \( \text{tr}(g) = -\frac{3}{2} + \frac{\sqrt{3}}{2} i \) exactly when \( \lambda_1 = \frac{2\pi i}{3}, \ \lambda_2 = -1 \).

**Proof.** If \( \lambda_1 = e^{\frac{2\pi i}{r_1}} \) then \( \lambda_2 = e^{-\frac{\pi i}{r_1}} e^{-\frac{2\pi i}{r_1}} \) and the trace

\[
\text{tr}(g) = \lambda_1 + \lambda_2 = (-\sqrt{3} + i) \sin \left( \frac{2\pi s}{r_1} - \frac{\pi}{3} \right)
\]

belongs to \( \mathcal{O}_3 = \mathbb{Z} + \frac{1+\sqrt{3}}{2} \mathbb{Z} \) if and only if \( \sin \left( \frac{2\pi s}{r_1} - \frac{\pi}{3} \right) \in \frac{\sqrt{3}}{2} \mathbb{Z} \). As a result, \( \sin \left( \frac{2\pi s}{r_1} = \frac{\pi}{3} \right) \in \frac{\sqrt{3}}{2} \mathbb{Z} \cap [-1, 1] = \left\{ 0, \pm \frac{\sqrt{3}}{2} \right\} \) and \( \text{tr}(g) \in \left\{ 0, \pm \left( \frac{3}{2} - \frac{\sqrt{3}}{2} i \right) \right\} \).

The equation \( \sin \left( \frac{2\pi s}{r_1} - \frac{\pi}{3} \right) = 0 \) for \( \frac{2\pi s}{r_1} - \frac{\pi}{3} \in \left( -\frac{\pi}{3}, \frac{5\pi}{3} \right) \) has solutions \( \frac{2\pi s}{r_1} - \frac{\pi}{3} = 0 \) and \( \frac{2\pi s}{r_1} - \frac{\pi}{3} = \pi. \)

If \( 6s = r_1 \) then \( s = 1, \ r_1 = 6 \) and \( \lambda_1 = e^{\frac{\pi i}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2} i, \ \lambda_2 = e^{\frac{\pi i}{6}} e^{-\frac{\pi i}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i. \) The automorphisms \( g \in GL(2, \mathcal{O}_3) \) with such eigenvalues are of order \( r = \text{LCM}(6, 3) = 6. \) For instance,

\[
\begin{pmatrix}
  e^{\frac{\pi i}{3}} & 0 \\
  0 & e^{-\frac{2\pi i}{3}}
\end{pmatrix} \in GL(2, \mathcal{O}_3)
\]

attains this case.

If \( \sin \left( \frac{2\pi s}{r_1} - \frac{\pi}{3} \right) = \frac{\sqrt{3}}{2} \) then \( \frac{2\pi s}{r_1} - \frac{\pi}{3} = \frac{\pi}{3} \) or \( \frac{2\pi s}{r_1} - \frac{\pi}{3} = \frac{2\pi}{3}. \) For \( 3s = r_1 \) one has \( s = 1, \ r_1 = 3 \) and \( \lambda_1 = e^{2\frac{\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i, \ \lambda_2 = e^{\frac{\pi i}{3}} e^{-\frac{2\pi i}{3}} = e^{-\pi i} = -1, \) attained by

\[
\begin{pmatrix}
  e^{\frac{\pi i}{3}} & 0 \\
  0 & -1
\end{pmatrix} \in GL(2, \mathcal{O}_3).\]
All $g \in GL(2, \mathcal{O}_{-3})$ with such eigenvalues are of order $r = LCM(3, 2) = 6$.

In the case of $2s = r_1$ there follows $s = 1$, $r_1 = 2$ and $\lambda_1 = e^{\pi i} = -1$, $\lambda_2 = e^{-\pi i}e^{-\pi i/6} = e^{-2\pi i/6}$, which is already discussed.

The equation $\sin\left(\frac{2\pi s}{r_1} - \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ for $\frac{2\pi s}{r_1} - \frac{\pi}{3} \in (-\frac{\pi}{3}, \frac{5\pi}{3})$ has solution $\frac{2\pi s}{r_1} - \frac{\pi}{3} = \frac{5\pi}{3}$. Therefore $6s = 5r_1$ and $s = 5$, $r_1 = 6$. As a result, $\lambda_1 = e^{\pi i} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, $\lambda_2 = e^{-\pi i/6}e^{\pi i/6} = 1$ and $g$ is of order $r = LCM(6, 1) = 6$. Note that

$$
\begin{pmatrix}
  e^{-\pi i/6} & 0 \\
  0 & 1
\end{pmatrix} \in GL(2, \mathcal{O}_{-3})
$$

attains this possibility and concludes the proof of the proposition.

\[\square\]

**Proposition 21.** If $g \in GL(2, \mathcal{O}_{-3})$ is of finite order $r$ and $\det(g) = e^{\frac{2\pi i}{3}}$ then

$$
\text{tr}(g) \in \left\{ 0, \pm\left(1 + \frac{\sqrt{-3}}{2}\right), \pm\left(1 + \sqrt{-3}\right) \right\}, \quad r \in \{3, 6, 12\}.
$$

More precisely,

(i) $\text{tr}(g) = 0$ or $\lambda_1 = e^{\frac{5\pi i}{6}}$, $\lambda_2 = e^{-\frac{\pi i}{6}}$ if and only if $g$ is of order 12;

(ii) if $\text{tr}(g) = \frac{1+\sqrt{3}}{2}$ or $\lambda_1 = e^{\frac{2\pi i}{3}}$, $\lambda_2 = 1$ then $g$ is of order 3;

(iii) if $\text{tr}(g) = -1 - \sqrt{3}i$ or $g = e^{-\frac{2\pi i}{3}}I_2$ then $g$ is of order 3;

(iv) if $\text{tr}(g) = \frac{-1-\sqrt{3}}{2}$ or $\lambda_1 = e^{-\frac{2\pi i}{3}}$, $\lambda_2 = -1$ then $g$ is of order 6;

(v) if $\text{tr}(g) = 1 + \sqrt{3}i$ or $g = e^{\frac{5\pi i}{6}}I_2$ then $g$ is of order 6.

Proof. If $\lambda_1 = e^{\frac{2\pi i}{3}}$ then $\lambda_2 = e^{\frac{2\pi i}{3}}e^{-\frac{2\pi i}{3}}$ and the trace

$$
\text{tr}(g) = \lambda_1 + \lambda_2 = (1 + \sqrt{3}i)\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{6}\right)
$$

belongs to $\mathcal{O}_{-3} = \mathbb{Z} + \frac{1+\sqrt{3}}{2}\mathbb{Z}$ if and only if $2\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{6}\right) \in \mathbb{Z}$. Combining with $\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{6}\right) \in [-1, 1]$, one obtains $2\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{6}\right) \in \mathbb{Z} \cap [-2, 2] = \{0, \pm 1, \pm 2\}$ and, respectively,

$$
\text{tr}(g) \in \left\{ 0, \pm\frac{1+\sqrt{3}i}{2}, \pm(1 + \sqrt{3}i) \right\}.
$$

If $\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{6}\right) = 0$ for $\frac{2\pi s}{r_1} + \frac{\pi}{6} \in \left(\frac{\pi}{3}, \frac{13\pi}{6}\right)$ then $\frac{2\pi s}{r_1} + \frac{\pi}{6} = \pi$ or $\frac{2\pi s}{r_1} + \frac{\pi}{6} = 2\pi$.

For $12s = 5r_1$ one has $s = 5$, $r_1 = 12$ and $\lambda_1 = e^{\frac{5\pi i}{6}} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$, $\lambda_2 = e^{\frac{2\pi i}{3}}e^{\frac{5\pi i}{6}} = e^{-\frac{\pi i}{6}} = \sqrt{2} - \frac{1}{2}i$. Therefore $g$ is of order $r = LCM(12, 12) = 12$. Note that

$$
\begin{pmatrix}
  e^{\frac{5\pi i}{6}} & 0 \\
  0 & e^{-\frac{\pi i}{6}}
\end{pmatrix} \in GL(2, \mathcal{O}_{-3})
$$
attains this possibility.

In the case of $12s = 11r_1$ there follows $s = 11, r_1 = 12$. As a result, $\lambda_1 = e^{\frac{11\pi i}{6}} = \sqrt{3} - \frac{1}{2}i$, $\lambda_2 = e^{\frac{21\pi i}{6}} = e^{\frac{5\pi i}{2}} = -\sqrt{3} + \frac{1}{2}i$, which was already obtained.

If $\sin \left( \frac{2\pi s}{r_1} + \frac{\pi}{6} \right) = \frac{1}{2}$ for $\frac{2\pi s}{r_1} + \frac{\pi}{6} \in \left( \frac{\pi}{6}, \frac{13\pi}{6} \right)$ then $\frac{2\pi s}{r_1} + \frac{\pi}{6} = \frac{5\pi}{6}$ and $3s = r_1$. Therefore $s = 1, r_1 = 3$ and $\lambda_1 = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\lambda_2 = e^{\frac{2\pi i}{3}}e^{-\frac{2\pi i}{3}} = 1$. The order of $g$ is $r = \text{LCM}(3, 1) = 3$. This possibility is attained by

\[
\left( e^{\frac{2\pi i}{3}}, 0 
\right) \in \text{GL}(2, \mathbb{O}_{-3}).
\]

The equation $\sin \left( \frac{2\pi s}{r_1} + \frac{\pi}{6} \right) = -\frac{1}{2}$ has solutions $\frac{2\pi s}{r_1} + \frac{\pi}{6} = \frac{7\pi}{6}$ and $\frac{2\pi s}{r_1} + \frac{\pi}{6} = \frac{11\pi}{6}$.

If $2s = r_1$ then $s = 1, r_1 = 2, \lambda_1 = e^{\pi i} = -1, \lambda_2 = e^{\frac{2\pi i}{3}}e^{-\pi i} = e^{-\frac{2\pi i}{3}} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ and $g$ is of order $r = \text{LCM}(2, 6) = 6$. For instance,

\[
\left( e^{-\frac{\pi i}{3}}, 0, -1 \right) \in \text{GL}(2, \mathbb{O}_{-3})
\]

attains these eigenvalues.

For $6s = 5r_1$ one has $s = 5, r_1 = 6 \lambda_1 = e^{\frac{5\pi i}{6}} = e^{-\frac{\pi i}{6}} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, $\lambda_2 = e^{\frac{2\pi i}{3}}e^{\frac{\pi i}{6}} = e^{\frac{\pi i}{6}} = e^{\pi i} = -1$, which is already obtained.

Note that $\sin \left( \frac{2\pi s}{r_1} + \frac{\pi}{6} \right) = 1$ is equivalent to $\frac{2\pi s}{r_1} + \frac{\pi}{6} = \frac{\pi}{2}$, whereas $6s = r_1$ and $s = 1, r_1 = 6$. The eigenvalues $\lambda_1 = e^{\frac{\pi i}{6}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\lambda_2 = e^{\frac{2\pi i}{3}}e^{-\frac{\pi i}{6}} = e^{\frac{\pi i}{6}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ are equal, so that $g = e^{\frac{\pi i}{6}}I_2$ and $r = \text{LCM}(6, 6) = 6$.

If $\sin \left( \frac{2\pi s}{r_1} + \frac{\pi}{6} \right) = -1$ then $\frac{2\pi s}{r_1} + \frac{\pi}{6} = \frac{3\pi}{2}$ and $3s = 2r_1, s = 2, r_1 = 3$. Then $\lambda_1 = e^{\frac{2\pi i}{3}} = e^{-\frac{\pi i}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, $\lambda_2 = e^{\frac{2\pi i}{3}}e^{\frac{2\pi i}{3}} = e^{-\frac{\pi i}{3}}$ determine uniquely $g = e^{-\frac{2\pi i}{3}}I_2$ of order $r = \text{LCM}(3, 3) = 3$. That concludes the description of $g \in \text{GL}(2, \mathbb{O}_{-3})$ of finite order and $\det(g) = e^{\frac{2\pi i}{3}}$.

\[\square\]

**Proposition 22.** If $g \in \text{GL}(2, \mathbb{O}_{-3})$ is of finite order $r$ and $\det(g) = e^{-\frac{2\pi i}{3}}$ then

\[
\text{tr}(g) \in \left\{ 0, \pm \frac{(1 - \sqrt{-3})}{2}, \pm (1 - \sqrt{-3}) \right\}, \quad r \in \{3, 6, 12\}.
\]

More precisely,

(i) $\text{tr}(g) = 0$ or $\lambda_1 = e^{\frac{2\pi i}{3}}$, $\lambda_2 = e^{-\frac{5\pi i}{6}}$ if and only if $g$ is of order 12;
(ii) if $\text{tr}(g) = \frac{1-\sqrt{3}i}{2}$ or $\lambda_1 = e^{\frac{4\pi i}{3}}$, $\lambda_2 = 1$ then $g$ is of order 3;
(iii) if $\text{tr}(g) = -1 + \sqrt{3}i$ or $g = e^{\frac{2\pi i}{3}}I_2$ then $g$ is of order 3;
(iv) if $\text{tr}(g) = \frac{1+\sqrt{3}i}{2}$ or $\lambda_1 = e^{\frac{2\pi i}{3}}$, $\lambda_2 = -1$ then $g$ is of order 6;
(v) if $\text{tr}(g) = 1 - \sqrt{3}i$ or $g = e^{-\frac{\pi i}{6}}I_2$ then $g$ is of order 6.

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Proof. If \( \lambda_1 = e^{\frac{2\pi i}{r'}} \) then \( \lambda_2 = e^{-\frac{2\pi i}{3}} e^{-\frac{2\pi i}{r'}} \) and the trace

\[
\text{tr}(g) = \lambda_1 + \lambda_2 = (-1 + \sqrt{3}i) \sin \left( \frac{2\pi s}{r_1} - \frac{\pi}{6} \right)
\]

belongs to \( \mathcal{O}_{-3} = \mathbb{Z} + \frac{1 + \sqrt{3}i}{2} \mathbb{Z} \) if and only if \( 2 \sin \left( \frac{2\pi s}{r_1} - \frac{\pi}{6} \right) \in \mathbb{Z} \). Combining with

\[
2 \sin \left( \frac{2\pi s}{r_1} - \frac{\pi}{6} \right) \in [-2, 2],
\]

one concludes that \( \sin \left( \frac{2\pi s}{r_1} - \frac{\pi}{6} \right) \in \{0, \pm \frac{1}{2}, \pm 1\} \) and \( \text{tr}(g) \in \{0, \pm \frac{1}{2}, \pm 1\} \).

If \( \sin \left( \frac{2\pi s}{r_1} - \frac{\pi}{6} \right) = 0 \) with \( \frac{2\pi s}{r_1} - \frac{\pi}{6} \in \left(-\frac{\pi}{6}, \frac{11\pi}{6}\right) \) then \( \frac{2\pi s}{r_1} - \frac{\pi}{6} = 0 \) or \( \frac{2\pi s}{r_1} - \frac{\pi}{6} = \pi \).

For \( 12s = r_1 \) one has \( s = 1, r_1 = 12, \lambda_1 = e^{\frac{\pi i}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \lambda_2 = e^{-\frac{2\pi i}{3}} e^{-\frac{\pi i}{3}} = e^{-\frac{5\pi i}{6}} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i \), so that \( g \) is of order \( r = \text{LCM}(12, 12) = 12 \). For instance,

\[
\begin{pmatrix}
    e^{\frac{\pi i}{6}} & 0 \\
    0 & e^{-\frac{5\pi i}{6}}
\end{pmatrix} \in \text{GL}(2, \mathcal{O}_{-3})
\]

attains this case.

For \( 12s = 7r_1 \) there follows \( s = 7, r_1 = 12, \lambda_1 = e^{\frac{7\pi i}{6}} = e^{-\frac{5\pi i}{6}}, \lambda_2 = e^{-\frac{\pi i}{3}} e^{\frac{5\pi i}{6}} = e^{\frac{\pi i}{6}}, \) which is already discussed.

In the case of \( \sin \left( \frac{2\pi s}{r_1} - \frac{\pi}{6} \right) = \frac{1}{2} \) note that \( \frac{2\pi s}{r_1} - \frac{\pi}{6} = \frac{\pi}{6} \) or \( \frac{2\pi s}{r_1} - \frac{\pi}{6} = \frac{5\pi}{6} \).

If \( 6s = r_1 \) then \( s = 1, r_1 = 6, \lambda_1 = e^{\frac{\pi i}{3}} = \frac{1}{2} + \frac{\sqrt{3}i}{2}, \lambda_2 = e^{-\frac{2\pi i}{3}} e^{-\frac{\pi i}{3}} = e^{-\frac{5\pi i}{6}} = -1 \) and \( g \) is of order \( r = \text{LCM}(6, 2) = 6 \). Note that

\[
\begin{pmatrix}
    e^{\frac{\pi i}{3}} & 0 \\
    0 & -1
\end{pmatrix} \in \text{GL}(2, \mathcal{O}_{-3})
\]

attains this case.

For \( 2s = r_1 \) there follows \( s = 1, r_1 = 2, \lambda_1 = e^{\pi i} = -1, \lambda_2 = e^{-\frac{2\pi i}{3}} e^{-\frac{\pi i}{3}} = e^{-\frac{5\pi i}{6}} = e^{\frac{\pi i}{3}}, \) which is already obtained.

Note that \( \sin \left( \frac{2\pi s}{r_1} - \frac{\pi}{6} \right) = -\frac{1}{2} \) for \( \frac{2\pi s}{r_1} - \frac{\pi}{6} \in \left(-\frac{\pi}{6}, \frac{11\pi}{6}\right) \) implies \( \frac{2\pi s}{r_1} - \frac{\pi}{6} = \frac{7\pi}{6}, \) whereas

\( 3s = 2r_1, s = 2 \) and \( r_1 = 3 \). Then \( \lambda_1 = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}i}{2}, \lambda_2 = e^{-\frac{\pi i}{3}} e^{-\frac{\pi i}{3}} = e^{-\frac{2\pi i}{3}} = 1 \) and \( g \) is of order \( r = \text{LCM}(3, 1) = 3 \), attained by

\[
\begin{pmatrix}
    e^{\frac{2\pi i}{3}} & 0 \\
    0 & 1
\end{pmatrix} \in \text{GL}(2, \mathcal{O}_{-3}).
\]

If \( \sin \left( \frac{2\pi s}{r_1} - \frac{\pi}{6} \right) = 1 \) then \( \frac{2\pi s}{r_1} - \frac{\pi}{6} = \frac{\pi}{2} \) or \( 3s = r_1 \). As a result, \( s = 1, r_1 = 3, \lambda_1 = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}i}{2}, \lambda_2 = e^{-\frac{\pi i}{3}} e^{-\frac{\pi i}{3}} = e^{\frac{\pi i}{3}}, \) whereas \( g = e^{\frac{2\pi i}{3}} I_2 \in \text{GL}(2, \mathcal{O}_{-3}) \) is a scalar matrix of order 3.

Finally, \( \sin \left( \frac{2\pi s}{r_1} - \frac{\pi}{6} \right) = -1 \) holds for \( \frac{2\pi s}{r_1} - \frac{\pi}{6} = \frac{3\pi}{2}, \) i.e., \( 6s = 5r_1 \) and \( s = 5, r_1 = 6 \).

Now \( \lambda_1 = e^{-\frac{\pi i}{3}} = \frac{1}{2} - \frac{\sqrt{3}i}{2}, \lambda_2 = e^{-\frac{2\pi i}{3}} e^{\frac{\pi i}{3}} = e^{-\frac{\pi i}{3}}, \) so that \( g = e^{-\frac{\pi i}{3}} I_2 \in \text{GL}(2, \mathcal{O}_{-3}) \) is a scalar matrix of order 6. That concludes the proof of the proposition.

\( \square \)
3 Finite linear automorphism groups of $E \times E$

The classification of the finite subgroups $K$ of $SL(2, R)$ for an endomorphism ring $R$ of an elliptic curve $E$ starts with a classification of the Sylow subgroups $H_p$ of $K$.

**Proposition 23.** If $K$ is a finite subgroup of $SL(2, R)$ then $K$ is of order $|K| = 2^a3^b$ for some integers $0 \leq a \leq 3$, $0 \leq b \leq 1$.

If $K$ is of even order then the Sylow 2-subgroup $H_{2^a}$ of $K$ is isomorphic to $\mathbb{C}_2$, $\mathbb{C}_4$ or the quaternion group $\mathbb{Q}_8 = \langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \ g_2g_1 = -g_1g_2 \rangle$ of order 8.

If the order of $K$ is divisible by 3 then the Sylow 3-subgroup $H_{3^b}$ of $K$ is isomorphic to the cyclic group $\mathbb{C}_3$ of the third roots of unity.

**Proof.** According to the First Sylow Theorem, if $|K| = p_1^{m_1} \ldots p_k^{m_k}$ for some rational primes $p_j \in \mathbb{N}$ and some $m_j \in \mathbb{N}$, then for any $1 \leq i \leq k$ there is a subgroup $H_{p_j^i} \leq K$ of order $|H_{p_j^i}| = p_j^i$. In particular, any $H_{p_j} = \langle g_{p_j} \rangle \simeq \mathbb{C}_{p_j}$ of prime order $p_j$, dividing $|K|$ is cyclic and there is an element $g_{p_j} \in K$ of order $p_j$. By Proposition 15, the order of an element $g \in SL(2, R)$ is $1, 2, 3, 4, 6$ or $\infty$. As a result, if $g \in SL(2, R)$ is of prime order $p$ then $p = 2$ or 3. In other words, $K$ is of order $|K| = 2^a3^b$ for some non-negative integers $a, b$.

Suppose that $b \geq 1$ and consider the Sylow subgroup $H_{3^b} \leq K$ of order $3^b$. Then any $h \in H_{3^b} \setminus \{I_2\}$ is of order 3 since there is no $g \in SL(2, R)$, whose order is divisible by 9. We claim that $H_{3^b} = \langle h_1 \rangle \simeq \mathbb{C}_3$ is a cyclic group of order 3. Otherwise, $b \geq 2$ and there exists $h_2 \in H_{3^b} \setminus \langle h_1 \rangle$. Note that $h_2h_2 = I_2$ implies $h_2 = h_1^{-j} \in \langle h_1 \rangle$, contrary to the choice of $h_2$. We are going to show that if $h_1, h_2, h_3h_2 \in SL(2, R)$ are of order 3 then $h_2^2h_2 = I_2$, so that there is no $h_2 \in H_{3^b} \setminus \langle h_1 \rangle$ and $H_{3^b} = \langle h_1 \rangle \simeq \mathbb{C}_3$. According to Proposition 15, $g \in SL(2, R)$ is of order 3 if and only if $\text{tr}(g) = -1$ and $g$ is conjugate to

$$D_g = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}.$$ 

Similarly, $g \in SL(2, R)$ coincides with the identity matrix $I_2$ exactly when $\text{tr}(g) = 2$. Thus, we have to check that if $h_1, h_2 \in SL(2, R)$ satisfy $\text{tr}(h_1) = \text{tr}(h_2) = \text{tr}(h_1h_2) = -1$ then $\text{tr}(h_1^2h_2) = 2$. Let

$$D_1 = S^{-1}h_1S = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}$$

be a diagonal form of $h_1$ for some $S \in GL(2, \mathbb{C})$ and

$$D_2 = S^{-1}h_2S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}).$$

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(More precisely, if \( Q(R) = \mathbb{Q} \) or \( Q(\sqrt{-d}) \) is the fraction field of \( R \) then the eigenvectors of \( h_1 \) have entries from \( Q(R)(\sqrt{-3}) \), so that \( S, D_2 \in Q(R)(\sqrt{-3})_{2 \times 2} \) have entries from \( Q(R)(\sqrt{-3}) = \mathbb{Q}(\sqrt{-3}) \) or \( Q(\sqrt{-d}, \sqrt{-3}) \).) Since the determinant and the trace of a matrix are invariant under conjugation, the statement is equivalent to the fact that if \( \det(D_2) = 1 \) and \( \text{tr}(D_2) = \text{tr}(D_1 D_2) = -1 \) then \( \text{tr}(D_1^2 D_2) = 2 \). Indeed, if \( d = -a - 1 \) and \( \text{tr}(D_1 D_2) = e^{2\pi i} a - e^{-2\pi i} (a + 1) = -1 \) then \( a = e^{2\pi i} \), \( d = e^{-2\pi i} \), whereas \( \text{tr}(D_1^2 D_2) = 2 \). That proves the non-existence of \( h_2 \in H_{3} \setminus \langle h_1 \rangle \) and \( H_{3e} = H_3 = \langle h_1 \rangle \cong \mathbb{C}_3 \).

Suppose that \( K \) is of even order and denote by \( H_{2^n} \) the Sylow 2-subgroup of \( K < SL(2, R) \) of order \( 2^n \geq 2 \). Then any \( g \in H_{2^n} \setminus \{I_2\} \) is of order

\[
r \in \{2^i \mid i \in \mathbb{N}\} \cap \{1, 2, 3, 4, 6\} = \{2, 4\}.
\]

Recall from Proposition 15 that there is a unique element \(-I_2\) of \( SL(2, R) \) of order 2 and \( g \in SL(2, R) \) is of order 4 if and only if the trace \( \text{tr}(g) = 0 \). For \( a = 1 \) the Sylow subgroup \( H_2 = \langle -I_2 \rangle \cong \mathbb{C}_2 \) is cyclic of order 2. If \( a = 2 \) then \( H_4 = \langle g \rangle \cong \mathbb{C}_4 \) is cyclic of order 4, since \( SL(2, R) \) has a unique element \(-I_2\) of order 2. From now on, let us assume that \( a \geq 3 \) and fix an element \( g_1 \in H_{2^n} \) of order 4. Due to \( g_1^2 = -I_2 \in \langle g_1 \rangle \), any \( g_2 \in H_{2^n} \setminus \langle g_1 \rangle \) is of order 4 and \( g_2^2 = -I_2 \). Moreover, \( g_1 g_2 \in H_{2^n} \) is of order 4, as far as \( g_1 g_2 = \pm I_2 \) requires \( g_2 = \mp g_1 \in \langle g_1 \rangle \), contrary to the choice of \( g_2 \). We claim that if \( g_1, g_2 \in SL(2, R) \) of order 4 have product \( g_1 g_2 \) of order 4 then they generate a quaternion group

\[
\langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \quad g_2 g_1 = -g_1 g_2 \rangle \cong \mathbb{Q}_8
\]

of order 8. In other words, if \( g_1, g_2 \in R_{2 \times 2} \) have \( \det(g_1) = \det(g_2) = 1 \) and \( \text{tr}(g_1) = \text{tr}(g_2) = \text{tr}(g_1 g_2) = 0 \) then \( g_2 g_1 = -g_1 g_2 \). In particular, if \( g_1, g_2 \in SL(2, R) \) of order 4 have product \( g_1 g_2 \) of order 4 then \( g_2 \not\in \langle g_1 \rangle = \{\pm I_2, \pm g_1\} \). To this end, let

\[
D_1 = S^{-1} g_1 S = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\]

be the diagonal form of \( g_1 \) and

\[
D_2 = S^{-1} g_2 S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

for appropriate matrices \( S \) and \( D_2 \) with entries from \( Q(R)(\sqrt{-1}) = \mathbb{Q}(\sqrt{-1}) \) or \( \mathbb{Q}(\sqrt{-d}, \sqrt{-1}) \). The determinant and the trace are invariant under conjugation, so that suffices to show that if \( \det(D_2) = 1 \) and \( \text{tr}(D_2) = \text{tr}(D_1 D_2) = 0 \) then \( D_2 D_1 = -D_1 D_2 \), whereas

\[
g_2 g_1 = (SD_2 S^{-1})(SD_1 S^{-1}) = S(D_2 D_1) S^{-1} =
\]

\[
= S(-D_1 D_2) S^{-1} = -(SD_1 S^{-1})(SD_2 S^{-1}) = -g_1 g_2.
\]

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Indeed, \( \text{tr}(D_2) = a + d = 0 \) and \( \text{tr}(D_1D_2) = i(a - d) = 0 \) require \( a = d = 0 \). Now, \( \det(D_2) = -bc = 1 \) determines \( c = -\frac{1}{b} \) for some \( b \in \mathbb{Q}(\sqrt{d}, \sqrt{-1}) \) and

\[
D_2D_1 = \begin{pmatrix} 0 & -ib \\ \frac{i}{b} & 0 \end{pmatrix} = -D_1D_2.
\]

Thus, if \( a = 3 \) then the Sylow 2-subgroup of \( K \) is isomorphic to the quaternion group \( \mathbb{Q}_8 \) of order 8,

\[
H_8 = \langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \ g_2g_1 = -g_1g_2 \rangle \simeq \mathbb{Q}_8.
\]

There remains to be rejected the case of \( a \geq 4 \). The assumption \( a \geq 4 \) implies the existence of \( g_3 \in H_{2^a} \setminus \langle g_1, g_2 \rangle \). Any such \( g_3 \) is of order 4, together with the products \( g_1g_3 \in H_{2^a} \) for \( 1 \leq j \leq 2 \), since \( g_3g_3 = \pm I_2 \) amounts to \( g_3 = \pm g_3^3 \in \langle g_j \rangle \) and contradicts the choice of \( g_3 \). Thus, the subgroups

\[
\langle g_1, g_3 \mid g_1^2 = g_3^2 = -I_2, \ g_3g_1 = -g_1g_3 \rangle \simeq
\]

\[
\langle g_2, g_3, \mid g_2^2 = g_3^2 = -I_2, \ g_3g_2 = -g_2g_3 \rangle \simeq \mathbb{Q}_8
\]

are also isomorphic to \( \mathbb{Q}_8 \). In particular,

\[
D_3 = S^{-1}g_3S = \begin{pmatrix} 0 & b_3 \\ -\frac{1}{b_3} & 0 \end{pmatrix}
\]

with \( b_3 \in \mathbb{Q}(\sqrt{d}, \sqrt{-1})^* \) is subject to

\[
D_3D_2 = \begin{pmatrix} -\frac{b}{b_3} & 0 \\ 0 & -\frac{b}{b_3} \end{pmatrix} = \begin{pmatrix} \frac{b}{b_3} & 0 \\ 0 & \frac{b}{b_3} \end{pmatrix} = -D_2D_3,
\]

whereas \( b_3^2 = -b^2 \) or \( b_3 = \pm ib \). As a result, \( D_3 = D_1D_2 \) and \( g_3 = g_1g_2 \), contrary to the choice of \( g_3 \notin \langle g_1, g_2 \rangle \). Therefore \( a < 4 \) and the Sylow 2-subgroup of a finite group \( K < SL(2, R) \) is \( H_2 \simeq \mathbb{C}_2 \), \( H_4 \simeq \mathbb{C}_4 \) or \( H_8 \simeq \mathbb{Q}_8 \).

\[\square\]

**Proposition 24.** Any finite subgroup \( K \) of \( SL(2, R) \) is isomorphic to one of the following:

\[
K_1 = \{I_2\},
\]

\[
K_2 = \langle -I_2 \rangle \simeq \mathbb{C}_2,
\]

\[
K_3 = \langle g_1 \rangle \simeq \mathbb{C}_4 \text{ for some } g_1 \in SL(2, R) \text{ with } \text{tr}(g_1) = 0,
\]

\[
K_4 = \langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \ g_2g_1g_2 = g_1 \rangle \simeq \mathbb{Q}_8,
\]

\[
K_5 = \langle g_3 \rangle \simeq \mathbb{C}_3 \text{ for some } g_3 \in SL(2, R) \text{ with } \text{tr}(g_3) = -1,
\]

\[
K_6 = \langle g_4 \rangle \simeq \mathbb{C}_6 \text{ for some } g_4 \in SL(2, R) \text{ with } \text{tr}(g_4) = 1
\]

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\[ K_7 = \langle g_1, g_4 \mid g_1^2 = g_4^3 = -I_2, \ g_4 g_1 g_4 = g_1 \rangle \cong \mathbb{Q}_{12} \]

for some \( g_1, g_4 \in SL(2, R) \) with \( \text{tr}(g_1) = 0, \ \text{tr}(g_4) = 1, \)

\[ K_8 = \langle g_1, g_2, g_3 \mid g_1^2 = g_2^2 = -I_2, \ g_3^3 = I_2, \ g_2 g_1 = -g_1 g_2, \]

\[ g_3 g_1 g_3^{-1} = g_2, \ g_3 g_2 g_3^{-1} = g_1 g_2 \rangle \cong \text{SL}(2, \mathbb{F}_3) \]

for some \( g_1, g_2, g_3 \in SL(2, R), \ \text{tr}(g_1) = \text{tr}(g_2) = 0, \ \text{tr}(g_3) = -1, \) where \( \mathbb{Q}_8 \) denotes the quaternion group of order 8, \( \mathbb{Q}_{12} \) stands for the dicyclic group of order 12 and \( \text{SL}(2, \mathbb{F}_3) \) is the special linear group over the field \( \mathbb{F}_3 \) with three elements.

**Proof.** By Proposition 23, \( K \) is of order 1, 2, 3, 6, 12 or 24. The only subgroup \( K < \text{SL}(2, R) \) of order 1 is \( K = K_1 = \{I_2\} \). Since \(-I_2\) is the only element of \( \text{SL}(2, R) \) of order 2, the group \( K = K_2 = \langle -I_2 \rangle \cong \mathbb{C}_2 \) is the only cyclic subgroup of \( \text{SL}(2, R) \) of order 2. Any subgroup \( K < \text{SL}(2, R) \) of order 4 is cyclic or \( K = K_3 = \langle g_1 \rangle \) for some \( g_1 \in \text{SL}(2, R) \) with \( \text{tr}(g_1) = 0 \), because \( \text{SL}(2, R) \) has a unique element \(-I_2\) of order 2. Proposition 15 has established the existence of elements \( g_1 \in \text{SL}(2, \mathbb{Z}) \leq \text{SL}(2, R) \) of order 4.

If \( K < \text{SL}(2, R) \) is a subgroup of order 8 then it coincides with its Sylow 2-subgroup

\[ K = H_8 = \langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \ g_2 g_1 = -g_1 g_2 \rangle = K_4 \cong \mathbb{Q}_8, \]

isomorphic to the quaternion group \( \mathbb{Q}_8 \) of order 8. Note that there is a realization

\[ \mathbb{Q}_8 \cong \langle D_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ D_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle < \text{SL}(2, \mathbb{Z}[i]) \]

as a subgroup of \( \text{SL}(2, \mathbb{Z}[i]) \). In general,

\[ D_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \in \text{SL}(2, R) \]

amount to \( a_j^2 + b_j c_j = -1 \). The anti-commuting relation \( g_2 g_1 = -g_1 g_2 \) is equivalent to \( 2a_1 a_2 + b_1 c_2 + b_2 c_1 = 0 \). Therefore \( K_4 = \langle g_1, g_2 \rangle < \text{SL}(2, R) \) is a realization of \( \mathbb{Q}_8 \) if and only if \( a_j, b_j, c_j \in R \) are subject to

\[
\begin{align*}
    a_1^2 + b_1 c_1 &= -1 \\
    a_2^2 + b_2 c_2 &= -1 \\
    2a_1 a_2 + b_1 c_2 + b_2 c_1 &= 0
\end{align*}
\]

The existence of a solution of (12) in an arbitrary \( R = R_{-d,f} = \mathbb{Z} + f\mathcal{O}_{-d} = \mathbb{Z} + f\omega_{-d}\mathbb{Z} \) is an open problem.

If \( |K| = 3 \) then \( K = K_5 = \langle g_3 \rangle \cong \mathbb{C}_3 \) for some \( g_3 \in \text{SL}(2, R) \) with \( \text{tr}(g_3) = -1 \).
From now on, let us assume that $K$ is of order $|K| = 2^a \cdot 3$ for some $1 \leq a \leq 3$ and consider some Sylow subgroups $H_2, H_3 = \langle g_4 \rangle \cong \mathbb{C}_3$ of $K$. We claim that the product

$$H_{2^a}H_3 = \{gg_4^i \mid g \in H_{2^a}, \ 0 \leq i \leq 2\}$$

depletes $K$. More precisely, $H_{2^a} \cap H_3 = \{I_2\}$, because $2^a$ and 3 are relatively prime. Therefore

$$H_{2^a}H_3/H_{2^a} = H_{2^a} \cup H_{2^a}g_4 \cup H_{2^a}g_4^2$$

is a right coset decomposition of the subset $H_{2^a}H_3 \subseteq K$ modulo $H_{2^a}$. Due to the disjointness of this decomposition, one has $|H_{2^a}H_3| = 3|H_{2^a}| = 3.2^a = |K|$. Therefore, the subset $H_{2^a}H_3$ of $K$ coincides with $K$ and $K = H_{2^a}H_3$ is a product of its Sylow subgroups.

If $K = H_2H_3 = \langle -I_2 \rangle \langle g_3 \rangle$ for some $g_3 \in SL(2, R)$ with $\text{tr}(g_3) = -1$ then $\pm I_2$ commute with $g_3^j$ for all $0 \leq j \leq 2$ and the group $K$ is abelian. Thus, $K = \langle -g_3 \rangle \cong \mathbb{C}_6$ is a cyclic group of order 6, generated by $-g_3 \in SL(2, R)$ with $\text{tr}(-g_3) = 1$.

For $K = H_4H_3 = \langle g_1 \rangle \langle g_3 \rangle$ with $g_1, g_3 \in SL(2, R)$ of $\text{tr}(g_1) = 0$, $\text{tr}(g_3) = -1$, note that $g_4 = -g_3 \in SL(2, R)$ is of order 6. Then $g_4^3 = -I_2 = g_1^2$, because $-I_2 \in SL(2, R)$ is the only element of order 2. We claim that $g_1, g_4 \in SL(2, R)$ are subject to $g_4g_1g_4 = g_1$. To this end, let $S \in \mathbb{Q}(R)(\sqrt{-d}, \sqrt{-3})_{2 \times 2}$ be a matrix, whose columns are eigenvectors of $g_1$. Then

$$D_4 = S^{-1}g_4S = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix}$$

and

$$D_1 = S^{-1}g_1S = \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} \text{ with } a_1^2 + b_1c_1 = -1$$

generate the subgroup $K^o = S^{-1}KS \cong K$. It suffices to check that $D_4D_1D_4 = D_1$, because then $g_4g_1g_4 = (SD_4S^{-1})(SD_1S^{-1})(SD_4S^{-1}) = S(D_4D_1D_4)S^{-1} = SD_1S^{-1} = g_1$ and

$$K = \langle g_1, g_3 \rangle = \langle g_1, g_4 = -g_3 \mid g_1^2 = g_4^3 = -I_2, \ g_4g_1g_4 = g_1 \rangle \cong \mathbb{Q}_{12}$$

is isomorphic to the dicyclic group $\mathbb{Q}_{12}$ of order 12. The group $K^o = \langle D_1, D_4 \rangle \cong K$ of order 12 has a cyclic subgroup $\langle D_4 \rangle \cong \mathbb{C}_6$ of order 6. The index $[K^o : \langle D_4 \rangle] = 2$, so that $\langle D_4 \rangle$ is a normal subgroup of $K^o$ and $D_1D_4D_4^{-1} \in \langle D_4 \rangle$ is an element of order 6. More precisely, $D_1D_4D_4^{-1} = D_4$ or $D_1D_4D_4^{-1} = D_4^{-1} = D_4$. If $D_1D_4 = D_4D_1$ then $D_1D_4 \in K^o$ is of order 12, as far as $(D_1D_4)^{12} = (D_1^4)^3(D_4^3)^2 = I_2^3I_2^2 = I_1$, $(D_1D_4)^6 = D_1^2 = -I_2 \neq I_2$, $(D_1D_4)^4 = D_4^3 = -D_4 \neq I_2$, whereas $D_1D_4, (D_1D_4)^2, (D_1D_4)^3 \notin \{I_2\}$. Consequently, $D_1D_4 = -D_4^3D_1$, so that $D_1D_4D_4 = -D_4^3D_1D_1$ and $K \cong K^o \cong \mathbb{Q}_{12}$. For instance, the subgroup

$$\langle D_1 \rangle = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ D_4 = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix} \text{ and } D_1^2 = D_4^3 = -I_2, \ D_1D_4D_4^{-1} = D_4^{-1}$$

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of $SL(2, O_{-3})$ realizes $Q_{12}$ as a subgroup of $SL(2, O_{-3})$. The existence of $Q_{12} \simeq K < \text{SL}(2, R)$ for an arbitrary $R$ is an open problem.

There remains to be shown that any subgroup $K = H_8H_3 = \langle g_1, g_2, g_3 \rangle \simeq \mathbb{Q}_8 \mathbb{C}_3$ of $\text{SL}(2, R)$ of order 24 is isomorphic to the special linear group $K_8 \simeq SL(2, \mathbb{F}_3)$ over $\mathbb{F}_3$. In other words, any $K < \text{SL}(2, R)$ of order $|K| = 24$ can be generated by such $g_1, g_2, g_3 \in \text{SL}(2, R)$ that the subgroup $\langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \ g_2g_1 = -g_1g_2 \rangle \simeq \mathbb{Q}_8$ is isomorphic to the quaternion group $Q_8$ of order 8, $g_3$ is of order 3 and $g_3g_1g_3^{-1} = g_1g_2g_3g_2^{-1} = g_1g_2$.

First of all, the Sylow 2-subgroup $H_8 \simeq \mathbb{Q}_8$ of $K$ is normal. More precisely, by the Third Sylow Theorem, the number $n_2 \in \mathbb{N}$ of the Sylow 2-subgroups of $K$ (i.e., the number $n_2$ of the subgroups of $K$ of order 8) divides $|K| = 24$ and $n_2 \equiv 1 \pmod{2}$. Therefore $n_2 = 1$ or $n_2 = 2$. By Second Sylow Theorem, all Sylow 2-subgroups are conjugate to each other, so that $n_2 = 1$ exactly when $H_8 = \langle g_1, g_2 \rangle \simeq \mathbb{Q}_8$ is a normal subgroup of $K$. Let us assume that $n_2 = 3$ and denote by $\nu_i$ the number of the elements $g \in K$ of order $i$. Due to $-I_2 \in H_8 = \langle g_1, g_2 \rangle < K$, one has $\nu_1 = 1$, $\nu_2 = 1$. Note that $g \in K$ is of order 3 if and only if $-g \in K$ is of order 6, so that $\nu_6 = \nu_3$. By the Third Sylow Theorem, the number $n_3 \in \mathbb{N}$ of the Sylow 3-subgroups of $K$ divides $|K| = 24$ and $n_3 \equiv 1 \pmod{3}$. Therefore $n_3 = 1$ or $n_3 = 4$.

If $n_3 = 1$ and there is a unique normal subgroup $H_3 = \langle g_3 \rangle \simeq \mathbb{C}_3$ of $K$ of order 3, then $g_3^jg_3^{-j} \in \langle g_3, g_3^{-1} \rangle \subset \langle g_3 \rangle$ for $j = 1$ and $j = 2$. If $g_3^jg_3^{-j} = g_3$ then $g_3g_3 = g_3g_3$ for $g_3$ of order 4 and $g_3$ of order 3, so that $g_3g_3 \in K$ is of order 12, contrary to the non-existence of an element of $SL(2, R)$ of order 12. Therefore $g_1g_3g_1^{-1} = g_3^2$, $g_2g_3g_2^{-1} = g_3$, whereas

$$(g_1g_2)g_3(g_1g_2)^{-1} = g_1(g_2g_3g_2^{-1})g_1^{-1} = g_1g_3^2g_1^{-1} = (g_1g_3g_1)^{-1}2 = (g_3^2)^2 = g_3$$

and $g_1g_2$ of order 4 commutes with $g_3$ of order 3. Thus, $\langle g_1, g_2, g_3 \rangle \simeq \mathbb{Q}_8$ is of order 12, which is an absurd. That assumes the assumption $n_3 = 1$ and proves that $n_3 = 4$.

Let $H_{3,j} = \langle g_{3,j} \rangle \simeq \mathbb{C}_3$, $1 \leq j \leq 4$ be the four subgroups of $K$ of order 3. Then $H_{3,i} \cap H_{3,j} = \{I_2\}$ for all $1 \leq i < j \leq 4$, as far as any $g \in H_{3,i} \setminus \{I_2\}$ generates $H_{3,i}$. As a result, $\bigcup_{i=1}^{4} H_{3,i}$ and $K$ contain 8 different elements $g_{3,i}, g_{3,i}^{-1}$, $1 \leq i \leq 4$ of order 3 and $\nu_6 = \nu_3 = 8$. Thus,

$$24 = |K| = \nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_6 = 18 + \nu_4,$$

so that $K$ has $\nu_4 = 6$ elements of order 4. Since any Sylow 2-subgroup

$$H_8 = \langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \ g_2g_1 = -g_1g_2 \rangle \simeq \mathbb{Q}_8$$

of $K$ contains six elements $\pm g_1, \pm g_3, \pm g_1g_2$ of order 4, there cannot be more than one $H_8$. In other words, $n_3 = 1$ and $H_8$ is a normal subgroup of $K$.

The above considerations show that

$$K = H_8 \times H_3 = \langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \ g_2g_1 = -g_1g_2 \rangle \times \langle g_3 \mid g_3^3 = I_2 \rangle \simeq \mathbb{Q}_8 \times \mathbb{C}_3$$

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is a semi-direct product of $\mathbb{Q}_8$ and $\mathbb{C}_3$. Up to an isomorphism, $K$ is uniquely determined by the group homomorphism

$$\varphi_K : H_3 \rightarrow \text{Aut}(H_8),$$

$$\varphi_K(g_j^k)(\pm g_l^k g_3^l) = g_j^k(\pm g_l^k g_3^l)g_3^{-1} \quad \text{for} \quad \forall \pm g_l^k g_3^l \in H_8, \quad 0 \leq k, l \leq 1.$$ 

Since $H_3 = \langle g_3 \rangle \cong \mathbb{C}_3$ is cyclic, $\varphi_K$ is uniquely determined by $\varphi_K(g_3) \in \text{Aut}(H_8)$. On the other hand, $H_3$ is generated by $g_1, g_2$, so that suffices to specify $\varphi_K(g_3)(g_j) = g_3g_jg_3^{-1} \in H_8$ for $1 \leq j \leq 2$, in order to determine $\varphi_K$. If the cyclic group $\langle g_1 \rangle \cong \mathbb{C}_4$ is normalized by $g_3$ then $g_3g_1g_3^{-1} \in \{ \pm g_1 \}$, as an element of order 4. In the case of $g_3g_1g_3^{-1} = g_1$, the element $g_1 \in K$ of order 4 commutes with the element $g_3 \in K$ of order 3 and their product $g_1g_3 \in K$ is of order 12. The lack of $g \in SL(2, R)$ of order 12 requires $g_3g_1g_3^{-1} = -g_1$. Now,

$$g_3^2g_1g_3^{-2} = g_3(g_3g_1g_3^{-1})g_3^{-1} = g_3(-g_1)g_3^{-1} = g_1$$

is equivalent to $g_3^2g_1 = g_1g_3^2$ and the product $g_1g_3^2 \in K$ of $g_1 \in K$ of order 4 with $g_3^2 \in K$ of order 3 is an element of order 12. The absurd justifies that neither of the cyclic subgroups $\langle g_1 \rangle \cong \langle g_2 \rangle \cong \langle g_1g_2 \rangle \cong \mathbb{C}_4$ of order 4 of $H_8$ is normalized by $g_3$. Thus, an arbitrary $g_1 \in H_8 \cong \mathbb{Q}_8$ of order 4 is completed by $g_2 := g_3g_1g_3^{-1} \in H_8 \setminus \langle g_1 \rangle$ of order 4 to a generating set of $H_8 \cong \mathbb{Q}_8$. Then

$$g_3^2g_1g_3^{-2} = g_3(g_3g_1g_3^{-1})g_3^{-1} = g_3g_2g_3^{-1} \in H_8 \setminus (\langle g_1 \rangle \cup \langle g_2 \rangle) = \{ g_1g_2, g_2g_1 \}$$

specifies that either $g_3g_2g_3^{-1} = g_1g_2$ or $g_3g_2g_3^{-1} = g_2g_1$. If $g_3g_2g_3^{-1} = g_2g_1$, we replace the generator $g_3$ of $K$ by $h_3 = g_3^2$ and note that $h_3g_1h_3^{-1} = g_2g_1$. Now, $h_1 := g_1$ and $h_2 := g_2g_1$ generate $H_8 = \langle h_1, h_2 \mid h_1^2 = h_2^2 = -I_2, h_2h_1 = -h_1h_2 \rangle$ and satisfy $h_3h_1h_3^{-1} = h_2$,

$$h_3h_2h_3^{-1} = g_3[(g_3g_2g_3^{-1})(g_3g_1g_3^{-1})]g_3^{-1} = g_3(g_2g_1g_2)g_3^{-1} = g_3g_1g_3^{-1} =$$

$$= g_2 = -(g_2g_1)g_1 = -h_2h_1 = h_1h_2.$$ 

Thus, the group

$$K' = \langle g_1, g_2, g_3 \mid g_1^2 = g_2^2 = -I_2, g_2g_1 = -g_1g_2, g_3^2 = I_2, g_3g_1g_3^{-1} = g_2, g_3g_2g_3^{-1} = g_2g_1 \rangle$$

is isomorphic ro the group

$$K = \langle g_1, g_2, g_3 \mid g_1^2 = g_2^2 = -I_2, g_2g_1 = -g_1g_2, g_3^2 = I_2, g_3g_1g_3^{-1} = g_2, g_3g_2g_3^{-1} = g_1g_2 \rangle.$$ 

We shall realize $SL(2, \mathbb{F}_3)$ as a subgroup $K_8' = \langle D_1, D_2, D_3 \rangle$ of $SL(2, \mathbb{Q}(\sqrt{-d}, \sqrt{-3}))$. The existence of subgroups $SL(2, \mathbb{F}_3) \cong K_8' \subset SL(2, R)$ is an open problem. Towards the construction of $K_8'$, let us choose

$$D_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \quad \text{with} \quad a_j^2 + b_jc_j = -1 \quad \text{for} \quad 1 \leq j \leq 2 \quad \text{and}$$

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$$D_3 = \begin{pmatrix}
e^{\frac{2\pi i}{3}} & 0 \\
0 & e^{-\frac{2\pi i}{3}}
\end{pmatrix}$$

from $SL(2, \mathbb{Q}(\sqrt{-d}, \sqrt{-3}))$. After computing

$$D_3D_jD_3^{-1} = \begin{pmatrix}a_j & e^{-\frac{2\pi i}{3}}b_j \\
e^{\frac{2\pi i}{3}}c_j & -a_j\end{pmatrix} \text{ for } 1 \leq j \leq 2,$$

observe that $D_3D_1D_3^{-1} = D_2$ reduces to

$$\begin{pmatrix}a_2 = a_1 \\
b_2 = e^{-\frac{2\pi i}{3}}b_1 \\
c_2 = e^{\frac{2\pi i}{3}}c_1\end{pmatrix}.$$

The relation $D_2D_1 = -D_1D_2$ is equivalent to $2a_1a_2 + b_1c_2 + b_2c_1 = 0$ and implies that $2a_1^2 = b_1c_1$. Now,

$$D_3D_2D_3^{-1} = \begin{pmatrix}a_1 & e^{\frac{2\pi i}{3}}b_1 \\
e^{-\frac{2\pi i}{3}}c_1 & -a_1\end{pmatrix} = \begin{pmatrix}\sqrt{-3}a_1 & \sqrt{-3}e^{\frac{2\pi i}{3}}a_1b_1 \\
\sqrt{-3}e^{-\frac{2\pi i}{3}}a_1c_1 & -\sqrt{-3}a_1^2\end{pmatrix} = D_1D_2$$

is tantamount to

$$\begin{pmatrix}a_1(1 - \sqrt{-3}a_1) = 0 \\
b_1(1 - \sqrt{-3}a_1) = 0 \\
c_1(1 - \sqrt{-3}a_1) = 0\end{pmatrix}$$

and specifies that $a_1 = \frac{\sqrt{-3}}{3}$. Namely, the assumption $a_1 \neq -\frac{\sqrt{-3}}{3}$ forces $a_1 = b_1 = c_1 = 0$, whereas $\det(D_1) = 0$, contrary to the choice of $D_1 \in SL(2, \mathbb{Q}(\sqrt{-d}, \sqrt{-3}))$.

As a result, $b_1 \neq 0$, $c_1 = -\frac{2}{3b_1}$ and

$$D_1 = \begin{pmatrix}-\frac{\sqrt{-3}}{3} & b_1 \\
-\frac{2}{3b_1} & \frac{\sqrt{-3}}{3}\end{pmatrix}, \quad D_2 = \begin{pmatrix}-\frac{\sqrt{-3}}{3} & e^{-\frac{2\pi i}{3}}b_1 \\
e^{\frac{2\pi i}{3}}c_1 & \frac{\sqrt{-3}}{3}\end{pmatrix}, \quad D_3 = \begin{pmatrix}e^{\frac{2\pi i}{3}} & 0 \\
0 & e^{-\frac{2\pi i}{3}}\end{pmatrix}$$

generate a subgroup $SL(2, \mathbb{F}_3) \simeq K_8^\circ < SL(2, \mathbb{Q}(\sqrt{-d}, \sqrt{-3})).$

\[\square\]

**Corollary 25.** If the finite subgroup $K$ of $SL(2, R)$ is not isomorphic to the dicyclic group

$$K_7 = \langle g_1, g_4 \mid g_1^2 = g_4^3 = -I_2, \quad g_4g_1g_4 = g_1 \rangle = \langle g_1, g_3 = -g_4 \mid g_1^2 = -I_2, \quad g_3^2 = I_2, \quad g_3g_1g_3^{-1} = g_1g_2 \rangle \simeq \mathbb{Q}_{12}$$

of order 12 then $K$ is isomorphic to a subgroup of the special linear group

$$K_8 = \langle g_1, g_2, g_3 \mid g_1^2 = g_2^2 = -I_2, \quad g_3^2 = I_2, \quad g_2g_1 = -g_1g_2, \quad g_3g_1g_3^{-1} = g_2, \quad g_3g_2g_3^{-1} = g_1g_2 \rangle \simeq SL(2, \mathbb{F}_3)$$

over the field $\mathbb{F}_3$ with three elements.
Proof. According to Proposition 24, any finite subgroup \( K < SL(2, R) \) is isomorphic to some of the groups \( K_1, \ldots, K_8 \). Thus, it suffices to establish that any \( K_j, 1 \leq j \leq 6 \) is isomorphic to a subgroup of \( K_8 \). Note that \( K_1 = \{I_2\} \subset K_8 \) and \( K_2 = \{-I_2\} \subset K_8 \) are subgroups of \( K_8 \). The generator \( g_1 \) of \( K_8 \) is of order 4, so that any subgroup \( K_3 \cong \mathbb{C}_4 \) of \( SL(2, R) \) is isomorphic to the subgroup \( \langle g_1 \rangle \) of \( K_8 \). In the proof of Proposition 24 we have seen that \( K_8 \) has a normal Sylow 2-subgroup

\[
H_8 = \langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \ g_2g_1 = -g_1g_2 \rangle \cong \mathbb{Q}_8,
\]
isomorphic to the quaternion group \( \mathbb{Q}_8 \cong \mathbb{K}_4 \) of order 8. The generator \( g_3 \) of \( K_8 \) provides a subgroup \( \langle g_3 \rangle \cong \mathbb{C}_3 \cong \mathbb{K}_5 \) of \( K_8 \). The product \( -I_2 g_3 \) of the commuting elements \(-I_2 \in K_8 \) or order 2 and \( g_3 \in K_8 \) of order 3 is an element \(-g_3 \in K_8 \) of order 6, so that \( K_6 \cong \mathbb{C}_6 \) is isomorphic to the subgroup \( \langle -g_3 \rangle \) of \( K_8 \).

\( \square \)

Towards the classification of the finite subgroups of \( GL(2, R) \), we proceed with the following:

**Lemma 26.** Let \( H \) be a finite subgroup of \( GL(2, R) \). Then

(i) \( \det(H) \) is a cyclic subgroup of \( R^* \);

(ii) \( H \) is a product \( H = [H \cap SL(2, R)] \langle h_o \rangle \) of its normal subgroup \( H \cap SL(2, R) \) and any \( \mathbb{C}_r \cong \langle h_o \rangle \subseteq H \) with \( \det(H) = \langle \det(h_o) \rangle \) \cong \( \mathbb{C}_s \);

(iii) the order \( s \) of \( \det(H) = \langle \det(h_o) \rangle \) divides the order \( r \) of \( h_o \in H \) and

\[
[H \cap SL(2, R)] \cap \langle h_o \rangle = \langle h_o^s \rangle \cong \mathbb{C}_r;
\]

(iv) \( H \) is of order \( s | [H \cap SL(2, R)] \);

(v) \( s = r \) if and only if \( H = [H \cap SL(2, R)] \ltimes \langle h_o \rangle \) is a semi-direct product.

Proof. (i) The image \( \det(H) \) of the group homomorphism \( \det : H \to R^* \) is a subgroup of \( R^* \). As far as the units group \( R^* \) of the endomorphism ring \( R \) of \( E \) is cyclic, its subgroup \( \det(H) \) is cyclic, as well.

(ii) If \( \det(h_o) \) is a generator of the cyclic subgroup \( \det(H) < R^* \) then one can represent \( H = [H \cap SL(2, R)] \langle h_o \rangle \). The inclusion \( [H \cap SL(2, R)] \langle h_o \rangle \subseteq H \) is clear by the choice of \( h_o \in H \). For the opposite inclusion, note that any \( h \in H \) with \( \det(h) = (\det(h_o))^m \) for some \( m \in \mathbb{Z} \) is associated with \( hh_o^{-m} \in H \cap SL(2, R) \), so that \( h = (hh_o^{-m})h_o^m \in [H \cap SL(2, R)] \langle h_o \rangle \) and \( H \subseteq [H \cap SL(2, R)] \langle h_o \rangle \).

(iii) If \( h_o \in H \) is of order \( r \) then \( h_o^s = I_2 \) and \( \det(h_o)^s = 1 \). Therefore the order \( s \) of \( \det(h_o) \in R^* \) divides \( s \). Note that \( h_o^s \in [H \cap SL(2, R)] \cap \langle h_o \rangle \), as far as \( \det(h_o^s) = \det(h_o)^s = 1 \). Therefore \( \langle h_o^s \rangle \) is a subgroup of \( [H \cap SL(2, R)] \cap \langle h_o \rangle \). Conversely, any \( h_o^x \in [H \cap SL(2, R)] \cap \langle h_o \rangle \) has \( \det(h_o^x) = \det(h_o)^x = 1 \), so that \( s \) divides \( x \) and \( h_o^x \in \langle h_o^s \rangle \). That justifies \( [H \cap SL(2, R)] \cap \langle h_o \rangle \subseteq \langle h_o^s \rangle \) and \( [H \cap SL(2, R)] \cap \langle h_o \rangle = \langle h_o^s \rangle \). The order of \( \langle h_o^s \rangle \) and \( h_o^s \) is \( \frac{s}{r} \), since \( s \) divides \( r \).
(iv) It suffices to show that

$$H = \cup_{j=0}^{s-1} [H \cap SL(2, R)] h_o^j$$

is the coset decomposition of $H$ with respect to its normal subgroup $H \cap SL(2, R)$, in order to conclude that the order $|H|$ of $H$ is $s$ times the order $|H \cap SL(2, R)|$ of $H \cap SL(2, R)$. The inclusion $H \supseteq \cup_{j=0}^{s-1} [H \cap SL(2, R)] h_o^j$ is clear by the choice of $h_o \in H$. According to $H = \langle H \cap SL(2, R) \rangle \langle h_o \rangle$, any element of $H$ is of the form $h = gh_o^m$ for some $g \in H \cap SL(2, R)$ and $m \in \mathbb{Z}$. If $m = sq + r_o$ is the division of $m$ by $s$ with residue $0 \leq r_o \leq s - 1$ then $h = [g(h_o^s)^q]h_o^r \in \langle H \cap SL(2, R) \rangle h_o^s$, due to $h_o \in H \cap SL(2, R)$. Therefore $H \subseteq \bigcup_{j=0}^{s-1} [H \cap SL(2, R)] h_o^j$ and $H = \bigcup_{j=0}^{s-1} [H \cap SL(2, R)] h_o^j$.

The cosets $[H \cap SL(2, R)] h_o^i$ and $[H \cap SL(2, R)] h_o^j$ are mutually disjoint for any $0 \leq i < j \leq s - 1$, because the assumption $g_1 h_i = g_2 h_o^j$ for $g_1, g_2 \in H \cap SL(2, R)$ implies that $h_o^{-i} = g_2^{-1} g_1 \in \langle H \cap SL(2, R) \rangle \langle h_o \rangle = \langle h_o^s \rangle$. As a result, $s$ divides $0 < j - i < s$, which is an absurd.

(v) According to (iii), the order $s$ of $\det(h_o)$ divides the order $r$ of $h_o$. On the other hand, $h_o^s = I_2$ exactly when $r$ divides $s$, so that $h_o^s = I_2$ is equivalent to $r = s$. Thus, $r = s$ exactly when

$$[H \cap SL(2, R)] \cap \langle h_o \rangle = \{I_2\}.$$

As far as the product of the normal subgroup $H \cap SL(2, R)$ and the subgroup $\langle h_o \rangle$ is the entire $H$, one has a semi-direct product $H = [H \cap SL(2, R)] \rtimes \langle h_o \rangle$ if and only if $r = s$.

\[\square\]

**Lemma 27.** Let $H = \langle H \cap SL(2, R) \rangle \langle h_o \rangle$ be a finite subgroup of $GL(2, R)$ for $h_o \in H$ of order $r$ with $\det(H) = \langle \det(h_o) \rangle \simeq \mathbb{C}_s$ and $H \cap SL(2, R)$ be generated by $g_0 = h_o^s, g_1, \ldots, g_t$. Then $H \cap SL(2, R)$, $r$ and

$$h_o g_i h_o^{-1} \in H \cap SL(2, R) \quad \text{for all} \quad 1 \leq i \leq t$$

determine $H$ up to an isomorphism.

**Proof.** By the proof of Lemma 26 (iv), $H$ has a coset decomposition

$$H = \cup_{j=0}^{s-1} [H \cap SL(2, R)] h_o^j$$

with respect to its normal subgroup $H \cap SL(2, R)$. Therefore, the group structures of $H \cap SL(2, R)$ and $\langle h_o \rangle \simeq \mathbb{C}_r$, together with the multiplication rule for $h_1 h_o^i, h_2 h_o^j \in H$ with $h_1, h_2 \in H \cap SL(2, R)$ and $0 \leq i, j \leq s - 1$ determine the group $H$ up to an isomorphism. Let us represent $h_1 = g_{j_1}^{a_1} g_{j_2}^{a_2} \cdots g_{j_k}^{a_k}$ and $h_2 = g_{j_1}^{b_1} g_{j_2}^{b_2} \cdots g_{j_k}^{b_k}$ as words in the alphabet $g_0 = h_o^s, g_1, \ldots, g_t$ with some integral exponents $a_p, b_q \in \mathbb{Z}$. (The group $H$ is finite, so that any $g_i$ is of finite order $r_i$ and one can reduce the exponent of $g_i$ to a residue modulo $r_i$.) In order to determine the product $(h_1 h_o^i)(h_2 h_o^j)$ as an element
of $H = \bigcup_{j=0}^{s-1} \langle g_0, g_1, \ldots, g_t \rangle h_o^j$, it suffices to specify $g'_i \in H \cap SL(2, R) = \langle g_0, g_1, \ldots, g_t \rangle$ with $h_o g_i = g'_i h_o$ for all $0 \leq i \leq t$. That allows to move gradually $h'_o$ to the end of $(h_1 h_o)(h_2 h'_o)^j$, producing $h_1 h_2 h_{o+j} \in [H \cap SL(2, R)] h_o^{i+j(\text{mod} s)}$ for an appropriate $h'_o \in H \cap SL(2, R)$. In other words, the group structures of $H \cap SL(2, R)$ and $\langle h_o \rangle \simeq \mathbb{C}_r$, together with the conjugates $g'_i = h_o g_i h_o^{-1}$ of $g_i$ determine the group multiplication in $H$. Note that $h_o g_i h_o^{-1} = g_o$, since $g_0 = h_o$ commutes with $h_o$. The conjugates $g'_i = h_o g_i h_o^{-1}$ with $1 \leq i \leq t$ belong to the normal subgroup $H \cap SL(2, R) \ni g_i$ of $H$ and have the same orders $r_i$ as $g_i$. 

Any finite subgroup $H = [H \cap SL(2, R)] \langle h_o \rangle$ of $GL(2, R)$ with determinant $\det(H) = \langle \det(h_o) \rangle \simeq \mathbb{C}_s$ has a conjugate

$$S^{-1}HS = \{S^{-1}[H \cap SL(2, R)]S\} \langle S^{-1}h_oS \rangle = [S^{-1}HS \cap SL(2, \mathbb{C})] \langle S^{-1}h_oS \rangle$$

with a diagonal matrix $S^{-1}h_oS$. More precisely, if $R$ is a subring of the integers ring $\mathcal{O}_{-d}$ of an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ and $\lambda_1 = \lambda_1(h_o)$, $\lambda_2 = \lambda_2(h_o)$ are the eigenvalues of $h_o$, then there exists a basis

$$v_1 = \begin{pmatrix} s_{11} \\ s_{21} \end{pmatrix}, \quad v_2 = \begin{pmatrix} s_{12} \\ s_{22} \end{pmatrix} \quad \text{of} \quad \mathbb{C}^2,$$

consisting of eigenvectors $v_j$ of $h_o$, associated with the eigenvalues $\lambda_j = \lambda_j(h_o)$. This is due to the finite order of $h_o$, because the Jordan block

$$J = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \quad \text{with} \quad \lambda_1 \in \mathbb{C}^*$$

is of infinite order in $GL(2, \mathbb{C})$. The matrix $S = (s_{ij})_{i,j=1}^2$ with columns $v_1, v_2$ is nonsingular and its entries belong to the extension $\mathbb{Q}((\sqrt{-d}, \lambda(h_o)) = \mathbb{Q}(\sqrt{-d}, \lambda_1(h_o))$ of $\mathbb{Q}(\sqrt{-d})$ by some of the eigenvalues of $h_o$. Making use of the classification of $h_o \in GL(2, R)$ of finite order $r$ and $\det(h_o) \in R^*$ of order $s$, done in section 2, one determines explicitly the field $F_{-d}^{(s,r)} = \mathbb{Q}(\sqrt{-d}, \lambda_1(h_o))$, obtained from $\mathbb{Q}(\sqrt{-d})$ by adjoining an eigenvalue $\lambda_1(h_o)$ of $h_o \in H$. The group

$$S^{-1}HS = [S^{-1}HS \cap SL(2, \mathbb{C})] \langle S^{-1}h_oS \rangle$$

has a diagonal generator $D_o = S^{-1}h_oS$ and the conjugates

$$(S^{-1}h_oS)(S^{-1}g_iS)(S^{-1}h_oS)^{-1} = S^{-1}(h_o g_i h_o^{-1})S$$

are easier to be computed.

The next lemma collects the fields $F_{-d}^{(s,r)}$. 

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Lemma 28. Let $H = [H \cap SL(2, R)](h_o)$ be a finite subgroup of $GL(2, R)$ with $h_o \in H$ of order $r$, $\det(h_o) \in R^*$ of order $s$ and $F_{-d}^{(s,r)}$ be the number field

$$F_{-d}^{(s,r)} = \begin{cases} 
\mathbb{Q}(\sqrt{-d}) & \text{for } s = r = 2, \\
\mathbb{Q}(i) & \text{for } s \in \{2, 4\}, r = 4, \\
\mathbb{Q}(\sqrt{-3}) & \text{for } (s, r) = (2,6) \text{ or } s \in \{3, 6\}, \\
\mathbb{Q}(\sqrt{2}, i) & \text{for } s \in \{2, 4\}, r = 8, \\
\mathbb{Q}(\sqrt{3}, i) & \text{for } s = 2, r = 12.
\end{cases}$$

Then there exists a matrix $S \in GL(2, F_{-d}^{(s,r)})$ such that

$$D_o = S^{-1}h_o S = \begin{pmatrix} \lambda_1(h_o) & 0 \\
0 & \lambda_2(h_o) \end{pmatrix}$$

is diagonal and

$$H^o = S^{-1}HS = [S^{-1}HS \cap SL(2, F_{-d}^{(s,r)})](D_o)$$

is a subgroup of $GL(2, F_{-d}^{(s,r)})$, isomorphic to $H$.

Summarizing the results of section 2, one obtains also the following

Corollary 29. If $h_o \in GL(2, R) \setminus SL(2, R)$ is of order $r$ with $\det(h_o) \in R^*$ of order $s$ and eigenvalues $\lambda_1(h_o), \lambda_2(h_o)$, then

$$\frac{\lambda_1(h_o)}{\lambda_2(h_o)} \in \{ \pm 1, \pm i, e^{\pm \frac{2\pi i}{4}}, e^{\pm \frac{2\pi i}{6}}, e^{\pm \frac{2\pi i}{8}} \}.$$ 

More precisely,

(i) $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = 1$ exactly when $h_o \in \{ \pm iI_2, e^{\pm \frac{2\pi i}{3}}I_2, e^{\pm \frac{2\pi i}{4}}I_2 \}$

is a scalar matrix;

(ii) $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = -1$ for

(a) $\lambda_1(h_o) = 1, \lambda_2(h_o) = -1$ and an arbitrary $R = R_{-d,f}$;

(b) $\lambda_1(h_o) = e^{\pm \frac{\pi i}{4}}, \lambda_2(h_o) = e^{\mp \frac{\pi i}{4}}, R = \mathbb{Z}[i], s = 4$;

(c) $\lambda_1(h_o) = e^{\pm \frac{\pi i}{4}}, \lambda_2(h_o) = e^{\mp \frac{\pi i}{4}}, R = \mathbb{O}_3, s = 3$

(d) $\lambda_1(h_o) = e^{\pm \frac{2\pi i}{3}}, \lambda_2(h_o) = e^{\mp \frac{2\pi i}{3}}, R = \mathbb{O}_3, s = 6$.

(iii) $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = \pm i$ for
Proposition 30. Let \( \lambda \) be an element of order \( s \) and eigenvalues \( \lambda_1(h_o), \lambda_2(h_o) \). Then \( r = s \) and \( H \) is isomorphic to \( H_{C_1(j)} \simeq \mathbb{C}_s \) for some \( 1 \leq j \leq 4 \), where

\[
H_{C_1(1)} = \langle h_o \rangle \simeq \mathbb{C}_2 \quad \text{with} \quad \lambda_1(h_o) = 1, \quad \lambda_2(h_o) = -1,
\]

\[
H_{C_1(2)} = \langle h_o \rangle \simeq \mathbb{C}_3 \quad \text{with} \quad R = \mathcal{O}_{-3}, \quad h_o = e^{-\frac{2\pi i}{3}}I_2 \quad \text{or} \quad \lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \quad \lambda_2(h_o) = 1,
\]

\[
H_{C_1(3)} = \langle h_o \rangle \simeq \mathbb{C}_4 \quad \text{with} \quad R = \mathbb{Z}[i], \quad \{\lambda_1(h_o), \lambda_2(h_o)\} = \{i, 1\} \quad \text{or} \quad \{-i, -1\},
\]

\[
H_{C_1(4)} = \langle h_o \rangle \simeq \mathbb{C}_6 \quad \text{with} \quad R = \mathcal{O}_{-3},
\]

\[
\{\lambda_1(h_o), \lambda_2(h_o)\} = \left\{ e^{\frac{2\pi i}{3}}, \frac{1}{\lambda_2(h_o)} \right\}, \quad \left\{ e^{-\frac{2\pi i}{3}}, -1 \right\} \quad \text{or} \quad \left\{ e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}} \right\}.
\]

**Proof.** By Lemma 26 (ii), the group \( H = \langle h_o \rangle \simeq \mathbb{C}_r \) is cyclic and generated by any \( h_o \in H \), whose determinant \( \det(h_o) \) generates \( \det(H) = \langle \det(h_o) \rangle \). Moreover, Lemma 26 (iii) specifies that \( \{I_2\} = [H \cap SL(2, R)] \cap \langle h_o \rangle = \langle h_o^s \rangle \) or the order \( r \) of \( h_o \) coincides with the order \( s \) of \( \det(h_o) \). For \( s \in \{3, 4, 6\} \) one can assume that \( \det(h_o) = e^{2\pi i} \), since the generators of \( \det(H) = \langle \det(h_o) \rangle \simeq \mathbb{C}_s \) are \( e^{\frac{2\pi i}{s}} \) and \( e^{-\frac{2\pi i}{s}} \). Making use of the classification of the elements \( h_o \in GL(2, R) \) of order \( s \) with \( \det(h_o) = e^{2\pi i} \), done in section 2, one concludes that \( H \simeq H_{C_1(j)} \) for some \( 1 \leq j \leq 4 \). \( \square \)
Proposition 31. Let $H$ be a finite subgroup of $GL(2, R)$, 

$$H \cap SL(2, R) = \langle -I_2 \rangle \simeq C_2$$

and $h_o \in H$ be an element of order $r$ with $\det(H) = \langle \det(h_o) \rangle \simeq C_s$ and eigenvalues $\lambda_1(h_o), \lambda_2(h_o)$. Then $H$ is isomorphic to $H_{C_2}(i)$ for some $1 \leq i \leq 6$, where

$$H_{C_2}(1) = \langle iI_2 \rangle \simeq C_4 \quad \text{with} \quad R = \mathbb{Z}[i],$$

$$H_{C_2}(2) = \langle -I_2 \rangle \times \langle h_o \rangle \simeq C_2 \times C_2 \quad \text{with} \quad \lambda_1(h_o) = 1, \quad \lambda_2(h_o) = -1,$$

$$H_{C_2}(3) = \langle h_o \rangle \simeq C_6 \quad \text{with} \quad R = \mathcal{O}_3, \quad h_o = e^{\frac{2\pi}{3}}I_2 \quad \text{or} \quad \lambda_1(h_o) = e^{-\frac{2\pi}{3}}, \quad \lambda_2(h_o) = -1,$$

$$H_{C_2}(4) = \langle h_o \rangle \simeq C_8 \quad \text{with} \quad R = \mathbb{Z}[i], \quad \lambda_1(h_o) = e^{\frac{2\pi}{3}}, \quad \lambda_2(h_o) = e^{-\frac{2\pi}{3}},$$

$$H_{C_2}(5) = \langle -I_2 \rangle \times \langle h_o \rangle \simeq C_2 \times C_4 \quad \text{with} \quad R = \mathbb{Z}[i], \quad \lambda_1(h_o) = i, \quad \lambda_2(h_o) = 1,$$

$$H_{C_2}(6) = \langle h_o \rangle \simeq C_8 \quad \text{with} \quad R = \mathbb{Z}[i], \quad \lambda_1(h_o) = e^{\frac{2\pi}{3}}, \quad \lambda_2(h_o) = e^{-\frac{2\pi}{3}},$$

$$H_{C_2}(7) = \langle -I_2 \rangle \times \langle h_o \rangle \simeq C_2 \times C_6 \quad \text{with} \quad R = \mathcal{O}_3,$$

$$\{\lambda_1(h_o), \lambda_2(h_o)\} = \{e^{\frac{2\pi}{3}}, e^{-\frac{2\pi}{3}}, 1\} \quad \text{or} \quad \{e^{\frac{2\pi}{3}}, -1\}.$$  \hfill (13)

Proof. By Lemma 26 (iii), one has $h_o^s \in H \cap SL(2, R) = \langle -I_2 \rangle$ for some $s \in \{2, 3, 4, 6\}$. If $h_o^s = I_2$ then $s = r$ and

$$H = \langle -I_2 \rangle \times \langle h_o \rangle \simeq C_2 \times C_s$$

is a direct product, as far as the scalar matrix $-I_2$ commutes with $h_o$. When $h_o$ is of odd order $s = 3$, its opposite matrix $-h_o \in H$ is of order 6 and $H = \langle -h_o \rangle \simeq C_6$. Without loss of generality, $h_1 := -h_o$ has $\det(h_1) = e^{\frac{2\pi}{3}}$ and Proposition 21 specifies that either $h_1 = e^{\frac{2\pi}{3}}I_2$ or $\lambda_1(h_1) = e^{-\frac{2\pi}{3}}, \lambda_2(h_o) = -1$. For $s = 2$ the group $H = \langle -I_2 \rangle \times \langle h_o \rangle = H_{C_2}(2) \simeq C_2 \times C_2$, where $h_o \in H$ has eigenvalues $\lambda_1(h_o) = 1, \lambda_2(h_o) = -1$. The case $s = 4$ occurs only for $R = \mathbb{Z}[i]$. Assuming $\det(h_o) = i$, one gets $\lambda_1(h_o) = \varepsilon i, \lambda_2(h_o) = \varepsilon$ for some $\varepsilon \in \{\pm 1\}$ by Proposition 17. Since $-I_2 \in H$, one can replace $h_o$ by $-h_o$ and reduce to the case of $\varepsilon = 1$. If $s = 6$, then Proposition 19 provides (13).

In the case of $h_o^s = -I_2$, the intersection $\langle h_o \rangle SL(2, R) = \langle -I_2 \rangle = H \cap SL(2, R)$ and the group

$$H = \langle h_o \rangle \simeq C_{2s}$$

is cyclic. More precisely, for $s = 2$ Proposition 16 implies that $h_o = \pm iI_2$ and $H \simeq H_{C_2}(1)$. If $s = 3$ and $\det(h_o) = e^{\frac{2\pi}{3}}$ then $H \simeq H_{C_2}(3)$ by Proposition 21. For $s = 4$ and $\det(h_o) = i$ one has $H \simeq H_{C_2}(6)$, according to Proposition 17. Making use of Proposition 19, one observes that there are no $h_o \in GL(2, R)$ of order 12 with $\det(h_o) = e^{\frac{2\pi}{3}}$ and concludes the proof of the proposition. \qed
Towards the description of the finite subgroups $H = [H \cap SL(2, R)] \langle h_0 \rangle$ of $GL(2, R)$ with $H \cap SL(2, R) \cong \mathbb{C}_t$ for some $t \in \{3, 4, 6\}$, one needs the following

**Lemma 32.** If $g \in GL(2, \mathbb{C})$ has different eigenvalues $\lambda_1 \neq \lambda_2$ then any $h \in GL(2, \mathbb{C})$ with $hg \neq gh$ and $h^2g = gh^2$ has vanishing trace $\text{tr}(h) = 0$.

*Proof.* The trace is invariant under conjugation, so that

$$g = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

can be assumed to be diagonal. If

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}),$$

then $h^2g = gh^2$ is equivalent to

$$\begin{vmatrix} (\lambda_1 - \lambda_2)b(a + d) = 0 \\ (\lambda_1 - \lambda_2)c(a + d) = 0 \end{vmatrix}.$$  

Due to $\lambda_1 \neq \lambda_2$, there follow $b(a + d) = 0$ and $c(a + d) = 0$. The assumption $\text{tr}(h) = a + d \neq 0$ leads to $b = c = 0$. As a result,

$$h = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

is a diagonal matrix and commutes with $g$. The contradiction justifies that $\text{tr}(h) = 0$. \hfill $\Box$

**Lemma 33.** Let $H = [H \cap SL(2, R)] \langle h_0 \rangle$ be a finite subgroup of $GL(2, R)$ with

$$H \cap SL(2, R) = \langle g \rangle \cong \mathbb{C}_t \quad \text{for some} \quad t \in \{3, 4, 6\} \quad \text{and}$$

$$\det(H) = \langle \det(h_0) \rangle = \langle e^{\frac{2\pi i}{s}} \rangle \cong \mathbb{C}_s, \quad s > 1$$

for some $h_0 \in H$ of order $r$. Then:

1. $\frac{r}{s}$ divides $t$;
2. $\frac{r}{s} = t$ if and only if $H = \langle h_0 \rangle \cong \mathbb{C}_r$ is cyclic and $H \cap SL(2, R) = \langle h_0^s \rangle$;
(iii) if \( \frac{\tau}{s} < t \) then \( H \) is isomorphic to the non-cyclic abelian group

\[
H' = \langle g, h_o \mid g^t = h_o^s = I_2, \ h_o g = g h_o \rangle
\]

or to the non-abelian group

\[
H'' = \langle g, h_o \mid g^t = h_o^s = I_2, \ h_o g h_o^{-1} = g^{-1} \rangle;
\]

(iv) if \( \frac{\tau}{s} < t \) and \( H \cong H'' \) is non-abelian then \( h_o \) has eigenvalues \( \lambda_1(h_o) = i e^{\frac{2\pi}{t}} \), \( \lambda_2(h_o) = -i e^{\frac{2\pi}{t}} \) and

\[
(r, s) \in \{(2, 2), \ (6, 6)\} \quad \text{for} \quad t = 3,
\]

\[
(r, s) \in \{(2, 2), \ (8, 4), \ (6, 6)\} \quad \text{for} \quad t = 4,
\]

\[
(r, s) \in \{(2, 2), \ (8, 4), \ (6, 6)\} \quad \text{for} \quad t = 6.
\]

Proof. (i) Note that if \( \det(h_o) \in R^* \) is of order \( s \) then \( \det(h_o^s) = \det(h_o)^s = 1 \) and \( h_o^s \in H \cap SL(2, R) = \langle g \rangle \) is an element of order \( \frac{\tau}{s} \). Since \( \langle g \rangle \cong C_t \) is of order \( t \), the ratio \( \frac{\tau}{s} \in \mathbb{N} \) divides \( t \). Proposition 16 provides the list of \( \frac{\tau}{s} = \frac{\tau}{t} \) for \( s = 2 \). If \( s = 3 \) then the values of \( \frac{\tau}{s} \) are taken from Propositions 21 and 22. Propositions 17 and 18 supply the range of \( \frac{\tau}{s} = \frac{\tau}{t} \) for \( s = 4 \), while Propositions 19 and 20 give account for \( \frac{\tau}{s} = \frac{\tau}{t} \) in the case of \( s = 6 \).

(ii) Note that \( h_o^s \in \langle g \rangle \) is of order \( \frac{\tau}{s} = t \) exactly when \( \langle g \rangle = \langle h_o^s \rangle \) and \( H = \langle h_o \rangle \cong C_r \) is a cyclic group.

(iii) According to Lemma 27, the group \( H = [H \cap SL(2, R)]\langle h_o \rangle = \langle g \rangle \langle h_o \rangle \) is completely determined by the order \( t \) of \( g \), the order \( r \) of \( h_o \) and the conjugate \( x = h_o g h_o^{-1} \in H \cap SL(2, R) = \langle g \rangle \) of \( g \) by \( h_o \). The order \( t \) of \( g \) is invariant under conjugation, so that \( x = g^m \) for some \( m \in \mathbb{Z}_t^* \). The Euler function \( \varphi(t) = 2 \) for \( t \in \{3, 4, 6\} \) and \( \mathbb{Z}_t^* = \{ \pm 1 (\text{mod} t) \} \). Therefore \( x = h_o g h_o^{-1} = g \) or \( x = h_o g h_o^{-1} = g^{-1} \).

(iv) If \( H \cong H'' \) is a non-abelian group then

\[
h_o^2 g h_o^{-2} = h_o (h_o g h_o^{-1}) h_o^{-1} = h_o g^{-1} h_o^{-1} = (h_o g h_o^{-1})^{-1} = (g^{-1})^{-1} = g,
\]

so that \( g \) commutes with \( h_o^2 \), but does not commute with \( h_o \). By Lemma 32 there follows \( \text{tr}(h_o) = 0 \). There exists a matrix \( S \in GL \left( 2, \mathbb{Q} \left( \sqrt{-d}, e^{\frac{2\pi i}{t}} \right) \right) \), such that

\[
D = S^{-1} g S = \begin{pmatrix} e^{\frac{2\pi i}{t}} & 0 \\ 0 & e^{-\frac{2\pi i}{t}} \end{pmatrix} \in SL \left( 2, \mathbb{Q} \left( \sqrt{-d}, e^{\frac{2\pi i}{t}} \right) \right)
\]

is diagonal. Since the trace is invariant under conjugation,

\[
D_o := S^{-1} h_o S = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in GL \left( 2, \mathbb{Q} \left( \sqrt{-d}, e^{\frac{2\pi i}{t}} \right) \right),
\]

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The relation \( h_o g = g^{-1} h_o \) implies the vanishing of \( a \). As a result, the characteristic polynomial
\[
X_{h_o}(\lambda) = \lambda^2 + \det(h_o) = \lambda^2 + e^{\frac{2\pi i}{r}} = 0
\]
has roots \( \lambda_1(h_o) = ie^{\frac{2\pi i}{r}} \), \( \lambda_2(h_o) = -ie^{\frac{2\pi i}{r}} \). More precisely, for \( s = 2 \) one has \( \lambda_1(h_o) = -1 \), \( \lambda_2(h_o) = 1 \), so that \( h_o \) and \( D_o \) are of order \( r = 2 \). The ratio \( \frac{r}{s} = 1 \) divides any \( t \in \{3, 4, 6\} \). If \( s = 3 \) then \( \lambda_1(h_o) = e^{\frac{3\pi i}{r}} \), \( \lambda_2(h_o) = e^{-\frac{3\pi i}{r}} \), so that \( h_o \) and \( D_o \) are of order \( r = 12 \). The quotient \( \frac{s}{r} = 4 \) divides only \( t = 4 \). Therefore \( \frac{s}{r} = t \) and \( H = \langle h_o \rangle \simeq \mathbb{C}_1 \), according to (ii). In the case of \( s = 4 \), one has \( \lambda_1(h_o) = e^{\frac{4\pi i}{r}} \), \( \lambda_2(h_o) = e^{-\frac{4\pi i}{r}} \), whereas \( h_o \) and \( D_o \) are of order \( r = 8 \). The quotient \( \frac{s}{r} = 3 \) divides only \( t \in \{4, 6\} \). Finally, for \( s = 6 \) the automorphism \( h_o \) has eigenvalues \( \lambda_1(h_o) = e^{\frac{6\pi i}{r}} \), \( \lambda_2(h_o) = e^{-\frac{6\pi i}{r}} \). Consequently, \( h_o \) and \( D_o \) are of order \( r = 6 \) and \( \frac{s}{r} = 1 \) divides all \( t \in \{3, 4, 6\} \).

\[ \square \]

**Lemma 34.** (i) For arbitrary \( d \in \mathbb{N} \) and \( t \in \{3, 4, 6\} \) there is a dihedral subgroup
\[
D_t = \langle g, h_o \mid g^t = h_o^t = I_2, \ h_o g h_o^{-1} = g^{-1} \rangle < GL(2, \mathbb{Q}(\sqrt{-d}))
\]
of order \( 2t \) with \( D_t \cap SL(2, \mathbb{Q}(\sqrt{-d})) = \langle g \rangle \simeq \mathbb{C}_t \), \( \det(D_t) = \det(h_o) = (-1) \simeq \mathbb{C}_2 \)
and eigenvalues \( \lambda_1(h_o) = -1 \), \( \lambda_2(h_o) = 1 \) of \( h_o \).

(ii) For an arbitrary \( t \in \{3, 4, 6\} \) there is a subgroup
\[
H_t = \langle g, h_o \mid g^t = h_o^t = I_2, \ h_o g h_o^{-1} = g^{-1} \rangle < GL(2, \mathbb{Q}(\sqrt{-3}))
\]
of order \( 6t \) with \( H_t \cap SL(2, \mathbb{Q}(\sqrt{-3})) = \langle g \rangle \simeq \mathbb{C}_t \), \( \det(H_t) = \det(h_o) = (e^{\frac{2\pi i}{6}}) \simeq \mathbb{C}_6 \)
and eigenvalues \( \lambda_1(h_o) = e^{\frac{2\pi i}{6}} \), \( \lambda_2(h_o) = e^{-\frac{2\pi i}{6}} \) of \( h_o \).

(iii) For an arbitrary \( t \in \{4, 6\} \) there is a subgroup
\[
H'_t = \langle g, h_o \mid g^t = h_o^4 = -I_2, \ h_o g h_o^{-1} = g^{-1} \rangle < GL(2, \mathbb{Q}(\sqrt{2}, i))
\]
of order \( 4t \) with \( H'_t \cap SL(2, \mathbb{Q}(\sqrt{2}, i)) = \langle g \rangle \simeq \mathbb{C}_t \), \( \det(H'_t) = \det(h_o) = \langle i \rangle \simeq \mathbb{C}_4 \)
and eigenvalues \( \lambda_1(h_o) = e^{\frac{4\pi i}{6}} \), \( \lambda_2(h_o) = e^{-\frac{4\pi i}{6}} \) of \( h_o \).

**Proof.** (i) Let us choose a diagonalizing matrix \( S \in GL(2, \mathbb{Q}(\sqrt{-d})) \) of \( h_o \), so that
\[
D_o = S^{-1} h_o S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Taking into account Proposition 15, one has to show the existence of
\[
D = S^{-1} g S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{-d}))
\]
with
\[
D_o D D_o^{-1} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = D^{-1}
\]
for any trace $\text{tr}(g) = \text{tr}(D) = a + d \in \{0, \pm 1\}$. More precisely, for $a = d = 0$, $b \neq 0$ and $c = -b^{-1}$, then the matrix

$$D = D_4 = \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}$$

of order 4 and the matrix $D_o$ of order 2 generate a dihedral group $D_4$ of order 8. If $a = d = -\frac{1}{2}$, $b \neq 0$ and $c = -\frac{3}{4}b^{-1}$ then

$$D = D_3 = \begin{pmatrix} -\frac{1}{2} & b \\ -\frac{3}{4}b^{-1} & -\frac{1}{2} \end{pmatrix}$$

of order 3 and $D_o$ of order 2 generate a symmetric group $D_3 \simeq S(3)$ of degree 3. In the case of $a = d = \frac{1}{2}$, $b \neq 0$ and $c = -\frac{3}{4}b^{-1}$, the matrix

$$D = D_6 = \begin{pmatrix} \frac{1}{2} & b \\ -\frac{3}{4}b^{-1} & \frac{1}{2} \end{pmatrix}$$

of order 6 and the matrix $D_o$ of order 2 generate a dihedral group $D_6$ of order 12.

(ii) By Proposition 19, if $h_o \in GL(2, R)$ has eigenvalues $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = e^{-\frac{2\pi i}{3}}$ then $R = O_{-3}$. Let us consider

$$D_o = S^{-1}h_oS = \begin{pmatrix} e^{\frac{2\pi i}{4}} & 0 \\ 0 & e^{-\frac{\pi i}{4}} \end{pmatrix} \in GL(2, \mathbb{Q}(\sqrt{-3}))$$

for some $S \in GL(2, \mathbb{Q}(\sqrt{-3}))$ and

$$D = S^{-1}gS = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{-3}))$$

with trace $\text{tr}(g) = \text{tr}(D) = a + d \in \{0, \pm 1\}$. Then

$$D_oDD_o^{-1} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = D^{-1}$$

is equivalent to $a = d$. Consequently, $D_3, D_4, D_6$ from the proof of (i) satisfy the required conditions.

(iii) Note that

$$D_o = S^{-1}h_oS = \begin{pmatrix} e^{\frac{3\pi i}{4}} & 0 \\ 0 & e^{-\frac{\pi i}{4}} \end{pmatrix} \in GL(2, \mathbb{Q}(\sqrt{2}, i))$$
for some $S \in GL(2, \mathbb{Q}(\sqrt{2}, i))$ and
\[
D = S^{-1}gS = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{2}, i))
\]
with trace $\text{tr}(g) = \text{tr}(D) = a + d \in \{0, 1\}$ satisfy
\[
D_oDD_o^{-1} = \begin{pmatrix} a & -b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = D^{-1}
\]
extactly when $a = d$. In the notations from the proof of (i), one has $\langle D_4, D_o \rangle \simeq \mathcal{H}_4'$ and $\langle D_6, D_o \rangle \simeq \mathcal{H}_6'$.

\[\square\]

Corollary 35. Let $H$ be a finite subgroup of $GL(2, R)$,
\[
H \cap SL(2, R) = \langle g \rangle \simeq \mathbb{C}_3
\]
and $h_o \in H$ be an element of order $r$ with $\det(H) = \langle \det(h_o) \rangle \simeq \mathbb{C}_s$ and eigenvalues $\lambda_1(h_o), \lambda_2(h_o)$. Then $H$ is isomorphic to some $H_{C3}(i)$, $1 \leq i \leq 5$, where
\[
H_{C3}(1) = \langle h_o \rangle \simeq \mathbb{C}_6
\]
with $R = R_{-3,f}$, $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = e^{\frac{4\pi i}{3}}$,
\[
H_{C3}(2) = \langle g, h_o \mid g^3 = h_o^2 = I_2, \ h_ogh_o^{-1} = g^{-1} \rangle \simeq S_3
\]
is the symmetric group of degree 3, $\lambda_1(h_o) = -1, \lambda_2(h_o) = 1,$
\[
H_{C3}(3) = \langle g \rangle \times \langle e^{\frac{2\pi i}{3}}I_2 \rangle \simeq \mathbb{C}_3 \times \mathbb{C}_3
\]
with $R = O_{-3}$ and any $g \in SL(2, O_{-3})$ of trace $\text{tr}(g) = -1$,
\[
H_{C3}(4) = \langle g \rangle \times \langle h_o \rangle \simeq \mathbb{C}_3 \times \mathbb{C}_6
\]
with $R = O_{-3}$, $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = e^{-\frac{2\pi i}{3}},$
\[
H_{C3}(5) = \langle g, h_o \mid g^3 = h_o^6 = I_2, \ h_ogh_o^{-1} = g^{-1} \rangle
\]
of order 18 with $R = O_{-3}, \lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = e^{-\frac{2\pi i}{3}}.$

There exist subgroups
\[
H_{C3}(1), H_{C3}(3), H_{C3}(4) < GL(2, O_{-3}),
\]
as well as subgroups
\[
H_{C3}(2)_{o} < GL(2, \mathbb{Q}(\sqrt{-d})) , \ H_{C3}(5)_{o} < GL(2, \mathbb{Q}(\sqrt{-3}))
\]
with $H_{C3}(j)_{o} \simeq H_{C3}(j)$ for $j \in \{2, 5\}$. 51
Proof. By Lemma 33 (i), the quotient \( \frac{\pi}{s} \) is a divisor of \( t = 3 \), so that either \( r = s \) or \( r = 3s = 6 \).

For \( s = 2 \), \( r = 6 \) one has a cyclic group \( H = \langle h_\circ \rangle \simeq \mathbb{C}_6 \) with \( \det(h_\circ) = -1 \). Up to an inversion \( h_\circ \mapsto h_\circ^{-1} \) of the generator, Proposition 16 specifies that \( \lambda_1(h_\circ) = e^{\frac{2\pi i}{3}} \), \( \lambda_2(h_\circ) = e^{\frac{2\pi i}{3}} \) and justifies the realization of \( H_{C3}(1) = \langle h_\circ \rangle \) over \( \mathbb{C}_3 \).

Form now on, let \( r = s \in \{2, 3, 46\} \). According to Lemma 33(iii) and (iv), the group \( H = \langle g, h_\circ \rangle \) is either abelian or isomorphic to some \( H_{C3}(j) \) for \( j \in \{2, 5\} \).

If \( H = \langle g, h_\circ \mid g^3 = h_\circ^r = I_2, \ gh_\circ = h_\circ g \rangle \) is an abelian group of order \( 3r \), then \( H = \langle g \rangle \times \langle h_\circ \rangle \simeq \mathbb{C}_3 \times \mathbb{C}_r \) is a direct product by Lemma 26 (iv). (Here we use that the semi-direct product \( H = [H \cap SL(2, \mathbb{R})] \rtimes \langle h_\circ \rangle = \langle g \rangle \rtimes \langle h_\circ \rangle \) is a direct product if and only if \( gh_\circ = h_\circ g \).

The order \( r = s = 2 \) of \( h_\circ \) is relatively prime to the order \( 3 \) of \( g \), so that \( gh_\circ \) is an element of order \( 6 \) and \( \langle g, h_\circ \rangle = \langle gh_\circ \rangle \simeq \mathbb{C}_6 \simeq H_{C3}(1) \).

The order \( r = s = 4 \) of \( h_\circ \) is relatively prime to the order \( 3 \) of \( g \) and \( gh_\circ \) is of order \( 12 \). By the classification of \( x \in GL(2, \mathbb{R}) \) of finite order, done in section 2, one has \( \det(gh_\circ) = -1 \). Therefore \( \det(h_\circ) = -1 \) and \( s = 2 \), contrary to the assumption \( s = 4 \).

For \( r = s = 3 \) one can assume \( \det(h_\circ) = e^{\frac{2\pi i}{3}} \), after an eventual inversion \( h_\circ \mapsto h_\circ^{-1} \). Then by Proposition 22 one has \( h_\circ = e^{\frac{2\pi i}{3}} I_2 \) or \( \lambda_1(h_\circ) = e^{\frac{2\pi i}{2}}, \lambda_2(h_\circ) = 1 \). Assume that \( \lambda_1(h_\circ) = e^{\frac{2\pi i}{3}}, \lambda_2(h_\circ) = 1 \) and note that the commuting \( g \) and \( h_\circ \) can be simultaneously diagonalized by an appropriate \( S \in GL(2, \mathbb{C}) \). Consequently,

\[
D = S^{-1} gS = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix} \quad \text{and} \quad D_0 = S^{-1} h_\circ S = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & 1 \end{pmatrix}
\]

are subject to \( D^2 D_0 = e^{\frac{2\pi i}{3}} I_2 \). As a result,

\[
g^2 h_\circ = (S D S^{-1} )^{-1} (SD_0 S^{-1} ) = S (D^2 D_0) S^{-1} = e^{\frac{2\pi i}{3}} I_2
\]

and \( H = \langle g, h_\circ \rangle \simeq \langle g, g^2 h_\circ \rangle \simeq H_{C3}(3) \).

Finally, for \( r = s = 6 \), let us assume that \( \det(h_\circ) = e^{-\frac{2\pi i}{3}} \). Then

\[
\{\lambda_1(h_\circ), \lambda_2(h_\circ)\} = \left\{ e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}} \right\}, \quad \left\{ e^{-\frac{2\pi i}{3}}, 1 \right\} \quad \text{or} \quad \left\{ e^{\frac{2\pi i}{3}}, -1 \right\}.
\]

Similarly to the case of \( r = s = 3 \), the commuting \( g \) and \( h_\circ \) admit a simultaneous diagonalization

\[
D = S^{-1} gS = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}, \quad D_0 = S^{-1} h_\circ S = \begin{pmatrix} \lambda_1(h_\circ) & 0 \\ 0 & \lambda_2(h_\circ) \end{pmatrix}.
\]

If \( \lambda_1(h_\circ) = e^{-\frac{2\pi i}{3}}, \lambda_2(h_\circ) = 1 \) then

\[
DD_0 = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix} \quad \text{and} \quad H \simeq \langle D, D_0 \rangle = \langle D, DD_0 \rangle \simeq H_{C3}(4).
\]
For $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$ and $\lambda_2(h_o) = -1$ note that

$$DD_o = \begin{pmatrix} e^{-\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix},$$

so that again $H \simeq \langle D, D_o \rangle = \langle D, DD_o \rangle \simeq H_{C3}(4)$.

Note that

$$g = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}, \quad h_o = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix} \in GL(2, \mathcal{O}_3)$$

generate a group, isomorphic to $H_{C3}(4)$.

\[\square\]

**Corollary 36.** Let $H$ be a finite subgroup of $GL(2, R)$,

$$H \cap SL(2, R) = \langle g \rangle \simeq C_4$$

and $h_o \in H$ be an element of order $r$ with $\det(H) = \langle \det(h_o) \rangle \simeq C_s$ and eigenvalues $\lambda_1(h_o), \lambda_2(h_o)$. Then $H$ is isomorphic to some $H_{C4}(i), 1 \leq i \leq 9$, where

$$H_{C4}(1) = \langle h_o \rangle \simeq C_8$$

with $R = \mathcal{O}_{-2}$, $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = e^{\frac{3\pi i}{3}},$

$$H_{C4}(2) = \langle g \rangle \times \langle h_o \rangle \simeq C_4 \times C_2$$

with $R = R_{-1,f}$, $\lambda_1(h_o) = -1$, $\lambda_2(h_o) = 1,$

$$H_{C4}(3) = \langle g, h_o \mid g^2 = -I_2, \ h_o^2 = I_2, \ h_oh_o^{-1} = g^{-1} \rangle \simeq D_4$$

is the dihedral group of order 8 with $\lambda_1(h_o) = -1$, $\lambda_2(h_o) = 1,$

$$H_{C4}(4) = \langle h_o \rangle \simeq C_{12}$$

with $R = \mathcal{O}_{-3}$, $\lambda_1(h_o) = e^{\frac{2\pi i}{6}}, \lambda_2(h_o) = e^{-\frac{2\pi i}{6}},$

$$H_{C4}(5) = \langle g \rangle \times \langle e^{\frac{2\pi i}{3}}I_2 \rangle \simeq C_4 \times C_3$$

for $R = \mathcal{O}_{-3}$ and $\forall g \in SL(2, \mathcal{O}_{-3})$ with $\mathrm{tr}(g) = 0,$

$$H_{C4}(6) = \langle g \rangle \times \langle h_o \rangle \simeq C_4 \times C_4$$

with $R = \mathbb{Z}[i]$, $\lambda_1(h_o) = i$, $\lambda_2(h_o) = 1,$

$$H_{C4}(7) = \langle ig \rangle \times \langle h_o \rangle \simeq C_2 \times C_8$$

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with \( R = \mathbb{Z}[i], \lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = e^{-\frac{2\pi i}{3}}, \)

\[
H_{C4}(8) = \langle g, h_o \mid g^2 = h_o^4 = -I_2, \ h_oh_o^{-1} = g^{-1} \rangle
\]

of order 16 with \( R = \mathbb{Z}[i], \lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = e^{-\frac{2\pi i}{3}}, \)

\[
H_{C4}(9) = \langle g, h_o \mid g^2 = -I_2, \ h_o^6 = I_2, \ h_oh_o^{-1} = g^{-1} \rangle
\]

of order 24 with \( R = O_3, \lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = e^{-\frac{2\pi i}{3}}. \)

There exist subgroups

\[
H_{C4}(1) < GL(2, O_{-2}), \ H_{C4}(4), H_{C4}(5) < GL(2, O_{-3}),
\]

\[
H_{C4}(2), H_{C4}(6) < GL(2, \mathbb{Z}[i]),
\]

as well as subgroups

\[
H_{C4}^0(7), H_{C4}^0(8) < GL(2, \mathbb{Q}(\sqrt{2}, i)), \ H_{C4}^0(3) < GL(2, \mathbb{Q}(\sqrt{-d})),
\]

\[
H_{C4}^0(9) < GL(2, \mathbb{Q}(\sqrt{-3})),
\]

with \( H_{C4}^0(j) \simeq H_{C4}(j) \) for \( j \in \{3, 7, 8, 9\}. \)

**Proof.** If \( \frac{r}{s} = 4 \) then either \( (s, r) = (2, 8) \) and \( H \simeq H_{C4}(1) \) or \( (s, r) = (3, 12) \) and \( H \simeq H_{C4}(4) \). By Proposition 16 there exists an element \( h_o \in GL(2, O_{-2}) \) of order 8 with \( \det(h_o) = -1 \). Proposition 21 provides an example of \( h_o \in GL(2, O_{-3}) \) of order 12 with \( \det(h_o) = e^{\frac{2\pi i}{3}} \). There remain to be considered the cases with \( \frac{r}{s} \in \{1, 2\} \). According to Lemma 33, the non-abelian \( H \) under consideration are isomorphic to \( H_{C4}(3), H_{C4}(8) \) or \( H_{C4}(9) \). By Lemma 34 (i) there is a subgroup \( H_{C4}^0(3) < GL(2, \mathbb{Q}(\sqrt{-d})) \), conjugate to \( H_{C4}(3) \). Lemma 34 (iii) provides an example of \( S^{-1}H_{C4}(8)S = H_{C4}^0(8) < GL(2, \mathbb{Q}(\sqrt{2}, i)) \), while Lemma 34(ii) justifies the existence of \( S^{-1}H_{C4}(9)S = H_{C4}^0(9) < GL(2, \mathbb{Q}(\sqrt{-3})) \).

There remain to be classified the non-cyclic abelian groups \( H = [H \cap SL(2, R)]\langle h_o \rangle \) with \( H \cap SL(2, R) \simeq C_4, \langle h_o \rangle \simeq C_r, \det(h_o) = e^{\frac{2\pi i}{3}} \) for \( s \in \{2, 3, 4, 6\}, r \in \{s, 2s\} \).

If \( r = s = 2 \) then by Proposition 16, the eigenvalues of \( h_o \) are \( \lambda_1(h_o) = -1 \) and \( \lambda_2(h_o) = 1 \). There exists a matrix \( S \in GL(2, \mathbb{Q}(\sqrt{-d})) \), such that

\[
D_o = S^{-1}h_oS = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Proposition 15 establishes that \( g \in SL(2, R) \) is of order 4 exactly when \( \text{tr}(g) = 0 \). The trace and the determinant are invariant under conjugation, so that

\[
D = S^{-1}gS = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{-d})).
\]

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The commutation
\[ DD_o = \begin{pmatrix} -a & b \\ -c & -a \end{pmatrix} = \begin{pmatrix} -a & -b \\ c & -a \end{pmatrix} = D_o D \]
holds only when \( b = c = 0 \) and
\[ D = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} . \]

Bearing in mind that \( D \in SL(2, \mathbb{Q}(\sqrt{-d})) \), one concludes that \( i \in \mathbb{Q}(\sqrt{-d}) \), whereas \( d = 1 \) and \( R = R_{-1.f} \). The matrices
\[ g = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} , \quad h_o = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Z}[i]) \]
generate a subgroup of \( GL(2, \mathbb{Z}[i]) \), isomorphic to \( H_{C4}(2) \).

For \( s = 2 \) and \( r = 4 \) one has \( R = \mathbb{Z}[i] \) and \( h_o = \pm I_2 \). Bearing in mind that \( g \in SL(2, R) \) is of order 4 if and only if \( \text{tr}(g) = 0 \), let
\[ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}[o]). \]

Then
\[ gh_o = \pm \begin{pmatrix} ai & bi \\ ci & -ai \end{pmatrix} \in \mathbb{Z}[i]_{2 \times 2} \]
has determinant \( \det(gh_o) = \det(g) \det(h_o) = \det(h_o) = -1 \) and trace \( \text{tr}(gh_o) = 0 \). By Proposition 16, \( gh_o \) has eigenvalues \( \lambda_1(gh_o) = -1, \lambda_2(gh_o) = 1 \) and \( H \cong H_{C4}(2) \).

If \( s = r = 3 \) then \( R = \mathcal{O}_{-3} \) and either \( h_o = e^{-\frac{2\pi i}{3}} \) or \( \lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = 1 \). Replacing \( e^{-\frac{2\pi i}{3}} I_2 \) by its inverse, one observes that \( H_{C4}(5) = \langle g, e^{-\frac{2\pi i}{3}} I_2 \rangle < GL(2, \mathcal{O}_{-3}) \).

If \( \lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = 1 \), then there exists \( S \in GL(2, \mathbb{Q}(\sqrt{-3})) \), such that
\[ D_o = S^{-1} h_o S = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & 1 \end{pmatrix} . \]

The determinant and the trace are invariant under conjugation, so that
\[ D = S^{-1} g S = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{-3})). \]

Note that
\[ DD_o = \begin{pmatrix} e^{\frac{2\pi i}{3}} a & b \\ e^{\frac{2\pi i}{3}} c & -a \end{pmatrix} = \begin{pmatrix} e^{\frac{2\pi i}{3}} a & e^{\frac{2\pi i}{3}} b \\ e^{\frac{2\pi i}{3}} c & -a \end{pmatrix} = D_o D \]
is equivalent to \( b = c = 0 \) and \( 1 = \det(g) = \det(D) = -a^2 \) specifies that
\[ D = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} . \]
That contradicts \( F \in SL(2, \mathbb{Q}(\sqrt{-3})) \) and justifies the non-existence of \( H \) with \( s = r = 3 \).

Let \( s = 3, \ r = 6 \). According to Proposition 21, there follows \( R = \mathcal{O}_{-3} \) with \( h_o = e^{\frac{\pi i}{3}} I_2 \) or \( \lambda_1(h_o) = e^{\frac{\pi i}{3}}, \lambda_2(h_o) = 1 \). If \( h_o = e^{\frac{\pi i}{3}} \) then \( H = \langle g, h_o \rangle = \langle g, g^2 h_o = -h_o = e^{-\frac{2\pi i}{3}} I_2 \rangle \simeq H_{C_4}(5) \). In the case of \( \lambda_1(h_o) = e^{\frac{\pi i}{3}}, \lambda_2(h_o) = 1 \) let us choose \( S \in GL(2, \mathbb{Q}(\sqrt{-3})) \) with

\[
D_o = S^{-1} h_o S = \begin{pmatrix} e^{-\frac{\pi i}{3}} & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Q}(\sqrt{-3})) \quad \text{and}
\]

\[
D = S^{-1} g S = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{-3})).
\]

Then

\[
DD_o = \begin{pmatrix} e^{-\frac{\pi i}{3}} a & b \\ e^{-\frac{\pi i}{3}} c & -a \end{pmatrix} = \begin{pmatrix} e^{-\frac{\pi i}{3}} a & e^{-\frac{\pi i}{3}} b \\ c & -a \end{pmatrix} = D_o D
\]

if and only if

\[
D = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{-3})),
\]

which is an absurd.

Let us suppose that \( s = r = 4 \). The Proposition 17 specifies that \( R = \mathbb{Z}[i] \) and \( \lambda_1(h_o) = \varepsilon i, \lambda_2(h_o) = \varepsilon \) for some \( \varepsilon \in \{\pm 1\} \). As far as \( g^2 = -I_2 \in H \), there is no loss of generality in assuming that \( \lambda_1(h_o) = i, \lambda_2(h_o) = 1 \) and \( H \simeq H_{C_4}(6) \). Note that

\[
g = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad h_o \in \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Z}[i])
\]

generate a subgroup, isomorphic to \( H_{C_4}(6) \).

For \( s = 4, \ r = 8 \), Proposition 17 implies that \( R = \mathbb{Z}[i] \) and \( \lambda_1(h_o) = e^{\frac{\pi i}{4}}, \lambda_2(h_o) = e^{-\frac{\pi i}{4}} \). Note that \((ig)^2 = -g^2 = I_2\), so that \( ig \in H = \langle g, h_o \rangle \) is of order 2 and \( h_o^6 = iI_2 \), according to \( \lambda_1(h_o^6) = \lambda_1(h_o)^6 = i, \lambda_2(h_o^6) = \lambda_2(h_o)^6 = i \). Consequently,

\[
H = \langle g, h_o \rangle = \langle h_o^6 g = ig, h_o \rangle = \langle ig \rangle \times \langle h_o \rangle \simeq \mathbb{C}_2 \times \mathbb{C}_8,
\]

as far as \( \langle ig \rangle \cap \langle h_o \rangle = \{I_2\} \). More precisely, if \( ig = h_o^m \), then the second eigenvalue

\[
1 = -i^2 = \lambda_2(ig) = \lambda_2(h_o^m) = e^{-\frac{\pi i m}{4}},
\]

whereas \( m \in 8\mathbb{Z} \) and the first eigenvalue

\[
-1 = \lambda_1(ig) = \lambda_1(h_o^m) = e^{\frac{3\pi i m}{4}} = 1,
\]

which is an absurd. Thus, \( H \simeq H_{C_4}(7) \) and there exists a subgroup

\[
H_{C_4}^0(7) = \langle \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right), \left( \begin{array}{cc} e^{\frac{3\pi i}{4}} & 0 \\ 0 & e^{-\frac{\pi i}{4}} \end{array} \right) \rangle \subset GL(2, \mathbb{Q}(\sqrt{2}, i)),
\]

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conjugate to $H_{C4}(7)$.

Let us assume that $s = r = 6$. Then Proposition 19 applies to provide $R = O_{-3}$ and

$$\{\lambda_1(h_o), \lambda_2(h_o)\} = \left\{e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\right\}, \left\{e^{\frac{2\pi i}{3}}, -1\right\}.$$

Choose a matrix $S \in GL(2, \mathbb{Q}(\sqrt{-3}))$ with

$$D_o = S^{-1}h_oS = \left(\begin{array}{cc} \lambda_1(h_o) & 0 \\ 0 & \lambda_2(h_o) \end{array} \right) \in GL(2, \mathbb{Q}(\sqrt{-3})), \quad D = S^{-1}gS = \left(\begin{array}{cc} a & b \\ c & -a \end{array} \right) \in SL(2, \mathbb{Q}(\sqrt{-3})).$$

If $\lambda_1(h_o) \neq \lambda_2(h_o)$ then

$$DD_o = \left(\begin{array}{cc} \lambda_1(h_o)a & \lambda_2(h_o)b \\ \lambda_1(h_o)c & -\lambda_2(h_o)a \end{array} \right) = \left(\begin{array}{cc} \lambda_1(h_o)a & \lambda_1(h_o)b \\ \lambda_2(h_o)c & -\lambda_2(h_o)a \end{array} \right) = D_oD$$

is tantamount to $b = c = 0$, $a = \pm i$ and

$$D = \pm \left(\begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right) \in SL(2, \mathbb{Q}(\sqrt{-3}))$$

is an absurd.

Similarly, in the case of $s = 6$, $r = 12$, Proposition 19 derives that $R = O_{-3}$ and

$$\{\lambda_1(h_o), \lambda_2(h_o)\} = \left\{e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\right\}, \left\{e^{\frac{2\pi i}{3}}, -1\right\}.$$

Note that $\lambda_1(h_o) \neq \lambda_2(h_o)$ for all the possibilities and apply the considerations for $s = r = 6$, in order to exclude the case $s = 6$, $r = 12$.

\[\square\]

**Corollary 37.** Let $H$ be a finite subgroup of $GL(2, R)$,

$$H \cap SL(2, R) = \langle g \rangle \cong C_6$$

and $h_o \in H$ be an element of order $r$ with $\det(H) = \langle \det(h_o) \rangle \cong C_s$ and eigenvalues $\lambda_1(h_o), \lambda_2(h_o)$. Then $H$ is isomorphic to some $H_{C6}(i)$, $1 \leq i \leq 7$, where

$$H_{C6}(1) = \langle h_o \rangle \cong C_{12}$$

with $R = \mathbb{Z}[i]$, $\lambda_1(h_o) = e^{\frac{\pi i}{6}}$, $\lambda_2(h_o) = e^{\frac{5\pi i}{6}}$,

$$H_{C6}(2) = \langle g \rangle \times \langle h_o \rangle \cong C_6 \times C_{12}$$

with $R = O_{-3}$ or $R = R_{-3,2}$, $\lambda_1(h_o) = -1$, $\lambda_2(h_o) = 1$,

$$H_{C6}(3) = \langle g, h_o \mid g^3 = -I_2, \quad h_o^2 = I_2, \quad h_oh_o^{-1} = g^{-1} \rangle \cong D_6$$

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is the dihedral group of order 12, \( \lambda_1(h_o) = -1, \lambda_2(h_o) = 1 \),

\[ H_{C6}(4) = \langle g \rangle \times \langle e^{\frac{2\pi i}{3}} I_2 \rangle \simeq \mathbb{C}_6 \times \mathbb{C}_3 \]

with \( R = O_{-3} \) and \( \forall g \in SL(2, O_{-3}) \) of \( \text{tr}(g) = 1 \),

\[ H_{C6}(5) = \langle g, h_o \mid g^3 = h_o^4 = -I_2, \ h_o g h_o^{-1} = g^{-1} \rangle \]

of order 24 with \( R = \mathbb{Z}[i] \), \( \lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = e^{-\frac{2\pi i}{3}} \),

\[ H_{C6}(6) = \langle g, h_o \mid g^3 = -I_2, \ h_o^6 = I_2, \ h_o g h_o^{-1} = g^{-1} \rangle \]

of order 36 with \( R = O_{-3} \), \( \lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = e^{-\frac{2\pi i}{3}} \),

\[ H_{C6}(7) = \langle g \rangle \times \langle h_o \rangle \simeq \mathbb{C}_6 \times \mathbb{C}_6 \]

of order 36 with \( R = O_{-3} \), \( \lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = e^{-\frac{2\pi i}{3}} \).

There exist subgroups

\[ H_{C6}(1) < GL(2, \mathbb{Z}[i]), \ H_{C6}(2), H_{C6}(4), H_{C6}(7) < GL(2, O_{-3}), \]

as well as subgroups

\[ H_{C6}^o(3) < GL(2, \mathbb{Q}(\sqrt{-d})), \ H_{C6}^o(5) < GL(2, \mathbb{Q}(\sqrt{2}, i)), \]

\[ H_{C6}^o(6) < GL(2, \mathbb{Q}(\sqrt{-3})) \]

with \( H_{C6}^o(j) \simeq H_{C6}(j) \) for \( j \in \{3, 5, 6\} \).

**Proof.** According to Lemma 33(i), the ratio \( \zeta \in \{1, 2, 3, 6\} \) is a divisor of \( t = 6 \). If \( r = 6s \) then \( s = 2 \) and \( H = \langle h_o \rangle \simeq \mathbb{C}_{12} \simeq H_{C6}(1) \) by Lemma 33 (i), (ii). According to Proposition 16, the existence of \( h_o \in GL(2, \mathbb{R}) \) of order 12 with \( \det(h_o) = -1 \) requires \( R = \mathbb{Z}[i] \) and there exist \( h_o \in GL(2, \mathbb{Z}[i]) \) of order 12 with \( \det(h_o) = -1 \).

For \( r = 3s \) Lemma 33(i) specifies that \( s = 2 \). Combining with Lemma 33(iv), one concludes that

\[ H = \langle g, h_o \mid g^3 = -I_2, \ h_o^6 = I_2, \ h_o g = g h_o \rangle \]

is a non-cyclic abelian group of order \( st = 12 \). By Proposition 16, \( R = O_{-3} \) or \( R = R_{-3,2} \) and \( h_o \) has eigenvalues \( \lambda_1(h_o) = e^{\frac{\pi i}{3}}, \lambda_2(h_o) = e^{\frac{2\pi i}{3}} \) for some \( \varepsilon \in \{ \pm 1 \} \). Due to \( \langle g, h_o \rangle = \langle g, h_o^{-1} = h_o^{5} \rangle \) by \( h_o = (h_o^{5})^5 \), one can assume that \( \lambda_1(h_o) = e^{\frac{\pi i}{3}}, \lambda_2(h_o) = e^{\frac{2\pi i}{3}} \). The commuting matrices \( g \) and \( h_o \) admit a simultaneous diagonalization

\[ D = S^{-1} g S = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{pmatrix}, \ D_o = S^{-1} h_o S = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{pmatrix} \]
by an appropriate $S \in GL(2, \mathbb{Q}(\sqrt{-3}))$. Then

$$D^2D_o = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

implies that $\lambda_1(g^2h_o) = -1$, $\lambda_2(g^2h_o) = 1$. As a result, $H = \langle g, h_o \rangle = \langle g, g^2h_o \rangle$ is a subgroup of $GL(2, \mathcal{O}_{-3})$, isomorphic to $H_{C6}(2)$.

Form now on, $\frac{r}{s} \in \{1, 2\}$. In particular, $\frac{r}{s} < t = 6$ and the non-abelian

$$H = \langle g, h_o \mid g^6 = h_o^r = I_2, \ h_o g h_o^{-1} = g^{-1} \rangle$$

occurs for $(r, s) \in \{(2, 2), (8, 4), (6, 6)\}$, according to Lemma 33(iv). Namely, for $r = s = 2$ one has a dihedral group $H \simeq \mathcal{D}_6 \simeq H_{C6}(3)$ of order 12, which is realized as a subgroup of $GL(2, \mathbb{Q}(\sqrt{-3})))$ by Lemma 34(i). In the case of $s = 4$ and $r = 8$ the group $H \simeq H_{C6}(5)$ of order 24 is embedded in $GL(2, \mathbb{Q}(\sqrt{2}, i))$ by Lemma 34(iii). In the case of $r = s = 6$ one has $H \simeq H_{C6}(6)$ of order 36, realized as a subgroup of $GL(2, \mathbb{Q}(\sqrt{-3}))$ by Lemma 34(ii).

There remain to be considered the non-cyclic abelian $H$ with $r = 2s$, $s \in \{2, 3, 4\}$ or $r = s \in \{2, 3, 4, 6\}$. If $s = 2$, $r = 4$ then Proposition 16 requires $R = \mathbb{Z}[i]$ and $h_o = \pm iI_2$. Up to an inversion of $h_o$, one can assume that $h_o = iI_2$. Then $H = \langle g, iI_2 \rangle = \langle -g = (iI_2)^2g, iI_2 \rangle$ is generated by the element $-g$ of order 3 and the scalar matrix $iI_2 \in H$ of order 4, so that $-ig = (iI_2)(-g) \in H$ of order 12 generates $H$, $H \simeq H_{C6}(1) \simeq \mathbb{C}_{12}$. (Note that for $g \in SL(2, \mathbb{Z}[i])$ of order 6 one has $g^3 = -I_2$, whereas $(-g)^3 = -g^3 = I_2$. The assumptions $-g = I_2$ and $(-g)^2 = g^2 = I_2$ lead to an absurd.)

Let us assume that $s = 3$ and $r = 6$. Then Proposition 21 implies that $R = \mathcal{O}_{-3}$ with $h_o = E^\frac{\pi i}{3} I_2$ or $\lambda_1(h_o) = e^{-\frac{\pi i}{6}}$, $\lambda_2(h_o) = -1$. Note that $H = \langle g, e^{\frac{\pi i}{6}} I_2 \rangle = \langle g, e^{-\frac{\pi i}{6}} I_2 \rangle$ by $e^{-\frac{\pi i}{6}} = \left(e^{\frac{\pi i}{6}}\right)^5$, $e^{\frac{\pi i}{6}} = \left(e^{-\frac{\pi i}{6}}\right)^5$. Further,

$$g^3 \left(e^{-\frac{\pi i}{6}} I_2 \right) = \left(e^{\frac{\pi i}{6}} I_2 \right) \left(e^{-\frac{\pi i}{6}} I_2 \right) = e^{2\pi i} I_2$$

implies that

$$H = \langle g, e^{-\frac{\pi i}{6}} I_2 \rangle = \langle g, g^3 \left(e^{-\frac{\pi i}{6}} I_2 \right) = e^{2\pi i} I_2 = \langle g \rangle \times \langle e^{2\pi i} \rangle \simeq \mathbb{C}_6 \times \mathbb{C}_3 \simeq H_{C6}(4).$$

For any $g \in SL(2, \mathcal{O}_{-3})$ of order 6, there is a subgroup $H_{C6}(4) = \langle g, e^{2\pi i} I_2 \rangle < GL(2, \mathcal{O}_{-3})$.

For $s = 4$, $r = 8$ there follow $R = \mathbb{Z}[i]$ and $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = e^{-\frac{2\pi i}{3}}$, according to Proposition 17. Suppose that $S \in GL(2, \mathbb{Q}(\sqrt{2}, i))$ diagonalizes $h_o$,

$$D_o = S^{-1} h_o S = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix} \in GL(2, \mathbb{Q}(\sqrt{2}, i)).$$

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By Proposition 15, \( g \in SL(2, \mathbb{Z}[i]) \) is of order 6 exactly when \( \text{tr}(g) = 1 \). Since the determinant and the trace are invariant under conjugation, one has

\[
D = S^{-1}gS = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{2}, i)).
\]

However,

\[
DD_o = \begin{pmatrix} e^{\frac{3\pi i}{4}}a & e^{-\frac{\pi i}{4}}b \\ e^{\frac{3\pi i}{4}}c & e^{-\frac{\pi i}{4}}(1-a) \end{pmatrix} = \begin{pmatrix} e^{\frac{3\pi i}{4}}a & e^{\frac{3\pi i}{4}}b \\ e^{-\frac{\pi i}{4}}c & e^{-\frac{\pi i}{4}}(1-a) \end{pmatrix} = D_oD
\]

if and only if \( b = c = 0 \) and \( a = e^{\frac{\pi i}{4}} \) for some \( \varepsilon \in \{\pm 1\} \). Now,

\[
D = \begin{pmatrix} e^{\frac{\pi i}{4}} & 0 \\ 0 & 1-e^{\frac{\pi i}{4}} \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{2}, i))
\]

is an absurd, justifying the non-existence of \( H \) with \( s = 4 \) and \( r = 8 \).

In the case of \( r = s = 2 \) Proposition 16 implies that \( \lambda_1(h_o) = -1 \), \( \lambda_2(h_o) = 1 \), so that \( H \simeq H_{C_6}(2) \simeq \mathbb{C}_6 \times \mathbb{C}_2 \).

For \( r = s = 3 \) Proposition 21 reveals that \( R \simeq O_{-3} \) with \( h_o = e^{-\frac{2\pi i}{3}}I_2 \) or \( \lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = 1 \). It is clear that

\[
H = \langle g, e^{-\frac{2\pi i}{3}}I_2 \rangle = \langle g, e^{\frac{2\pi i}{3}}I_2 \rangle = \langle e^{-\frac{2\pi i}{3}}I_2 \rangle \simeq H_{C_6}(4) \simeq \mathbb{C}_3 \times \mathbb{C}_3.
\]

If \( \lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = 1 \) then the commuting matrices \( g \) and \( h_o \) admit a simultaneous diagonalization

\[
D = S^{-1}gS = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix}, \quad D_o = S^{-1}h_oS = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Q}(\sqrt{-3}))
\]

by an appropriate \( S \in GL(2, \mathbb{Q}(\sqrt{-3})) \). Then \( D^2D_o = e^{-\frac{2\pi i}{3}}I_2 \), whereas \( g^2h_o = S\left(e^{\frac{2\pi i}{3}}I_2\right)S^{-1} = e^{\frac{2\pi i}{3}}I_2 \) and

\[
H = \langle g, h_o \rangle = \langle g, g^2h_o = e^{-\frac{2\pi i}{3}}I_2 \rangle \simeq H_{C_6}(4) \simeq \mathbb{C}_6 \times \mathbb{C}_3.
\]

The assumption \( r = s = 4 \) implies that \( R = \mathbb{Z}[i] \) and \( \lambda_1(h_o) = \varepsilon i, \lambda_2(h_o) = \varepsilon \) for some \( \varepsilon \in \{\pm 1\} \), according to Proposition 17. Due to \( g^3 = -I_2 \), one has \( \langle g, h_o \rangle = \langle g, -h_o = g^3h_o \rangle \), so that there is no loss of generality in assuming \( \varepsilon = 1 \). If \( S \in GL(2, \mathbb{Q}(i)) \) conjugates \( h_o \) to its diagonal form

\[
D_o = S^{-1}h_oS = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Q}(9i)),
\]

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then
\[ D = S^{-1}gS = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix} \in SL(2, \mathbb{Q}(i)). \]

The relation
\[ DD_o = \begin{pmatrix} ia & b \\ ic & 1-a \end{pmatrix} = \begin{pmatrix} ia & ib \\ c & 1-a \end{pmatrix} = D_oD \]
implies that
\[ D = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix} \in SL(2, \mathbb{Q}(i)) \text{ for some } \varepsilon \in \{\pm\}. \]

The contradiction proves the non-existence of \( H \) with \( r = s = 4 \).

Finally, for \( r = s = 6 \) Proposition 19 specifies that \( R = \mathcal{O}_{-3} \) and
\[ \{\lambda_1(h_o), \lambda_2(h_o)\} = \left\{ e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}} \right\}, \quad \left\{ 1, e^{\frac{2\pi i}{3}} \right\} \quad \text{or} \quad \left\{ e^{-\frac{2\pi i}{3}}, -1 \right\}. \]

The commuting matrices \( g \) and \( h_o \) admit simultaneous diagonalization
\[ D = S^{-1}gS = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix}, \]
\[ D_o = S^{-1}h_oS = \begin{pmatrix} \lambda_1(h_o) & 0 \\ 0 & \lambda_2(h_o) \end{pmatrix} \in GL(2, \mathbb{Q}(\sqrt{-3})) \]
by an appropriate \( S \in GL(2, \mathbb{Q}(\sqrt{-3})) \). Let us denote
\[ D_o := \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix}, \quad D'_o := \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{\pi i}{3}} \end{pmatrix}, \quad D''_o := \begin{pmatrix} e^{-\frac{\pi i}{3}} & 0 \\ 0 & -1 \end{pmatrix} \in GL(2, \mathcal{O}_{-3}) \]
and observe that
\[ D^2D_o = D''_o, \quad D62D''_o = D'_o. \]

By its very definition,
\[ H = \langle D, D_o \rangle < GL(2, \mathcal{O}_{-3}) \]
is isomorphic to \( H_{C6}(7) \). The equalities \( \langle D, D'_o = D^2D''_o \rangle = \langle D, D''_o \rangle \) and \( \langle D, D''_o = D^2D_o \rangle = \langle D, D_o \rangle \) conclude the proof of the proposition.

\[ \square \]

**Proposition 38.** Let \( H \) be a finite subgroup of \( GL(2, R) \),
\[ H \cap SL(2, R) = \langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \quad g_2g_1 = -g_1g_2 \rangle \cong \mathbb{Q}_8, \]
and \( h_o \in H \) be an element of order \( r \) with \( \det(H) = \langle \det(h_o) \rangle \cong \mathbb{C}_s \) and eigenvalues \( \lambda_1(h_o), \lambda_2(h_o) \). Then \( H \) is isomorphic to some \( H_{\mathbb{Q}_8}(i), \quad 1 \leq i \leq 9 \), where
\[ H_{\mathbb{Q}_8}(1) = \langle g_1, g_2, iI_2 \mid g_1^2 = g_2^2 = -I_2, \quad g_2g_1 = -g_1g_2 \rangle \]
is of order 16 with $R = \mathbb{Z}[i]$,
\[ H_{Q_8}(2) = \langle g_1, g_2, h_0 \mid g_1^2 = g_2^2 = -I_2, \ h_0^2 = I_2, \ g_2g_1 = -g_1g_2, \ h_0g_1h_0^{-1} = -g_1, \ h_0g_2h_0^{-1} = -g_2 \rangle \]
is of order 16 with $R = \mathbb{Z}[i]$, $\lambda_1(h_0) = -1$, $\lambda_2(h_0) = 1$,
\[ H_{Q_8}(3) = \langle g_1, g_2, h_0 \mid g_1^2 = g_2^2 = h_0^4 = -I_2, \ g_2g_1 = -g_1g_2, \ h_0g_1h_0^{-1} = g_2, \ h_0g_2h_0^{-1} = g_1 \rangle \]
is of order 16 with $R = O_2$, $\lambda_1(h_0) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_0) = e^{\frac{4\pi i}{3}}$, $h_0^2 = \pm g_1g_2$,
\[ H_{Q_8}(4) = \langle g_1, g_2, h_0 \mid g_1^2 = g_2^2 = -I_2, \ h_0^2 = I_2, \ g_2g_1 = -g_1g_2, \ h_0g_1h_0^{-1} = g_2, \ h_0g_2h_0^{-1} = g_1 \rangle \]
is of order 16 with $R = R_{-2}, \lambda_1(h_0) = -1$, $\lambda_2(h_0) = 1$,
\[ H_{Q_8}(5) = \langle g_1, g_2 \rangle \times \langle e^{\frac{2\pi i}{3}} \rangle \simeq \mathbb{Q}_8 \times C_3 \]
is of order 24 with $R = O_3$,
\[ H_{Q_8}(6) = \langle g_1, g_2, h_0 \mid g_1^2 = g_2^2 = -I_2, \ h_0^3 = I_2, \ g_2g_1 = -g_1g_2, \ h_0g_1h_0^{-1} = g_2, \ h_0g_2h_0^{-1} = g_1 \rangle \]
is of order 24 with $R = O_{-3}$, $\lambda_1(h_0) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_0) = 1$,
\[ H_{Q_8}(7) = \langle g_1, g_2, h_0 \mid g_1^2 = g_2^2 = h_0^4 = -I_2, \ g_2g_1 = -g_1g_2, \ h_0g_1h_0^{-1} = -g_1, \ h_0g_2h_0^{-1} = -g_2 \rangle \]
is of order 32 with $R = \mathbb{Z}[i]$, $\lambda_1(h_0) = e^{\frac{3\pi i}{4}}$, $\lambda_2(h_0) = e^{-\frac{3\pi i}{4}}$,
\[ H_{Q_8}(8) = \langle g_1, g_2, h_0 \mid g_1^2 = g_2^2 = h_0^4 = -I_2, \ g_2g_1 = -g_1g_2, \ h_0g_1h_0^{-1} = g_2, \ h_0g_2h_0^{-1} = g_1 \rangle \]
is of order 32 with $R = \mathbb{Z}[i]$, $\lambda_1(h_0) = e^{\frac{3\pi i}{4}}$, $\lambda_2(h_0) = e^{-\frac{3\pi i}{4}}$,
\[ H_{Q_8}(9) = \langle g_1, g_2, h_0 \mid g_1^2 = g_2^2 = -I_2, \ h_0^4 = I_2, \ g_2g_1 = -g_1g_2, \ h_0g_1h_0^{-1} = g_2, \ h_0g_1h_0^{-1} = g_2 \rangle \]
is of order 32 with $R = \mathbb{Z}[i]$, $\lambda_1(h_0) = i$, $\lambda_2(h_0) = 1$.

There exist subgroups
\[ H_{Q_8}(1), \ H_{Q_8}(2), \ H_{Q_8}(9) < GL(2, \mathbb{Z}[i]), \ q_{8}(5) < GL(2, O_{-3}), \]
as well as subgroups
\[ H_{Q_8}^o(4) < GL(2, \mathbb{Q}(\sqrt{-2})), \ H_{Q_8}^o(6) < GL(2, \mathbb{Q}(\sqrt{-3})), \]
\[ H_{Q_8}^o(3), \ H_{Q_8}^o(7), \ H_{Q_8}^o(8) < GL(2, \mathbb{Q}(\sqrt{2}, i)), \]
such that $H_{Q_8}^o(j) \simeq H_{Q_8}(j)$ for $j \in \{3, 4, 6, 7, 8\}.$
Proof. According to Lemmas 26 and 27, the group \( H = \langle g_1, g_2 \rangle \) with \( \det(H) = \langle \det(h_o) \rangle \simeq C_8 \) is completely determined by the order \( r \) of \( h_o \) and the elements \( x_j = h_og_jh_o^{-1} \in \langle g_1, g_2 \rangle \), \( 1 \leq j \leq 2 \) of order 4. Bearing in mind that \( \langle g_1, g_2 \rangle^{(4)} = \{ \pm g_1, \pm g_2, \pm g_1g_2 \} \), let us split the considerations into Case A with \( x_j \in \{ \pm g_j \} \) for \( 1 \leq j \leq 2 \), Case B with \( h_o g_1 h_o^{-1} = g_2 \), \( h_o g_2 h_o^{-1} = \varepsilon g_1 \) for some \( \varepsilon = \pm 1 \) and Case C with \( h_o g_1 h_o^{-1} = g_2 \), \( h_o g_2 h_o^{-1} = \varepsilon g_1 g_2 \) for some \( \varepsilon = \pm 1 \).

In the case A, let us distinguish between Case A1 with \( x_j = h_og_jh_o^{-1} = g_j \) for \( \forall 1 \leq j \leq 2 \) and Case A2 with \( x_k = h_og_kh_o^{-1} = -g_k \) for some \( k \in \{1, 2\} \). Note that if \( h_og_j = g_jh_o \) for \( \forall 1 \leq j \leq 2 \) then \( h_o \in H \) is a scalar matrix. Indeed, if \( h_o \) has diagonal form

\[
D_o = S^{-1}h_oS = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}
\]

for some \( S \in GL(2, \mathbb{Q}(\sqrt{-d}, \lambda_1)) \) and

\[
D_j = S^{-1}g_jS = \begin{pmatrix}
a_j & b_j \\
c_j & -a_j
\end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{-d}, \lambda_1)) \quad \text{for} \quad 1 \leq j \leq 2
\]

then

\[
D_oD_jD_o^{-1} = \begin{pmatrix}
a_j & \frac{\lambda_1 b_j}{\lambda_2} \\
\frac{\lambda_2 c_j}{\lambda_1} & -a_j
\end{pmatrix}
\]

(14)

coincides with \( D_j \) if and only if

\[
\begin{vmatrix}
\frac{\lambda_1}{\lambda_2} & 1 \\
\frac{\lambda_2}{\lambda_1} & -1
\end{vmatrix} b_j = 0 \quad \text{and} \quad \begin{vmatrix}
\frac{\lambda_1}{\lambda_2} & 1 \\
\frac{\lambda_2}{\lambda_1} & -1
\end{vmatrix} c_j = 0.
\]

The assumption \( \lambda_1(h_o) = \lambda_1 \neq \lambda_2(h_o) \) implies \( b_j = c_j = 0 \) for \( \forall 1 \leq j \leq 2 \), so that

\[
D_1 = \pm i \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \quad \text{and} \quad D_2 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]

are diagonal. In particular, \( D_1 \) commutes with \( D_2 \), contrary to \( D_2D_1 = -D_1D_2 \). Thus, in the Case A1 with \( h_og_j = g_jh_o \) for \( \forall 1 \leq j \leq 2 \) the matrix \( h_o \in H \) is to be scalar. By Propositions 16, 17, 18, 19, 20, 21, 22, the scalar matrices \( h_o \in GL(2, R) \setminus SL(2, R) \) are

\[
h_o = iI_2 \in GL(2, \mathbb{Z}[i]) \quad \text{of order} \quad 4,
\]

\[
h_o = e^{\pm 2\pi i}I_2 \in GL(2, \mathbb{Z}[i]) \quad \text{of order} \quad 3 \quad \text{and}
\]

\[
h_o = e^{\pm \pi i}I_2 \in GL(2, \mathbb{Z}[i]) \quad \text{of order} \quad 6.
\]

For any subgroup

\[
\mathbb{Q}_8 \simeq \langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \quad g_2g_1 = -g_1g_2 \rangle \subset SL(2, \mathbb{Z}[i])
\]

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one obtains a group
\[ H_{Q8}(1) = \langle g_1, g_2, i I_2 \mid g_1^2 = g_2^2 = -I_2, \quad g_2 g_1 = -g_1 g_2 \rangle < GL(2, \mathbb{Z}[i]) \]
of order 16. As far as \(-I_2 \in H \cap SL(2, R)\), the group \(H\) contains \(e^{2\pi i} I_2\) if and only if it contains \(-e^{2\pi i} I_2 = e^{-2\pi i} I_2\). Since \(\langle g_1, g_2 \rangle \cap \langle e^{2\pi i} I_2 \rangle = \{I_2\}\), any finite group \(H\) with \(e^{2\pi i} I_2 \in H\) is a subgroup of \(GL(4, \mathbb{O}_{-3})\) of the form
\[ H_{Q8}(5) = \langle g_1, g_2 \rangle \times \langle e^{2\pi i} I_2 \rangle \cong \mathbb{Q}_8 \times \mathbb{C}_3. \]

These deplete \(H = [H \cap SL(2, R)]\langle h_o \rangle = \langle g_1, g_2 \rangle \langle h_o \rangle \simeq \mathbb{Q}_8 \mathbb{C}_s\) of Case A1.

In the Case A2, one can assume that \(h_o g_1 h_o^{-1} = -g_1\). If \(h_o g_2 h_o = g_2\) then \(h_o (g_1 g_2) h_o^{-1} = -g_1 g_2\), so that there is no loss of generality in supposing \(h_o g_2 h_o^{-1} = -g_2\).

By Lemma 33(iv), \(h_o g_1 h_o^{-1} = -g_1\) requires \(\lambda_1(h_o) = i e^{\frac{2\pi i}{s}}, \lambda(h_o) = -i e^{\frac{2\pi i}{s}}\), whereas \(\frac{\lambda_1(h_o)}{\lambda_2(h_o)} + 1 = \frac{\lambda_1(h_o)}{\lambda_2(h_o)} + 1 = 0\). If
\[ D_o = S^{-1} h_o S = \begin{pmatrix} i e^{\frac{2\pi i}{s}} & 0 \\ 0 & -i e^{\frac{2\pi i}{s}} \end{pmatrix} \in GL(2, \mathbb{Q}(\sqrt{-d}, i e^{\frac{2\pi i}{s}})) \]
is a diagonal form of \(h_o \in H\) and
\[ D_j = S^{-1} g_j S = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \in GL(2, \mathbb{Q}(\sqrt{-d}, i e^{\frac{2\pi i}{s}})) \quad \text{for} \quad 1 \leq j \leq 2, \]
then \(D_o D_j D_o^{-1} = -D_j\) for \(1 \leq j \leq 2\) is equivalent to \(a_1 = a_2 = 0\). As a result, \(b_j \neq 0\) and \(c_j = -\frac{1}{b_j}\).Straightforwardly, \(D_2 D_1 = -D_1 D_2\) amounts to \(2a_1 a_2 + b_1 c_2 + b_2 c_1 = 0\), whereas \(\frac{b_1}{a_1} + \frac{b_2}{a_2} = 0\). Denoting \(\beta := \frac{b_1}{a_1} \in \mathbb{Q}(\sqrt{-d}, i e^{\frac{2\pi i}{s}})\), one computes that \(\beta = \pm i \in \mathbb{Q}(\sqrt{-d}, i e^{\frac{2\pi i}{s}})\). Then by Lemma 28 there follows \(s = 2\) with \(d = 1\) or \(s = 4\). For \(s = 2\) one has \(\lambda_1(h_o) = -1, \lambda_2(h_o) = 1\), so that \(h_o \in H\) is of order 2 and
\[ H = H_{Q8}(2) = \langle g_1, g_2, h_o \mid g_1^2 = g_2^2 = -I_2, \quad h_o^2 = I_2, \]
\[ g_2 g_1 = -g_1 g_2, \quad h_o g_1 h_o^{-1} = -g_1, \quad h_o g_2 h_o^{-1} = -g_2 \]
is a subgroup of \(GL(2, R_{-1,f})\) of order 16. Note that
\[ h_o = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \]
generate a subgroup of \(GL(2, \mathbb{Z}[i])\), isomorphic to \(H_{Q8}(2)\). In the case of \(s = 4\), the element \(h_o \in H\) with eigenvalues \(\lambda_1(h_o) = e^{\frac{2\pi i}{4}}, \lambda_2(h_o) = e^{-\frac{2\pi i}{4}}\) is of order 8 and
\[ H = H_{Q8}(7) = \langle g_1, g_2, h_o \mid g_1^2 = g_2^4 = h_o^4 = -I_2, \quad g_2 g_1 = -g_1 g_2 \]
\[ h_o g_1 h_o^{-1} = -g_1, \quad h_o g_2 h_o^{-1} = -g_2 \]
is a subgroup of $GL(2, \mathbb{Z}[i])$ of order 32. The matrices

$$D_o = \begin{pmatrix} e^{\frac{3\pi i}{4}} & 0 \\ 0 & e^{-\frac{\pi i}{4}} \end{pmatrix}, \ D_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ D_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

generate a subgroup $H_{Q8}^o(7)$ of $GL(2, \mathbb{Q}(\sqrt{2}, i))$, isomorphic to $H_{Q8}(7)$. That concludes the Case A.

In the Case B, let us observe that $h_o g_1 h_o^{-1} = g_2$ and $h_o g_2 h_o^{-1} = \varepsilon g_1$ imply $h_o^2 g_1 h_o^{-2} = \varepsilon g_1$ and $h_o^2 g_2 h_o^{-2} = \varepsilon g_2$. If $h_o^2 \in H \cap SL(2, R)$ then $\det(h_o) = \lambda_1(h_o) \lambda_2(h_o) = -1$. The matrices

$$D_o = S^{-1} h_o S = \begin{pmatrix} \lambda_1(h_o) & 0 \\ 0 & \lambda_2(h_o) \end{pmatrix} \quad \text{and} \quad D_j = S^{-1} g_j S = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix}$$

with $a_j^2 + b_j c_j = -1, 2a_1 a_2 + b_1 c_2 + b_2 c_1 = 0$ satisfy $D_o D_1 D_o^{-1} = D_2$ if and only if

$$D_2 = \begin{pmatrix} a_1 & b_1 \\ -c_1 \lambda_1^2(h_o) & -a_1 \end{pmatrix}.$$ 

Then $D_o D_2 D_o^{-1} = \varepsilon D_1$ is equivalent to

$$\begin{vmatrix} (\varepsilon - 1) a_1 & 0 \\ (\varepsilon - \lambda_1^4(h_o)) b_1 & 0 \\ (\varepsilon - \frac{1}{\lambda_1^4(h_o)}) c_1 & 0 \end{vmatrix} = 0.$$

According to $\det(D_1) = 1 \neq 0$, there follows $(\varepsilon - 1)(\varepsilon - \lambda_1^4(h_o)) = 0$. In the case of $-1 = \varepsilon = \lambda_1^4(h_o)$, Proposition 16 implies that $R = \mathcal{O}_{-2}, h_o$ is of order 8 and

$$D_o = S^{-1} h_o S = \begin{pmatrix} e^{\frac{\pi i}{4}} & 0 \\ 0 & e^{\frac{3\pi i}{4}} \end{pmatrix} \in GL(2, \mathbb{Q}(\sqrt{2}, i)).$$

Moreover,

$$D_1 = \begin{pmatrix} 0 & b_1 \\ -\frac{1}{b_1} & 0 \end{pmatrix}, \ D_2 = \begin{pmatrix} 0 & -ib_1 \\ -\frac{1}{ib_1} & 0 \end{pmatrix},$$

so that the subgroup

$$H_{Q8}(3) = \langle g_1, g_2, h_o \mid g_1^2 = g_2^3 = h_o^4 = -I_2, \ g_2 g_1 = -g_1 g_2, \ h_o g_1 h_o^{-1} = g_2, \ h_o g_2 h_o^{-1} = -g_1 \rangle < GL(2, \mathcal{O}_{-2})$$

of order 16 is conjugate to the subgroup

$$H_{Q8}^o(3) = \langle D_o = \begin{pmatrix} e^{\frac{\pi i}{4}} & 0 \\ 0 & e^{\frac{3\pi i}{4}} \end{pmatrix}, \ D_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle,$$
\[ D_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} < GL(2, \mathbb{Q}(\sqrt{2}, i)). \]

For \( \varepsilon = 1 \) and \( \lambda_4^i(h_o) \neq 1 \) there follows
\[ D_2 = D_1 = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \]
which contradicts \( D_2D_1 = -D_1D_2 \). Therefore \( \varepsilon = 1 \) implies \( \lambda_4^i(h_o) = 1 \) and
\[ D_o = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
is of order 2, since all \( h_o \in H \) of order 4 with \( \det(h_o) = -1 \) are scalar matrices and commute with \( g_1, g_2 \). In such a way, one obtains the group
\[ H_{Q_8}(4) = \langle g_1, g_2, h_o \mid g_1^2 = g_2^2 = -I_2, \ h_o^2 = I_2, \ g_2g_1 = -g_1g_2, \ h_o g_1 h_o^{-1} = g_2, \ h_o g_2 h_o^{-1} = g_1 \rangle \]
of order 16. The matrices
\[ D_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} \quad \text{and} \quad D_2 = \begin{pmatrix} a_1 & -b_1 \\ -c_1 & -a_1 \end{pmatrix} \]
generate a subgroup of \( GL(2, \mathbb{Q}(\sqrt{-d})) \), isomorphic to \( Q_8 \) exactly when \( a_1 = \pm \frac{\sqrt{-1}}{2} \in \mathbb{Q}(\sqrt{-d}) \) and \( c_1 = -\frac{1}{b_1} \) for some \( b_1 \in \mathbb{Q}(\sqrt{-d})^* \). Therefore \( H_{Q_8}(4) \) occurs only as a subgroup of \( GL(2, R_{-2, f}) \) and
\[ D_o = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D_1 = \begin{pmatrix} \frac{\sqrt{-1}}{2} & 1 \\ -\frac{1}{2} & -\frac{\sqrt{-1}}{2} \end{pmatrix}, \quad D_2 = \begin{pmatrix} \frac{\sqrt{-1}}{2} & 1 \\ \frac{1}{2} & -\frac{\sqrt{-1}}{2} \end{pmatrix} \]
generate a subgroup \( H_{Q_8}^o(4) \) of \( GL(2, \mathbb{Q}(\sqrt{-2})) \), isomorphic to \( H_{Q_8}(4) \). That concludes the Case B with \( h_o^2 \in H \cap SL(2, R) \).

Let us suppose that \( h_o g_1 h_o^{-1} = g_2, \ h_o g_2 h_o^{-1} = \varepsilon g_1 \) with \( \det(h_o) \in R^* \) of order \( s > 2 \). Note that \( h_o^s \in H \cap SL(2, R) = \langle g_1, g_2 \rangle \) implies \( h_o^j g_1 h_o^{-j} \in \{ \pm g_1 \} \) for \( \forall 1 \leq j \leq 2 \), so that \( s \in \{ 4, 6 \} \) has to be an even natural number. The group
\[ H' = \langle g_1, g_2, h_o^2 \mid g_1^2 = g_2^2 = -I_2, \ h_o^2 = I_2, \ g_2g_1 = -g_1g_2, \ h_o^2 g_1 h_o^{-2} = \varepsilon g_1, \ h_o^2 g_2 h_o^{-2} = \varepsilon g_2 \rangle \]
with \( h_o^2 \in GL(2, R) \setminus SL(2, R), \ H' \cap SL(2, R) = \langle g_1, g_2 \rangle \simeq Q_8 \) is of order \( 8^s \in \{ 16, 24 \} \) and satisfies the assumptions of Case A. Thus, for \( \varepsilon = 1 \) one has \( h_o^2 = i I_2 \) or \( h_o^2 = e^{\frac{2\pi i}{s}} I_2 \). If \( h_o^2 = i I_2 \) then \( h_o \in H \) is of order 8 with \( \det(h_o) = \pm i \). Therefore \( R = \mathbb{Z}[i] \)
and $h_o$ has eigenvalues $\lambda_1(h_o) = e^{\frac{4\pi}{3}}$, $\lambda_2(h_o) = e^{-\frac{4\pi}{3}}$ with $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = \frac{\lambda_2(h_o)}{\lambda_1(h_o)} = -1$. The relations $D_oD_1D_o^{-1} = D_2$, $D_oD_2D_o^{-1} = D_1$ on the diagonal form $D_o$ of $h_o$ hold for

$$D_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} a_1 & -b_1 \\ -c_1 & -a_1 \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{2}, i)).$$

The group $\langle D_1, D_2 \rangle$ is isomorphic to $\mathbb{Q}_8$ if and only if $a_1 = \pm \frac{\sqrt{2}}{2}$ and $c_1 = -\frac{1}{b_1}$ for some $b_1 \in \mathbb{Q}(\sqrt{2}, i)$. In such a way, one obtains the group

$$H_{Q8}(8) = \langle g_1, g_2, h_o | g_1^2 = g_2^2 = h_o^4 = -I_2, \ g_2g_1 = -g_1g_2, \ h_o g_1 h_o^{-1} = g_2, \ h_o g_2 h_o^{-1} = g_1 \rangle$$

for $R = \mathbb{Z}[i]$. Note that $H_{Q8}(8)$ is of order 32 and has a conjugate $H_{Q8}^\prime(8) = \langle D_1, D_2, D_o \rangle < GL(2, \mathbb{Q}(\sqrt{2}, i))$. If $h_o = e^{\frac{2\pi i}{3}}I_2$ then $R = \mathcal{O}_{-3}$ and $h_o \in H$ is of order 6 with $\det(h_o) = e^{\pm \frac{2\pi i}{3}}$. According to $h_o g_1 h_o^{-1} = g_2 \neq g_1$, $h_o$ is not a scalar matrix, so that $\lambda_1(h_o) = e^{-\frac{4\pi}{3}}$, $\lambda_2(h_o) = -1$ for $\det(h_o) = e^{\frac{4\pi i}{3}}$. Now, $D_oD_1D_o^{-1} = D_2$ is tantamount to

$$D_2 = \begin{pmatrix} a_1 & e^{\frac{2\pi i}{3}} b_1 \\ e^{-\frac{2\pi i}{3}} c_1 & -a_1 \end{pmatrix}$$

and $D_oD_2D_o^{-1} = D_1$ reduces to

$$\begin{vmatrix} 1 - e^{\frac{2\pi i}{3}} & b_1 \\ 1 - e^{-\frac{2\pi i}{3}} & c_1 \end{vmatrix} = 0.$$

As a result, $b_1 = c_1$ and

$$D_1 = D_2 = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

commute with each other. Thus, there is no group $H$ of Case B with $h_o^2 = e^{\frac{2\pi i}{3}}I_2$. If $h_o g_1 h_o^{-1} = g_2$, $h_o g_2 h_o^{-1} = -g_1$ and $h_o^2 \not\in \langle g_1, g_2 \rangle$ then

$$H' = \langle g_1, g_2, h_o^2 | g_1^2 = g_2^2 = -I_2, \ h_o^2 = I_2, \ g_2g_1 = -g_1g_2, \ h_o^2 g_1 h_o^{-2} = -g_1, \ h_o^2 g_2 h_o^{-2} = -g_2 \rangle$$

is isomorphic to $H_{Q8}(2)$ or $H_{Q8}(7)$, according to the considerations for Case A. More precisely, if $H' \simeq H_{Q8}(2)$ then $h_o$ of order 4 has $\det(h_o) = \pm i$ and $R = \mathbb{Z}[i]$. Due to $-I_2 \in \langle g_1, g_2 \rangle$, one can assume that

$$D_o = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}.$$
Then $D_oD_1D_o^{-1} = D_2$ requires

$$D_2 = \begin{pmatrix} a_1 & ib_1 \\ -ic_1 & -a_1 \end{pmatrix},$$

so that $D_oD_2D_o^{-1} = -D_1$ results in $a_1 = 0$. Bearing in mind that det($D_1$) = det($D_2$) = 1, one concludes that

$$D_1 = \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & ib_1 \\ 0 & 0 \end{pmatrix}.$$  

For $b_1 = 1$, one obtains a subgroup $\langle D_1, D_2, D_o \rangle$ of $GL(2, \mathbb{Z}[i])$, isomorphic to

$$H_{Q8}(9) = \langle g_1, g_2, h_o \mid g_1^2 = g_2^2 = -I_2, \ h_o^4 = I_2, \ g_2g_1 = -g_1g_2, \ h_o g_1 h_o^{-1} = g_2, \ h_o g_2 h_o^{-1} = -g_1 \rangle < GL(2, \mathbb{Z}[i]).$$

Since det($h_o$) = $i$ is of order $s = 4$, the group $H_{Q8}(9)$ is of order 32. If $H' = \langle g_1, g_2, h_o^2 \rangle \simeq H_{Q8}(7)$ then $h_o \in H$ is to be of order 16, since $h_o^2$ is of order 8. The lack of $h_o \in GL(2, R)$ of order 16 reveals that the groups $H_{Q8}(3), H_{Q8}(4), H_{Q8}(8), H_{Q8}(9)$ deplete Case B.

There remains to be considered Case C with $h_o g_1 h_o^{-1} = g_2, h_o g_2 h_o^{-1} = \varepsilon g_1 g_2, h_o(g_1 g_2) h_o^{-1} = \varepsilon g_1$ for some $\varepsilon = \pm 1$. Note that $h_o^2 g_1 h_o^{-2} = \varepsilon g_1 g_2, h_o^2 g_2 h_o^{-2} = g_1, h_o^3 g_1 h_o^{-3} = g_1, h_o^3 g_2 h_o^{-3} = g_2$ require the divisibility of $s$ by 3, as far as $\langle g_j \rangle$ are normal subgroups of $\langle g_1, g_2 \rangle$ and $h_o^s \in \langle g_1, g_2 \rangle$. In other words, $s \in \{3, 6\}$ and $R = \mathcal{O}_{-3}$. The non-scalar matrices $h_o \in GL(2, \mathcal{O}_{-3})$ with det($h_o$) = $e^{\frac{2\pi i}{3}}$ have eigenvalues $\{\lambda_1(h_o), \lambda_2(h_o)\} = \{e^{\frac{2\pi i}{3}}, 1\}, \{e^{-\frac{2\pi i}{3}}, -1\}$ or $\{e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\}$. If $h_o$ is of order 3 or 6 then $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = e^{\frac{2\pi i}{3}}$ and $D_oD_1D_o^{-1} = D_2$ specifies that

$$D_2 = \begin{pmatrix} a_1 & e^{\frac{2\pi i}{3}}b_1 \\ e^{-\frac{2\pi i}{3}c_1} & -a_1 \end{pmatrix}.$$

Now, $2a_1a_2 + b_1c_2 + b_2c_1 = 0$ reduces to $2a_2^2 = b_1c_1$ and $a_2^2 + b_1c_1 = -1$ requires $a_1 = \pm \frac{3}{3}, c_1 = -\frac{2}{3b_1}$ for some $b_1 \in \mathbb{Q}(\sqrt{-3})^*$. Replacing, eventually, $D_j$ by $D_j^3$, one has

$$D_1 = \begin{pmatrix} \frac{\sqrt{-3}}{3} & b_1 \\ -\frac{2}{3b_1} & -\frac{\sqrt{-3}}{3} \end{pmatrix}, \quad D_2 = \begin{pmatrix} \frac{\sqrt{-3}}{3} & e^{\frac{2\pi i}{3}}b_1 \\ -2e^{-\frac{2\pi i}{3}}\frac{1}{3b_1} & -\frac{\sqrt{-3}}{3} \end{pmatrix}.$$  

Now,

$$D_1D_2 = \begin{pmatrix} \frac{\sqrt{-3}}{3} & e^{\frac{2\pi i}{3}}b_1 \\ -2e^{-\frac{2\pi i}{3}}\frac{1}{3b_1} & -\frac{\sqrt{-3}}{3} \end{pmatrix}$$
and \( D_oD_o^{-1} = \varepsilon D_1 D_2 \) holds for \( \varepsilon = 1 \). Thus,

\[
H^o_{Q_8}(6) = \langle D_1, D_2, D_o \rangle < GL(2, \mathbb{Q}(\sqrt{-3}))
\]

is conjugate to

\[
H_{Q_8}(6) = \langle g_1, g_2, h_o \mid g_1^2 = g_2^2 = -I_2, \ h_o^3 = I_2, \ g_2g_1 = -g_1g_2 \rangle \\
\text{with } h_o g_1 h_o^{-1} = g_2, \ h_o g_2 h_o^{-1} = g_1 g_2 < GL(2, O_{-3})
\]
of order 24 with \( \lambda_1(h_o) = e^{\frac{2\pi i}{3}} \), \( \lambda_2(h_o) = 1 \) or to

\[
H = \langle g_1, g_2, h_o \mid g_1^2 = g_2^2 = -I_2, \ h_o^3 = -I_2, \ g_2g_1 = -g_1g_2 \rangle \\ (15)
\]

of order 24 with \( \lambda_1(h_o) = e^{-\frac{2\pi i}{3}}, \ \lambda_3(h_o) = -1 \). Due to \( -I_2 \notin \langle g_1, g_2 \rangle \), the presence of \( h_o \in H \) of order 6 with \( \det(H) = \langle \det(h_o) \rangle \simeq \mathbb{C}_3 \) is equivalent to the existence of \( -h_o \in H \) of order 3 with \( \det(H) = \langle \det(-h_o) \rangle \simeq \mathbb{C}_3 \) and \( H \) from (15) is isomorphic to \( H_{Q_8}(6) \). If \( h_o \) has diagonal form

\[
D_o = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix} \in GL(2, \mathbb{Q}(\sqrt{-3}))
\]
of order 12 with \( \det(D_o) = e^{\frac{2\pi i}{3}}, \ \frac{\lambda_1(h_o)}{\lambda_2(h_o)} = -1 \), then \( D_o D_1 D_o^{-1} = D_2 \) implies that

\[
D_2 = \begin{pmatrix} a_1 & -b_1 \\ c_1 & a_1 \end{pmatrix}
\]

with \( a_1^2 = b_1 c_1 = -\frac{1}{2} \). Therefore, \( a_1 = \pm \frac{\sqrt{-7}}{2} \in GL(2, \mathbb{Q}(\sqrt{-3})) \), which is an absurd. If \( h_o g_1 h_o^{-1} = g_2, \ h_o g_2 h_o^{-1} = \varepsilon g_1 g_2 \) and \( s = 6 \) then \( h_o \in H \) is of order 6, according to Proposition 19. Now \( H'' = \langle g_1, g_2, h_o^3 \rangle < GL(2, R) \) with \( h_o^3 \notin \langle g_1, g_2 \rangle \) is subject to Case A with a scalar matrix \( h_o \in H \), according to \( h_o^3 g_1 h_o^{-3} = g_1, \ h_o^3 g_2 h_o^{-3} = g_2 \). If \( h_o^3 = iI_2 \) then \( h_o \) is of order \( r = 12 \). The assumption \( h_o^3 = e^{\frac{2\pi i}{3}} I_2 \) holds for \( h_o \) of order \( r = 9 \). Both contradict to \( r = 6 \) and establish that any subgroup \( H < GL(2, R) \) with \( H \cap SL(2, R) \simeq Q_8 \) is isomorphic to \( H_{Q_8}(i) \) for some \( 1 \leq i \leq 9 \).

\( \square \)

**Proposition 39.** Let \( H \) be a finite subgroup of \( GL(2, R) \),

\[
H \cap SL(2, R) = K_7 = \langle g_1, g_4, \ g_1^2 = g_4^3 = -I_2, \ g_1 g_4 g_1^{-1} = g_4^{-1} \rangle \simeq Q_{12}
\]

and \( h_o \in H \) be an element of order \( r \) with \( \det(H) = \langle \det(h_o) \rangle \simeq \mathbb{C}_s \) and eigenvalues \( \lambda_1(h_o), \lambda_2(h_o) \). Then \( H \) is isomorphic to \( H_{Q_{12}}(i) \) for some \( 1 \leq i \leq 10 \), where

\[
H_{Q_{12}}(i) = \langle g_1, g_4, h_o = iI_2 \mid g_1^2 = g_4^3 = -I_2, \ g_1 g_4 g_1^{-1} = g_4^{-1} \rangle
\]

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is of order 24 with $R - \mathbb{Z}[i]$,

$$H_{Q12}(2) = \langle g_1, g_4, h_o \mid g_1^2 = g_4^3 = -I_2, \quad h_o^6 = I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1},$$

$$h_o g_1 h_o^{-1} = g_1 g_4, \quad h_o g_4 h_o^{-1} = g_4$$

of order 24, with $R = \mathcal{O}_{-3}$, $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = e^{\frac{4\pi i}{3}}$,

$$H_{Q12}(3) = \langle g_1, g_4, h_o \mid g_1^2 = g_4^3 = h_o^6 = -I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1},$$

$$h_o g_1 h_o^{-1} = g_1 g_4^2, \quad h_o g_4 h_o^{-1} = g_4$$

is of order 24 with $R = \mathcal{O}_{-3}, \lambda_1(h_o) = e^{\frac{4\pi i}{3}}, \lambda_2(h_o) = e^{\frac{5\pi i}{3}}$,

$$H_{Q12}(4) = \langle g_1, g_4, h_o \mid g_1^2 = g_4^3 = -I_2, \quad h_o^2 = I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1},$$

$$h_o g_1 h_o^{-1} = -g_1, \quad h_o g_4 h_o^{-1} = g_4$$

is of order 24 with $\lambda_1(h_o) = -1, \lambda_2(h_o) = 1$,

$$H_{Q12}(5) = \langle g_1, g_4, h_o = e^{\frac{2\pi i}{3}} I_2 \mid g_1^2 = g_4^3 = -I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1}$$

is of order 36 with $R = \mathcal{O}_{-3}$,

$$H_{Q12}(6) = \langle g_1, g_4, h_o \mid g_1^2 = g_4^3 = -I_2, \quad h_o^3 = I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1},$$

$$h_o g_1 h_o^{-1} g_4 g_4^2, \quad h_o g_4 h_o^{-1} = g_4$$

is of order 36 with $R = \mathcal{O}_{-3}, \lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = 1$,

$$H_{Q12}(7) = \langle g_1, g_4, h_o \mid g_1^2 = g_4^3 = h_o^6 = -I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1},$$

$$h_o g_1 h_o^{-1} = -g_1, \quad h_o g_4 h_o^{-1} = g_4$$

is of order 36 with $R = \mathcal{O}_{-3}, \lambda_1(h_o) = e^{-\frac{\pi i}{3}}, \lambda_2(h_o) = e^{\frac{5\pi i}{3}}$,

$$H_{Q12}(8) = \langle g_1, g_4, h_o \mid g_1^2 = g_4^3 = h_o^4 = -I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1},$$

$$h_o g_1 h_o^{-1} = g_1, \quad h_o g_4 h_o^{-1} = g_4$$

is of order 48 with $R = \mathbb{Z}[i], \lambda_1(h_o) = e^{\frac{3\pi i}{4}}, \lambda_2(h_o) = e^{-\frac{3\pi i}{4}}$,

$$H_{Q12}(9) = \langle g_1, g_4, h_o \mid g_1^2 = g_4^3 = -I_2, \quad h_o^6 = I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1},$$

$$h_o g_1 h_o^{-1} = g_1 g_4, \quad h_o g_4 h_o^{-1} = g_4$$

is of order 72 with $R = \mathcal{O}_{-3}, \lambda_1(h_o) = 1, \lambda_2(h_o) = e^{\frac{\pi i}{3}}$,

$$H_{Q12}(10) = \langle g_1, g_4, h_o \mid g_1^2 = g_4^3 = -I_2, \quad h_o^6 = I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1},$$
\[ h_oh_1h_o^{-1} = -g_1, \quad h_og_4h_o^{-1} = g_4 \]
is of order 72 with \( R = \mathcal{O}_{-3}, \lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = e^{-\frac{2\pi i}{3}}. \)

There exist subgroups
\[ H_{Q12}(2), H_{Q12}(4), H_{Q12}(5), H_{Q12}(6), H_{Q12}(9), H_{Q12}(10) < GL(2, \mathcal{O}_{-3}), \]
as well as subgroups
\[ H_{Q12}^o(1), H_{Q12}^o(3), H_{Q12}^o(7) < GL(2, \mathbb{Q}(\sqrt{3}, i)), \quad H_{Q12}^o(8) < GL(2, \mathbb{Q}(\sqrt{2}, \sqrt{3}, i)) \]
with \( H_{Q12}^o(j) \simeq H_{Q12}(j) \) for \( j \in \{1, 3, 7, 8\}. \)

**Proof.** According to Lemma 27, the groups \( H = K_7\langle h_0 \rangle \) with \( \det(H) = \langle \det(h_0) \rangle \simeq \mathbb{C}_s \) are determined up to an isomorphism by the order \( r \) of \( h_0 \), the element \( h_0g_1h_o^{-1} \in K_7 \) of order 4 and the element \( h_0g_4h_o^{-1} \in K_7 \) of order 6. Let us denote by \( K_7^{(m)} \) the set of the elements of \( K_7 \) of order \( m \). Straightforwardly,

\[ K_7^{(6)} = \{ g_4, g_4^{-1} \}, \quad K_7^{(4)} = \{ \pm g_1g_4 \mid 0 \leq i \leq 3 \}. \]

Inverting \( g_1g_4g_1^{-1} = g_4^{-1} \), one obtains \( g_1g_4^{-1}g_1^{-1} = g_4 \). If \( h_0g_4h_o^{-1} = g_4^{-1} \) then

\[ (g_1h_o)g_4(g_1h_o^{-1}) = g_1(h_0g_4h_o^{-1})g_1^{-1} = g_1g_4^{-1}g_1^{-1} = g_4. \]

As far as \( K_7 = \langle g_1, g_4, h_o \rangle = \langle g_1, g_4, g_1h_o \rangle \), there is no loss of generality in assuming \( h_0g_1h_o^{-1} = g_4 \).

We start the study of \( H \) by a realization of \( K_7 \) as a subgroup of the special linear group \( SL(2, \mathbb{Q}(\sqrt{-d}, \sqrt{-3})) \). Let

\[ D_4 = S^{-1}g_4S = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix} \]

be a diagonal form of \( g_4 \) for some \( S \in GL(2, \mathbb{Q}(\sqrt{-d}, \sqrt{-3})) \) and

\[ D_1 = S^{-1}g_1S = \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} \quad \text{with} \quad a_1^2 + b_1c_1 = -1. \]

Then

\[ D_1D_4D_1^{-1} = \begin{pmatrix} -\sqrt{-3}a_1^2 + e^{\frac{2\pi i}{3}} & -\sqrt{-3}a_1b_1 \\ -\sqrt{-3}a_1c_1 & \sqrt{-3}a_1^2 + E^{\frac{2\pi i}{3}} \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{-d}, \sqrt{-3})) \]

coincides with \( D_4^{-1} \) if and only if

\[ D_1 = \begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix} \quad \text{for some} \quad b_1 \in \mathbb{Q}(\sqrt{-d}, \sqrt{-3})^*. \]
That allows to compute explicitly

\[
K_7^{(4)} = \left\{ \pm D_1 = \pm \begin{pmatrix} 0 & b_1 \\ -b_1^{-1} & 0 \end{pmatrix}, \quad \pm D_1 D_4 = \pm \begin{pmatrix} 0 & \left(-e^{\frac{2\pi i}{3}} b_1\right)^{-1} \\ -\left(e^{\frac{2\pi i}{3}} b_1\right)^{-1} & 0 \end{pmatrix} \right\},
\]

\[
\pm D_1 D_4^2 = \pm \begin{pmatrix} 0 & e^{-\frac{2\pi i}{3}} b_1 \\ -e^{-\frac{2\pi i}{3}} b_1 & 0 \end{pmatrix},
\]

\[
K_7^{(4)} = \left\{ D_1 D_4^j = \begin{pmatrix} 0 & e^{-\frac{2\pi i}{3}} b_1 \\ -\left(e^{-\frac{2\pi i}{3}} b_1\right)^{-1} & 0 \end{pmatrix} \quad 0 \leq j \leq 5 \right\}.
\]

Now, \(D_oD_4D_o^{-1} = D_4\) amounts to

\[
D_o = \begin{pmatrix} \lambda_1(h_o) & 0 \\ 0 & \lambda_2(h_o) \end{pmatrix}
\]

and

\[
D_oD_1D_o^{-1} = \begin{pmatrix} -\left[\frac{\lambda_1(h_o)}{\lambda_2(h_o)}\right]^{-1} & 0 \\ 0 & \frac{\lambda_1(h_o)}{\lambda_2(h_o)} b_1 \end{pmatrix} = \begin{pmatrix} 0 & e^{-\frac{2\pi i}{3}} b_1 \\ -\left(e^{-\frac{2\pi i}{3}} b_1\right)^{-1} & 0 \end{pmatrix} = D_1 D_4^j
\]

if and only if \(\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = e^{-\frac{2\pi i}{3}}\). Note that the ratio \(\frac{\lambda_1(h_o)}{\lambda_2(h_o)}\) of the eigenvalues of \(h_o\) is determined up to an inversion and

\[
\left\{ e^{-\frac{2\pi i}{3}} \mid 0 \leq j \leq 5 \right\} = \left\{ 1 = e^0, \quad e^{\frac{2\pi i}{3}}, \quad -1 = e^{\pi i} \mid 1 \leq j \leq 2 \right\}.
\]

For any \(h_o \in H\) with \(\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = e^{\frac{2\pi i}{3}}, 0 \leq j \leq 3\) the group

\[
H = \langle g_1, g_4, h_o \mid g_1^2 = g_4^3 = -I_2, \quad h_o = I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1},
\]

\[
\lambda_1(h_o) = 1 \quad \lambda_2(h_o) = 0
\]

Note that \(\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = 1\) exactly when \(h_o \in H \setminus SL(2,R)\) is a scalar matrix. According to Propositions 16, 17, 18, 19, 20, 21, 22, the only scalar matrices \(h_o \in GL(2,R) \setminus SL(2,R)\) are \(h_o = \pm i I_2\) for \(R = \mathbb{Z}[i]\) and \(h_o = e^{\frac{2\pi i}{3}} I_2\) or \(e^{\frac{4\pi i}{3}} I_2\) with \(R = \mathbb{O}_{-3}\). Replacing, eventually, \(h_o = -i I_2\) by \(h_o^{-1} = i I_2\), one obtains the group \(H_{Q/12}(1) = \langle g_1, g_4, i I_2 \rangle\) with \(R = \mathbb{Z}[i]\). Note that \(H_{Q/12}(1) = \langle D_1, D_4, h_o = i I_2 \rangle\) is a realization of \(H_{Q/12}(1)\) as a subgroup of \(GL(2, \mathbb{Q}(\sqrt{3}, i))\). Bearing in mind that \(-I_2 \in K_7\), one observes that \(e^{-\frac{2\pi i}{3}} I_2 \in H\) if and only if \(-e^{-\frac{2\pi i}{3}} I_2 = e^{\frac{2\pi i}{3}} I_2 \in H\). Replacing, eventually, \(e^{\frac{2\pi i}{3}} I_2\) and \(e^{\frac{4\pi i}{3}} I_2\) by their inverse matrices, one observes that \(h_o = e^{\frac{2\pi i}{3}} I_2 \in H\) whenever \(H\) contains a scalar matrix of order 3 or 6. That provides the group \(H_{Q/12}(5) = \langle g_1, g_4, e^{\frac{2\pi i}{3}} I_2 \rangle\). Note that

\[
\langle D_1 \rangle = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D_4 = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & -e^{-\frac{2\pi i}{3}} \end{pmatrix}, \quad D_o = e^{\frac{2\pi i}{3}} I_2 \rangle < GL(2, \mathbb{O}_{-3})
\]

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is a realization of $H_{Q12}(5)$ as a subgroup of $GL(2, \mathcal{O}_3)$.

For $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = e^{\frac{2\pi i}{3}}$, Corollary 29 specifies that either $R = \mathcal{O}_3$, $s = 2$, $r = 6$, $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = e^{\frac{2\pi i}{3}}$ and $H \cong H_{Q12}(2)$ or $R = \mathcal{O}_3$, $s = 6$, $r = 6$, $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = e$. In the second case, one can restrict to $e = 1$, due to $I_2 \subset K_7 \subset H$. The corresponding group $H \cong H_{Q12}(9)$. Both, $H_{Q12}(2)$ and $H_{Q12}(9)$ can be realized as subgroups of $GL(2, \mathcal{O}_3)$, setting

$$g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad g_4 = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}.$$

$$h_o = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{pmatrix} \quad \text{or, respectively,} \quad h_o = \begin{pmatrix} e^{-\frac{2\pi i}{3}} & 0 \\ 0 & 1 \end{pmatrix}.$$

If $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = e^{\frac{2\pi i}{3}}$ then, eventually, replacing $h_o$ by $h_o^{-1}$, one has $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = e^{\frac{2\pi i}{3}}$, $s = 2$, $r = 12$, $R = \mathbb{Z}[i]$ and $H \cong H_{Q12}(3)$ or $\lambda_1(h_o) = e$, $\lambda_2(h_o) = e^{\frac{2\pi i}{3}}$, $s = 3$, $R = \mathcal{O}_3$, by Corollary 29. Note that $-I_2 \subset K_7 \subset H$ reduces the second case to $\lambda_1(h_o) = 1$, $\lambda_2(h_o) = e^{\frac{2\pi i}{3}}$, $s = 3$, $r = 3$, $R = \mathcal{O}_3$ and $H \cong H_{Q12}(6)$. Note that

$$g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad g_4 = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}, \quad h_o = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{pmatrix}.$$

generate a subgroup of $GL(2, \mathcal{O}_3)$, isomorphic to $H_{Q12}(6)$. In the case of $H \cong H_{Q12}(3)$ the eigenvalues of $h_o$ are primitive twelfth roots of unity, so that

$$D_1 = \begin{pmatrix} 0 & b_1 \\ -b_1^{-1} & 0 \end{pmatrix}, \quad D_4 = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}, \quad D_o = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{pmatrix}.$$

generate a subgroup $H_{Q12}(3) < GL(2, \mathbb{Q}(\sqrt{3}, i))$, isomorphic to $H_{Q12}(3)$.

For $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = -1$ there are four non-equivalent possibilities for the eigenvalues $\lambda_1(h_o), \lambda_2(h_o)$ of $h_o$. The first one is $\lambda_1(h_o) = 1, \lambda_2(h_o) = -1$ with $s = 2, r = 2$ for any $R = R_{-d,f}$ and $H \cong H_{Q12}(4)$ of order 24. Note that

$$D_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D_4 = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}, \quad h_o = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

realizes $H_{Q12}(4)$ as a subgroup of $GL(2, \mathcal{O}_3)$. The second one is $\lambda_1(h_o) = e^{\frac{3\pi i}{4}}, \lambda_2(h_o) = e^{-\frac{3\pi i}{4}}$ with $s = 4, r = 8, R = \mathbb{Z}[i]$ and $H \cong H_{Q12}(8)$ of order 48. Observe that

$$D_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D_4 = \begin{pmatrix} e^{\frac{3\pi i}{4}} & 0 \\ 0 & e^{-\frac{3\pi i}{4}} \end{pmatrix}, \quad D_o = \begin{pmatrix} e^{\frac{3\pi i}{4}} & 0 \\ 0 & e^{-\frac{3\pi i}{4}} \end{pmatrix}.$$
generate a subgroup of $GL(2, \mathbb{Q}(\sqrt{2}, \sqrt{3}, i))$, isomorphic to $H_{Q12}(8)$. In the third case, 
$\lambda_1(h_o) = e^{-\frac{2\pi i}{6}}$, $\lambda_2(h_o) = e^{\frac{2\pi i}{6}}$ with $s = 3$, $r = 12$, $R = \mathcal{O}_3$ and $H \simeq H_{Q12}(7)$ of order 36, realized by

$$D_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D_4 = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}, \quad D_o = \begin{pmatrix} e^{-\frac{2\pi i}{6}} & 0 \\ 0 & e^{\frac{2\pi i}{6}} \end{pmatrix}$$

as a subgroup of $GL(2, \mathbb{Q}(\sqrt{3}, i))$. In the fourth case, $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = e^{-\frac{2\pi i}{3}}$ with $s = 6$, $r = 6$, $R = \mathcal{O}_3$ and $H \simeq H_{Q12}(10)$ of order 72. The matrices

$$g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad g_4 = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}, \quad h_o = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}$$

generate a subgroup of $GL(2, \mathcal{O}_3)$, isomorphic to $H_{Q12}(10)$. The groups $H_{Q12}(4)$, $H_{Q12}(7)$, $H_{Q12}(8)$, $H_{Q12}(10)$ with $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = -1$ are non-isomorphic, as far as they are of different orders.

\[\square\]

**Proposition 40.** Let $H$ be a finite subgroup of $GL(2, R)$,

$$H \cap SL(2, R) = K_8 = \langle g_1, g_2, g_3 \mid g_1^2 = g_2^2 = -I_2, \quad g_3^2 = I_2, \quad g_2g_1 = g_1g_2, \rangle$$

$$g_3g_1g_3^{-1} = g_2, \quad g_3g_2g_3^{-1} = g_1g_2 \simeq SL(2, \mathbb{F}_3)$$

and $h_o \in H$ be an element of order $r$ with $\det(H) = \det(h_o) \simeq \mathbb{C}_s$ and eigenvalues $\lambda_1(h_o), \lambda_2(h_o)$. Then $H$ is isomorphic to $H_{SL(2,3)}(i)$ for some $1 \leq i \leq 9$, where

$$H_{SL(2,3)}(1) = \langle g_1, g_2, g_3, iI_2 \mid g_1^2 = g_2^2 = -I_2, \quad g_3^2 = I_2, \quad g_2g_1 = g_1g_2, \rangle$$

$$g_3g_1g_3^{-1} = g_2, \quad g_3g_2g_3^{-1} = g_1g_2$$

of order 48 with $R = \mathbb{Z}[i]$,

$$H_{SL(2,3)}(2) = \langle g_1, g_2, g_3, h_o \mid g_1^2 = g_2^2 = -I_2, \quad g_3^2 = I_2, \quad h_o^2 = I_2, \quad g_2g_1 = g_1g_2 \rangle$$

$g_3g_1g_3^{-1} = g_2, \quad g_3g_2g_3^{-1} = g_1g_2, \quad h_o^2g_1h_o^{-1} = g_1, \quad h_o^2g_2h_o^{-1} = g_2, \quad h_o^2g_3h_o^{-1} = g_2g_3$ of order 48 with $R = \mathbb{Z}[i], \lambda_1(h_o) = -1, \lambda_2(h_o) = 1$,

$$H_{SL(2,3)}(3) = \langle g_1, g_2, g_3, h_o \mid g_1^2 = g_2^4 = h_o^2 = -I_2, \quad g_3^3 = I_2, \quad g_2g_1 = g_1g_2, \rangle$$

$g_3g_1g_3^{-1} = g_2, \quad g_3g_2g_3^{-1} = g_1g_2, \quad h_o^2g_1h_o^{-1} = g_2, \quad h_o^2g_2h_o^{-1} = g_1, \quad h_o^2g_3h_o^{-1} = g_2g_3$ of order 48 with $R = \mathcal{O}_2, \lambda_1(h_o) = e^{\frac{2\pi i}{3}}, \lambda_2(h_o) = e^{\frac{2\pi i}{3}},$ $H_{SL(2,3)}(4) = \langle g_1, g_2, g_3, h_o \mid g_1^2 = g_2^2 = -I_2, \quad g_3^3 = I_2, \quad h_o^2 = I_2, \quad g_2g_1 = g_1g_2, \rangle$
of order 48 with \( R = R_{-2, f} \), \( \lambda_1(h_o) = -1 \), \( \lambda_2(h_o) = 1 \),

\[
H_{SL(2,3)}(5) = K_8 \times \langle e^{\frac{2\pi i}{7}} I_2 \rangle \simeq SL(2, \mathbb{F}_3) \times \mathbb{C}_3
\]
of order 72 with \( R = \mathcal{O}_{-3} \),

\[
H_{SL(2,3)}(6) = \langle g_1, g_2, g_3, h_o \mid g_1^2 = g_2^2 = g_3^3 = I_2, \quad g_2 g_1 = -g_1 g_2, \quad g_3 g_1^{-1} = g_2, \quad g_3 g_2^{-1} = g_1 g_2, \quad h_o g_1 h_o^{-1} = g_2, \quad h_o g_2 h_o^{-1} = g_1 g_2, \quad h_o g_3 h_o^{-1} = g_3 \rangle
\]
of order 72 with \( R = \mathcal{O}_{-3} \), \( \lambda_1(h_o) = e^{\frac{2\pi i}{7}}, \lambda_2(h_o) = 1 \),

\[
H_{SL(2,3)}(7) = \langle g_1, g_2, g_3, h_o \mid g_1^2 = g_2^2 = h_o^3 = I_2, \quad g_2 g_1 = -g_1 g_2, \quad g_3 g_1^{-1} = g_2, \quad g_3 g_2^{-1} = g_1, \quad h_o g_1 h_o^{-1} = -g_1, \quad h_o g_2 h_o^{-1} = -g_2, \quad h_o g_3 h_o^{-1} = -g_2 g_3 \rangle
\]
of order 96 with \( R = \mathbb{Z}[i] \), \( \lambda_1(h_o) = e^{\frac{\pi i}{4}}, \lambda_2(h_o) = e^{-\frac{\pi i}{4}} \),

\[
H_{SL(2,3)}(8) = \langle g_1, g_2, g_3, h_o \mid g_1^2 = g_2^2 = h_o^4 = I_2, \quad g_2 g_1 = -g_1 g_2, \quad g_3 g_1^{-1} = g_2, \quad g_3 g_2^{-1} = g_1 g_2, \quad h_o g_1 h_o^{-1} = g_2, \quad h_o g_2 h_o^{-1} = g_1, \quad h_o g_3 h_o^{-1} = g_1 g_3 \rangle
\]
of order 96 with \( R = \mathbb{Z}[i] \), \( \lambda_1(h_o) = e^{\frac{3\pi i}{4}}, \lambda_2(h_o) = e^{-\frac{\pi i}{4}} \),

\[
H_{SL(2,3)}(9) = \langle g_1, g_2, g_3, h_o \mid g_1^2 = g_2^2 = h_o^4 = I_2, \quad g_2 g_1 = -g_1 g_2, \quad g_3 g_1^{-1} = g_2, \quad g_3 g_2^{-1} = g_1, \quad h_o g_1 h_o^{-1} = -g_1, \quad h_o g_2 h_o^{-1} = -g_1 g_2, \quad h_o g_3 h_o^{-1} = g_2 g_3 \rangle
\]
of order 96 with \( R = \mathbb{Z}[i] \), \( \lambda_1(h_o) = i, \lambda_2(h_o) = 1 \).

There exists a subgroup

\[
H_{SL(2,3)}(5) < GL(2, \mathcal{O}_{-3}),
\]
as well as subgroups

\[
H^0_{SL(2,3)}(1), H^0_{SL(2,3)}(2), H^0_{SL(2,3)}(9) < GL(2, \mathbb{Q}(\sqrt{3}, i)),
\]
\[
H^0_{SL(2,3)}(4) < GL(2, \mathbb{Q}(\sqrt{-2}, \sqrt{-3})),
\]
\[
H^0_{SL(2,3)}(3), H^0_{SL(2,3)}(7), H^0_{SL(2,3)}(8) < GL(2, \mathbb{Q}(\sqrt{2}, \sqrt{3}, i))
\]
with \( H^0_{SL(2,3)}(j) \simeq H_{SL(2,3)}(j) \) for \( 1' \leq j \leq 4 \) or \( 6 \leq j \leq 9 \).
Proof. According to Lemma 27, the groups $H$ under consideration are uniquely determined up to an isomorphism by the order $r$ of $h_o$ and by the elements $h_o g_j h_o^{-1} \in K_8^{(4)}, 1 \leq j \leq 2$, $x_3 := h_o g_3 h_o^{-1} \in K_8^{(3)}$. (Throughout, $G^{(\nu)}$ denotes the set of the elements of order $\nu$ from a group $G$.) Recall by Proposition 24 the realization of $K_8 \cong SL(2, \mathbb{F}_3)$ as a subgroup $\mathcal{K}_8$ of $GL(2, \mathbb{Q}(\sqrt{-d}, \sqrt{-3}))$, generated by the matrices

$$
D_1 = \begin{pmatrix}
-\frac{\sqrt{-3}}{3} & b_1 \\
-2 & \sqrt{-3} \\
\frac{\sqrt{-3}}{3} & 0
\end{pmatrix},
D_2 = \begin{pmatrix}
-\frac{\sqrt{-3}}{3} & e^{-\frac{2\pi i}{3}} b_1 \\
-2 & \sqrt{-3} \\
\frac{2\pi i}{3} & 0
\end{pmatrix},
D_3 = \begin{pmatrix}
-\frac{2\pi i}{3} & 0 \\
-2 & \sqrt{-3} \\
e^{-\frac{2\pi i}{3}} & 0
\end{pmatrix}
$$

with some $b_1 \in \mathbb{Q}(\sqrt{-d}, \sqrt{-3})^\times$. After computing

$$
D_1 D_2 = \begin{pmatrix}
-\frac{\sqrt{-3}}{3} & e^{-\frac{2\pi i}{3}} b_1 \\
-2 & \sqrt{-3} \\
\frac{2\pi i}{3} & 0
\end{pmatrix},
$$

one puts

$$
\delta_j := \begin{pmatrix}
-\frac{\sqrt{-3}}{3} & e^{-\frac{2\pi i}{3}} b_1 \\
-2 & \sqrt{-3} \\
\frac{2\pi i}{3} & 0
\end{pmatrix} \text{ for } 0 \leq j \leq 2
$$

and observes that $\delta_0 = D_1$, $\delta_1 = D_2$, $\delta_2 = D_1 D_2$. The elements of $\mathcal{K}_8$ of order 4 constitute the subset

$$
\mathcal{K}_8^{(4)} = \{ \pm \delta_j \mid 0 \leq j \leq 2 \}.
$$

In order to list the elements of $\mathcal{K}_8$ of order 3, let us note that $D_3 D_1 D_3^{-1} = D_2$ and $D_3 D_2 D_3^{-1} = D_1 D_2$ imply $D_3 (D_1 D_2) D_3^{-1} = D_1$. Thus, for any even permutation $j, l, m$ of $0, 1, 2$, one has

$$
\left| \begin{array}{c}
D_3 \delta_j D_3^{-1} = \delta_l \\
D_3 \delta_l D_3^{-1} = \delta_m \\
D_3 \delta_m D_3^{-1} = \delta_j
\end{array} \right| \text{ or, equivalently, } \left| \begin{array}{c}
D_3 \delta_j = \delta_l D_3 \\
D_3 \delta_l = \delta_m D_3 \\
D_3 \delta_m = \delta_j D_3
\end{array} \right. \quad (16)
$$

Making use of (16, one computes that

$$
(-\delta_j D_3)^2 = \delta_m D_3^2, \quad (-\delta_j D_3)^3 = (-\delta_j D_3) (-\delta_j D_3)^2 = I_2 \quad \text{for all } 0 \leq j \leq 2,
$$

so that $-\delta_j D_3 \in \mathcal{K}_8^{(3)}$. As a result, $\delta_j D_3^2 = (-\delta_l D_m)^2 \in \mathcal{K}_8^{(3)}$ for all $0 \leq j \leq 2$ and

$$
\mathcal{K}_8^{(3)} = \{ D_3, \ -\delta_j D_3, \ D_3^2, \ \delta_j D_3^2 \mid 0 \leq j \leq 2 \}.
$$

Proposition 24 has established that $\mathcal{K}_8$ has a unique Sylow 2-subgroup

$$\mathcal{H}_8 = \langle \delta_0, \delta_1 \mid \delta_0^2 = \delta_1^2 = -I_2, \ \delta_1 \delta_0 = -\delta_0 \delta_1 \rangle = \{ \pm I_2, \pm \delta_j \mid 0 \leq j \leq 2 \}.$$
so that the set $K_8^{(4)} = H_8^{(4)}$ of the elements of $K_8$ of order 4 are contained in $H_8 \cong Q_8$. In other words, $x_j := h_o \delta_j h_o^{-1} \in H_8$ and $H' = \langle g_1, g_2, h_o \rangle \cong H' = \langle \delta_0, \delta_1, D_0 \rangle$ is a subgroup of $H$ with $H \cap SL(2, R) \cong Q_8$. Proposition 38 establishes that any such $H'$ is isomorphic to $H_{Q_8(i)}$ for some $1 \leq i \leq 9$.

We claim that for any $1 \leq i \leq 9$ there is (at most) a unique finite subgroup $H = \langle g_1, g_2, g_3, h_o \rangle$ of $GL(2, R)$ with $\langle g_1, g_2, h_o \rangle \cong H_{Q_8(i)}$, $H \cap SL(2, R) = \langle g_1, g_2, g_3 \rangle \cong SL(2, \mathbb{F}_3)$ and $\det(H) = \langle \det(h_o) \rangle$. To this end, let us consider the adjoint representation

\[
Ad : K_8 \longrightarrow S(K_8^{(4)}) \cong S_6
\]

and its restriction

\[
Ad : K_8^{(3)} \longrightarrow S(K_8^{(4)}) \cong S_6
\]

to the elements of $K_8$ of order 3. Note that

\[
\langle x_0, x_1 \rangle = h_o \langle \delta_0, \delta_1 \rangle h_o^{-1} = h_o H_8 h_o^{-1} = H_8,
\]

as far as $H_8 \cong Q_8$ is normal subgroup of $H' = H_8 \langle h_o \rangle$. The adjoint action

\[
Ad_{h_o} : K_8 \longrightarrow K_8
\]

\[
Ad_{h_o}(x) = h_o x h_o^{-1} \quad \text{for} \quad \forall x \in K_8
\]

of $h_o$ is a group homomorphism and transforms the relations $D_3 \delta_s D_3^{-1} = \delta_{s+1}$ for $0 \leq s \leq 1$ into the relations $x_3 x_s x_3^{-1} = x_{s+1}$ for $0 \leq s \leq 1$. For any $1 \leq i \leq 9$ the subgroup $H' \cong H_{Q_8(i)}$ of $H$ determines uniquely $x_0, x_1 \in H_8$. We claim that for any such $x_0, x_1$ there is a unique $x_3 \in K_8^{(3)}$ with

\[
Ad_{x_3}(x_0) = x_1, \quad Ad_{x_3}(x_1) = x_0 x_1.
\]  

(17)

Indeed, Proposition 38 specifies the following five possibilities:

Case 1 \quad $x_0 = \delta_0$, \quad $x_1 = \delta_1$;

Case 2 \quad $x_0 = \delta_0$, \quad $x_1 = -\delta_1$;

Case 3 \quad $x_0 = \delta_1$, \quad $x_1 = -\delta_0$;

Case 4 \quad $x_0 = -\delta_1$, \quad $x_1 = \delta_0$;

Case 5 \quad $x_0 = -\delta_1$, \quad $x_1 = \delta_2$.

For any $0 \leq s \neq t \leq 2$ and $\varepsilon, \eta \in \{\pm 1\}$ note that

\[
Ad_{\varepsilon \delta_s}(\eta \delta_s) = \eta \delta_s, \quad Ad_{\varepsilon \delta_t}(\eta \delta_t) = -\eta \delta_t.
\]
Combining with (14), one concludes that
\[
\text{Ad}_{D_2}(\langle \delta_i \rangle) = \text{Ad}_{(-\delta_i D_2)}(\langle \delta_i \rangle) = \langle \delta_i \rangle,
\]
\[
\text{Ad}_{D_3}(\langle \delta_i \rangle) = \text{Ad}_{(-\delta_i D_3)}(\langle \delta_i \rangle) = \langle \delta_i \rangle,
\]
\[
\text{Ad}_{D_3}(\langle \delta_m \rangle) = \text{Ad}_{(-\delta_i D_3)}(\langle \delta_m \rangle) = \langle \delta_j \rangle
\]
for any \(0 \leq s \leq 2\) and any even permutation \(j, l, m\) of \(0, 1, 2\). Similarly,
\[
\text{Ad}_{D_3}(\langle \delta_j \rangle) = \text{Ad}_{\delta_i D_3}(\langle \delta_j \rangle) = \langle \delta_m \rangle,
\]
\[
\text{Ad}_{D_3}(\langle \delta_l \rangle) = \text{Ad}_{\delta_i D_3}(\langle \delta_l \rangle) = \langle \delta_j \rangle,
\]
\[
\text{Ad}_{D_3}(\langle \delta_m \rangle) = \text{Ad}_{\delta_i D_3}(\langle \delta_m \rangle) = \langle \delta_l \rangle
\]
for any \(0 \leq s \leq 2\) and any even permutation \(j, l, m\) of \(0, 1, 2\). In the case 1, (17) read as \(\text{Ad}_{x_3}(\delta_0) = \delta_1, \text{Ad}_{x_3}(\delta_1) = \delta_2\) and imply that \(x_3 = D_3\), according to (16) and \(\text{Ad}_{(-\delta_i)} \neq Id_{K_8}\) for all \(0 \leq s \leq 2\). In the Case 2, \(\text{Ad}_{x_3}(\delta_0) = \delta_1\) and \(\text{Ad}_{x_3}(\delta_1) = -\delta_2\) specify that \(x_3 = -\delta_1 D_3 = -D_2 D_3\). In the next Case 3, the relations \(\text{Ad}_{x_3}(\delta_1) = -\delta_0\), \(\text{Ad}_{x_3}(\delta_0) = \delta_2\) hold if and only if \(x_3 = \delta_1 D_3^2 = D_2 D_3^2\). Further, \(\text{Ad}_{x_3}(\delta_1) = \delta_0\), \(\text{Ad}_{x_3}(\delta_0) = -\delta_2\) in Case 4 are satisfied by \(x_3 = \delta_0 D_3^2 = D_1 D_3^2\) and \(\text{Ad}_{x_3}(\delta_1) = \delta_2\), \(\text{Ad}_{x_3}(\delta_2) = \delta_0\) in Case 5 are valid for \(x_3 = D_3\). Given a presentation of \(H' \simeq H_{Q8}(i)\) with generators \(g_1, g_2, h_o\), one adjoins a generator \(g_3 \in SL(2, R)\) of order 3 and the relation \(h_o g_3 h_o^{-1} = x_3\), in order to obtain a presentation of \(H \simeq H_{SL(2,3)}(i), 1 \leq i \leq 9\).

4 Explicit Galois groups for \(A/H\) of fixed Kodaira-Enriques type

In order to classify the finite subgroups \(H\) of \(Aut(A)\), for which \(A/H\) is of a fixed Kodaira-Enriques classification type, one needs to describe the finite subgroups \(H\) of \(Aut(A)\) for \(A = E \times E\). Making use of the classification of the finite subgroups \(\mathcal{L}(H)\) of \(GL(2, R)\), done in section 3, let \(\det \mathcal{L}(H) = \langle \det \mathcal{L}(h_o) = e^{2\pi i s} \rangle \simeq \mathbb{C}_s\) for some \(s \in \{1, 2, 3, 4, 6\}\), \(h_o \in H\). (In the case of \(s = 1\), we choose \(h_o = Id_A\).) By Proposition 24 one has \(\mathcal{L}(H) \cap SL(2, R) = \langle \mathcal{L}(h_1), \ldots, \mathcal{L}(h_t) \rangle\) for some \(0 \leq t \leq 3\). (Assume \(\mathcal{L}(H) \cap SL(2, R) = \{I_2\}\) for \(t = 0\).) The linear part
\[
\mathcal{L}(H) = \langle \mathcal{L}(h) \rangle, \langle SL(2, R) \rangle \langle \mathcal{L}(h_o) \rangle = \langle \mathcal{L}(h_1), \ldots, \mathcal{L}(h_t) \rangle \langle \mathcal{L}(h_o) \rangle
\]
of \(H\) is a product of its normal subgroup \(\langle \mathcal{L}(h_1), \ldots, \mathcal{L}(h_t) \rangle\) and the cyclic group \(\langle \mathcal{L}(h_o) \rangle\). The translation part \(\mathcal{T}(H) = \ker(\mathcal{L}|_H)\) of \(H\) is a finite subgroup of \((\mathcal{T}_A, +) \simeq (A, +)\). The lifting \(\mathcal{T}(H) = \langle \mathcal{T}(h_i) \rangle \langle \mathcal{L}(h_o) \rangle\) of \(H\) is a free \(\mathbb{Z}\)-module of rank 4. Therefore \(\overline{\mathcal{T}(H)} = \langle \mathcal{T}(h_i) \rangle \langle \mathcal{L}(h_o) \rangle\) has at most four generators and
\[
\mathcal{T}(H) = \langle \mathcal{T}(h_i) \rangle \langle \mathcal{L}(h_o) \rangle = \langle \mathcal{T}(h_i) \rangle \langle \mathcal{L}(h_o) \rangle
\]
for some \(0 \leq m \leq 4\).
(In the case of \( m = 0 \) one has \( \mathcal{T}(H) = \{Id_A\} \).) We claim that

\[
H = \mathcal{T}(H)(h_1, \ldots, h_t, h_o) = \langle \tau(P,Q_i), h_j, h_o \mid 1 \leq i \leq m, 1 \leq j \leq t \rangle
\]

for some \( 0 \leq m \leq 4, 0 \leq t \leq 3 \). The choice of \( \langle \tau(P,Q_i), h_j, h_o \rangle \in H \) justifies the inclusion \( \langle \tau(P,Q_i), h_j, h_o \mid 1 \leq i \leq m, 1 \leq j \leq t \rangle \subseteq H \). For the opposite inclusion, an arbitrary element \( h \in H \) with \( \mathcal{L}(h) = \mathcal{L}(h_1)^{k_1} \cdots \mathcal{L}(h_t)^{k_t} \mathcal{L}(h_o)^{k_o} \) for some \( k_i \in \mathbb{Z} \) produces a translation \( \tau_{U,V} := h_{1}^{k_o}h_{t}^{k_t} \cdots h_{1}^{k_1} \in \ker(\mathcal{L}|_H) = \mathcal{T}(H) = \langle \tau(P,Q_i) \mid 1 \leq i \leq m \rangle \), so that \( h = \tau_{U,V}h_1^{k_1} \cdots h_t^{k_t}h_o^{k_o} \in \langle \tau(P,Q_i), h_j, h_o \mid 1 \leq i \leq m, 1 \leq j \leq t \rangle \) and \( H \subseteq \langle \tau(P,Q_i), h_j, h_o \mid 1 \leq i \leq m, 1 \leq j \leq t \rangle \). In such a way, we have derived the following

**Lemma 41.** If \( H \) is a finite subgroup of \( \text{Aut}(A) \), \( A = E \times E \) with

\[
\det \mathcal{L}(H) = \langle \det \mathcal{L}(h_o) = e^{2\pi i} \rangle \simeq \mathbb{C}_{s} \quad \text{and}
\]

\[
\mathcal{L}(H) \cap \text{SL}(2,R) = \langle \mathcal{L}(h_1), \ldots, \mathcal{L}(h_t) \rangle \quad \text{for some} \quad 0 \leq t \leq 3 \quad \text{then}
\]

\( H = \langle \tau(P,Q_i), h_j, h_o \mid 1 \leq i \leq m, 1 \leq j \leq t \rangle \)

is generated by \( 0 \leq m \leq 3 \) translations and at most four non-translation elements.

Bearing in mind that \( A/H \) is birational to a K3 surface exactly when \( \mathcal{L}(H) \) is a subgroup of \( \text{SL}(2,R) \), one obtains the following

**Corollary 42.** The quotient \( A/H \) by a finite subgroup \( H \) of \( \text{Aut}(A) \) has a smooth K3 model if and only if \( H \) is isomorphic to some \( H^{K3}(j,m) \) with \( 1 \leq j \leq 8, 0 \leq m \leq 3 \), where

\[
H^{K3}(1,m) = \langle \tau(P,Q_i), \tau_{U,V}(-I_2) \mid 1 \leq i \leq m \rangle
\]

\[
H^{K3}(2,m) = \langle \tau(P,Q_i), \ h_1 \mid 1 \leq i \leq m \rangle
\]

for \( \mathcal{L}(h_1) \in \text{SL}(2,R) \), \( \text{tr} \mathcal{L}(h_1) = 0 \),

\[
H^{K3}(3,m) = \langle \tau(P,Q_i), \ h_1, h_2 \mid 1 \leq i \leq m \rangle
\]

for \( \mathcal{L}(h_1), \mathcal{L}(h_2) \in \text{SL}(2,R) \), \( \text{tr} \mathcal{L}(h_1) = \text{tr} \mathcal{L}(h_2) = 0 \), \( \mathcal{L}(h_2)\mathcal{L}(h_1) = -\mathcal{L}(h_1)\mathcal{L}(h_2) \),

\[
H^{K3}(4,m) = \langle \tau(P,Q_i), \ h_3 \mid 1 \leq i \leq m \rangle
\]

for \( \mathcal{L}(h_3) \in \text{SL}(2,R) \), \( \text{tr} \mathcal{L}(h_3) = -1 \),

\[
H^{K3}(5,m) = \langle \tau(P,Q_i), \ h_4 \mid 1 \leq i \leq m \rangle
\]

for \( \mathcal{L}(h_4) \in \text{SL}(2,R) \), \( \text{tr} \mathcal{L}(h_4) = 1 \),

\[
H^{K3}(6,m) = \langle \tau(P,Q_i), \ h_1, h_4 \mid 1 \leq i \leq m \rangle
\]

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for $\mathcal{L}(h_1), \mathcal{L}(h_4) \in SL(2, R)$, $\text{tr}\mathcal{L}(h_1) = 0$, $\text{tr}\mathcal{L}(h_4) = 1$, $\mathcal{L}(h_1)\mathcal{L}(h_4)[\mathcal{L}(h_1)]^{-1} = [\mathcal{L}(h_4)]^{-1},$

$$H^{K_3}(7, m) = \langle \tau(P, Q_i), \ h_1, h_2, h_3 \ | \ 1 \leq i \leq m \rangle$$

for $\mathcal{L}(h_1), \mathcal{L}(h_2), \mathcal{L}(h_3) \in SL(2, R)$, $\text{tr}\mathcal{L}(h_1) = \text{tr}\mathcal{L}(h_2) = 0$, $\text{tr}\mathcal{L}(h_3) = -1,

$$\mathcal{L}(h_2)\mathcal{L}(h_1) = -\mathcal{L}(h_1)\mathcal{L}(h_2),$$

$$\mathcal{L}(h_3)\mathcal{L}(h_1)[\mathcal{L}(h_3)]^{-1} = \mathcal{L}(h_2), \ \mathcal{L}(h_3)\mathcal{L}(h_2)[\mathcal{L}(h_3)]^{-1} = \mathcal{L}(h_1)\mathcal{L}(h_2),$$

$$H^{K_3}(8, m) = \langle \tau(P, Q_i), \ h_1, h_2, h_3 \ | \ 1 \leq i \leq m \rangle$$

for $\mathcal{L}(h_1), \mathcal{L}(h_2), \mathcal{L}(h_3) \in SL(2, R)$, $\text{tr}\mathcal{L}(h_1) = \text{tr}\mathcal{L}(h_2) = 0$, $\text{tr}\mathcal{L}(h_3) = -1,

$$\mathcal{L}(h_2)\mathcal{L}(h_1) = -\mathcal{L}(h_1)\mathcal{L}(h_2),$$

$$\mathcal{L}(h_3)\mathcal{L}(h_1)[\mathcal{L}(h_3)]^{-1} = \mathcal{L}(h_2), \ \mathcal{L}(h_3)\mathcal{L}(h_2)[\mathcal{L}(h_3)]^{-1} = \mathcal{L}(h_1)\mathcal{L}(h_2).$$

We are going to show that for an arbitrary finite subgroup $H < \text{Aut}(A)$ with an abelian linear part $\mathcal{L}(H) < GL(2, R)$, there exist an isomorphic model $F_1 \times F_2$ of $A$ and a normal subgroup $N_1$ of $H$, embedded in $\text{Aut}(F_1)$, such that the quotient group $H/N_1$ is an automorphism group of $F_2$. This result can be viewed as a generalization of Bombieri-Mumford’s classification [3] of the hyper-elliptic surfaces. More precisely, if $H = \mathcal{T}(H) \langle h_o \rangle$ for some $h_o \in H$ with eigenvalues $\lambda_1 \mathcal{L}(h_o) = 1$, $\lambda_2 \mathcal{L}(h_o) = \det \mathcal{L}(h_o) = e^{2\pi i s}, s \in \{2, 3, 4, 6\}$, then there is a translation subgroup $N_1$ of $\text{Aut}(F_1)$, such that $G \simeq H/N_1$ is a non-translation group, acting on the split abelian surface $F'_1 \times F_2 = (F_1 \times N_1) \times F_2$. According to Proposition 5, the quotient $A/H$ is hyper-elliptic (respectively, ruled with elliptic base) exactly when the finite Galois covering $A \to A/H$ is unramified (respectively, ramified). Since $F_1 \to F_1/N_1 = F'_1$ is unramified for a translation subgroup $N_1 \mathcal{T}_{F_1} < \text{Aut}(F_1)$, the covering $A \to A/H$ is unramified and only if the covering $F'_1 \times F_2 \to (F'_1 \times F_2)/G$ is unramified for $G = H/N_1$. In particular, the first canonical projection $\text{pr}_1 : G \to \text{Aut}(F'_1)$ is a group monomorphism and $G$ is an abelian group with at most two generators, according to the classification of the finite translation groups of $F'_1$. Thus, Bombieri-Mumford’s classification of the hyper-elliptic surfaces $(F'_1 \times F_2)/G$ reduces to the classification of the split, fixed point free abelian subgroups $G < \text{Aut}(F'_1 \times F_2)$ with at most two generators, for which the canonical projections $\text{pr}_1 : G \to \text{Aut}(F'_1)$ and $\text{pr}_2 : G \to \text{Aut}(F_2)$ are injective group homomorphisms.

Towards the classification of the finite subgroups of $\text{Aut}(E)$, let us recall that the semi-direct products $\langle a \rangle \rtimes \langle b \rangle \simeq \mathbb{C}_m \rtimes \mathbb{C}_s$ of cyclic groups are completely determined by the adjoint action of $b$ on $a$. Namely, $\text{Ad}_b(a) = bab^{-1} = a^j$ for some residue $j \in \mathbb{Z}_m^*$ modulo $m$, relatively prime to $m$. Now $\text{Ad}_b(a) = a^j = a$ requires $j^s \equiv 1 \pmod{m}$. In other words, $j \in \mathbb{Z}_m^*$ is of order $r$, dividing $s$ and $\langle a \rangle \rtimes \langle b \rangle$ is isomorphic to

$$G_s^{(j)}(m) := \mathbb{C}_m \rtimes_j \mathbb{C}_s = \langle a, b \ | \ a^m = 1, b^s = 1, bab^{-1} = a^j \rangle \quad (18)$$
for some \( j \in \mathbb{Z}_m^* \) of order \( r \), dividing \( s \). Form now on, we use the notation (18) without further reference. Note that the only \( j \in \mathbb{Z}_m^* \) of order 1 is \( j \equiv 1(\text{mod} m) \) and 
\[ G_m^{(l)}(m) = \langle a, b, c \rangle \simeq \mathbb{C}_m \times \mathbb{C}_s \] is the direct product of \( \langle a \rangle = \mathbb{C}_m \) and \( \langle b \rangle = \mathbb{C}_s \).

**Lemma 43.** Let \( G \) be a finite subgroup of the automorphism group \( \text{Aut}(E) \) of an elliptic curve \( E \) with endomorphism ring \( \text{End}(E) = R \). Then \( G \) is isomorphic to some of the groups \( G_1(m, n), G_2^{(-1,-1)}(m, n), G_3^{(j)}(m), s \in \{3, 4, 6\} \), where

\[ G_1(m, n) = \langle \tau_{P_1}, \tau_{P_2} \rangle \simeq \mathbb{C}_m \times \mathbb{C}_n, \quad m, n \in \mathbb{N} \]

is a translation group with at most two generators,

\[ G_2^{(-1,-1)}(m, n) = \langle \tau_{P_1}, \tau_{P_2} \rangle \times \langle -1 \rangle \simeq (\mathbb{C}_m \times \mathbb{C}_n) \times \langle -1 \rangle \]

\[ G_3^{(j)}(m) = \langle \tau_{P_1} \rangle \times \langle e^{\frac{2\pi i}{j}} \rangle \simeq \mathbb{C}_m \times \mathbb{C}_3 \]

\[ G_4^{(j)}(m) = \langle \tau_{P_1} \rangle \times \langle i \rangle \simeq \mathbb{C}_m \times \mathbb{C}_4 \]

\[ G_5^{(j)}(m) = \langle \tau_{P_1} \rangle \times \langle e^{\frac{\pi i}{j}} \rangle \simeq \mathbb{C}_m \times \mathbb{C}_6 \]

for some \( j \in \mathbb{Z}_m^* \) of order 1 or 3, \( R = \mathcal{O}_{-3} \),

\[ G_6^{(j)}(m) = \langle \tau_{P_1} \rangle \times \langle i \rangle \simeq \mathbb{C}_m \times \mathbb{C}_6 \]

for some \( j \in \mathbb{Z}_m^* \) of order 1, 2, 4 or 6.

**Proof.** Any finite translation group \( G \triangleleft (\mathcal{L}_E, +) \) lifts to a lattice \( \tilde{G} \triangleleft (\tilde{E} = \mathbb{C}, +) \) of rank 2, containing \( \pi_1(E) \). By the Structure Theorem for finitely generated modules over the principal ideal domain \( \mathbb{Z} \), there exists a \( \mathbb{Z} \)-basis \( \lambda_1, \lambda_2 \) of \( \tilde{G} \) and natural numbers \( m, n \in \mathbb{N} \), such that

\[ \tilde{G} = \lambda_1 \mathbb{Z} + \lambda_2 \mathbb{Z}, \quad \pi_1(E) = m\lambda_1 \mathbb{Z} + mn\lambda_2 \mathbb{Z}. \]

As a result, \( P_1 = \lambda_1 + \pi_1(E) \in (\mathbb{E}, +) \) of order \( m \) and \( P_2 = \lambda_2 + \pi_1(E) \in (\mathbb{E}, +) \) of order \( mn \) generate the finite translation group \( G = \tilde{G}/\pi_1(E) \simeq \mathbb{C}_m \times \mathbb{C}_mn \).

If \( G \) is a finite non-translation subgroup of \( \text{Aut}(E) \) then the linear part \( \mathcal{L}(G) \) of \( G \) is a non-trivial subgroup of the units group \( R^* \). Bearing in mind that

\[ R^* = \begin{cases} 
\langle -1 \rangle \simeq \mathbb{C}_2 & \text{for } R \neq \mathbb{Z}[i], \mathcal{O}_{-3}, \\
\langle i \rangle \simeq \mathbb{C}_4 & \text{for } R = \mathbb{Z}[i], \\
\langle e^{\frac{\pi i}{j}} \rangle & \text{for } R = \mathcal{O}_{-3}, 
\end{cases} \]

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one concludes that $G = \langle e^{2\pi i/s} \rangle \simeq \mathbb{C}_s$ for some $s \in \{2, 3, 4, 6\}$. Any lifting $g_0 = \tau_U e^{2\pi i} \in G$ of $\mathcal{L}(g_0) = e^{2\pi i}$ has a fixed point $P_0 \in E$. After moving the origin of $E$ at $P_0$, one can assume that $g_0 = e^{2\pi i}$. Bearing in mind that the translation part $T(G) = \ker(|_G)$, one observes that $G = \mathcal{T}(G) \langle e^{2\pi i/s} \rangle$. The inclusion $\mathcal{T}(G) \langle e^{2\pi i/s} \rangle \subseteq G$ is clear. For any $g \in G$ with $\mathcal{L}(g) = e^{2\pi i/j}$ for some $0 \leq j \leq s - 1$, one has $g \left( e^{2\pi i} \right)^{-j} \in \ker(\mathcal{L}|_G) = T(G)$, so that $G \subseteq \mathcal{T}(G) \langle e^{2\pi i/s} \rangle$ and $G = \mathcal{T}(G) \langle e^{2\pi i/s} \rangle$. Note that $\mathcal{T}(G)$ is a normal subgroup of $G$ with $\mathcal{T}(G) \cap \langle e^{2\pi i/s} \rangle = \{\text{Id}_E\}$, so that

$$G = \mathcal{T}(G) \rtimes \langle e^{2\pi i/s} \rangle$$

is a semi-direct product. As a result, there is an adjoint action

$$\text{Ad} : \langle e^{2\pi i/s} \rangle \longrightarrow \text{Aut}(\mathcal{T}(G)),$$

$$\text{Ad}_{e^{2\pi i/s}}(\tau_{P_1}) = e^{2\pi i/s} \tau_{P_1} e^{-2\pi i/s} = \tau_s e^{2\pi i/s} P_1$$

of $\langle e^{2\pi i/s} \rangle$ on $\mathcal{T}(G)$, which is equivalent to the invariance of $\mathcal{T}(G)$ under a multiplication by $e^{2\pi i/s} \in \mathbb{R}^*$. The translation group $\mathcal{T}(G) = \langle \tau_{P_1}, \tau_{P_2} \rangle$ has at most two generators, so that

$$G = \langle \tau_{P_1}, \tau_{P_2} \rangle \rtimes \langle e^{2\pi i/s} \rangle$$

for some $s \in \{2, 3, 4, 6\}$. If $s = 2$ and $\langle \tau_{P_1}, \tau_{P_2} \rangle \simeq \mathbb{C}_m \times \mathbb{C}_n = \langle \tau_{Q_1} \rangle \times \langle \tau_{Q_2} \rangle$, then $\tau_{Q_1} = \tau_{Q_2} = \tau_{Q_k}$ for $1 \leq k \leq 2$. The residue classes $-1(\text{mod}m) \in \mathbb{Z}_m^*$ and $-1(\text{mod}n) \in \mathbb{Z}_n^*$ are order 1 or 2.

We claim that $G = \langle \tau_{P_1}, \tau_{P_2} \rangle \rtimes \langle e^{2\pi i/s} \rangle$ has at most two generators for $s \in \{3, 4, 6\}$. Indeed, $\tau_{P_1} \in \mathcal{T}(G)$ implies that $\text{Ad}_{e^{2\pi i/s}}(\tau_{P_1}) = \tau_s e^{2\pi i/s} P_1 \in \mathcal{T}(G)$. For $s \in \{3, 4, 6\}$ the points $P_1, e^{2\pi i/j} P_1$ have $\mathbb{Z}$-linearly independent liftings from $\widetilde{\mathcal{T}(G)}$, so that $\mathcal{T}(G) = \langle \tau_{P_1}, \tau_{P_2} \rangle = \langle \tau_{P_1}, \tau_s e^{2\pi i/j} P_1 \rangle$. As a result,

$$G = \langle \tau_{P_1}, \tau_{P_2} \rangle \rtimes \langle e^{2\pi i/s} \rangle$$

$$= \langle \tau_{P_1}, e^{2\pi i/j} \tau_{P_1} e^{-2\pi i/j} \rangle \rtimes \langle e^{2\pi i/s} \rangle = \langle \tau_{P_1} \rangle \rtimes \langle e^{2\pi i/s} \rangle \simeq \mathbb{C}_s \times_j \mathbb{C}_s = \langle a \rangle \rtimes_j \langle c \rangle = \langle a, c | a^m = 1, c^s = 1, cac^{-1} = a^j \rangle$$

for some $j \in \mathbb{Z}_m^*$ of order $r$, dividing $s \in \{3, 4, 6\}$.

Let us put $G_{1}^{(1)}(m, n) := G_{1}(m, n)$, in order to list the finite subgroups of $\text{Aut}(E)$ as $G_{s}^{(1,2)}(m, n)$ with $s \in \{1, 2\}$ and $G_{s}^{(j)}(m)$ with $s \in \{3, 4, 6\}$.

**Lemma 44.** Let $H$ be a finite subgroup of $\text{Aut}(A)$ with abelian linear part $\mathcal{L}(H)$. Then:

(i) there exists $S \in \text{GL}(2, \mathbb{C})$, such that all the elements of

$$S^{-1}HS = \{S^{-1}hs = (\tau_{U_1} \lambda_1 \mathcal{L}(h), \tau_{U_2} \lambda_2 \mathcal{L}) | h \in H\} < \text{Aut}(S^{-1}A)$$

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have diagonal linear parts;
(ii) if \( F_1 = S^{-1}(E \times \delta_E) \), \( F_2 = S^{-1}(\delta_E \times E) \) then \( S^{-1}A = F_1 \times F_2 \) and the canonical projections
\[
\text{pr}_k : S^{-1}HS \to Aut(F_k),
\]
\[
\text{pr}_k(\tau_{U_1}, \lambda_1L(h), \tau_{U_2}, \lambda_2L(h)) = \tau_{U_k}\lambda_kL(h),
\]
are group homomorphisms with \( \text{pr}_k(S^{-1}HS) \simeq G_s^{(j_1,j_2)}(m,n) \), \( s \in \{1, 2\} \) or \( G_s^{(j)} \), \( s \in \{3, 4, 6\} \);
(iii) \( S^{-1}HS = \ker(\text{pr}_2)\langle h_1, \ldots, h_t \rangle \) for any liftings \( h_j = (\alpha_j, \beta_j) \in S^{-1}HS \) of the generators \( \beta_1, \ldots, \beta_t \) of \( \text{pr}_2(S^{-1}HS) \), \( 1 \leq t \leq 3 \);
(iv) \( S^{-1}A/\ker(\text{pr}_2) = C_1 \times F_2 \), where \( C_1 \) is an elliptic curve for a translation subgroup \( \ker(\text{pr}_2) < (\mathcal{T}_{F_1}, +) < Aut(F_1) \) or a rational curve for a non-translation subgroup \( \ker(\text{pr}_2) < Aut(F_1) \), \( \ker(\text{pr}_2) \setminus (\mathcal{T}_{F_1}, +) \neq \emptyset \);
(v) \( A/H \simeq (C_1 \times F_2)/G \) for :
\[
G := \langle h_1, \ldots, h_t \rangle / (\langle h_1, \ldots, h_t \rangle \cap \ker(\text{pr}_2))
\]
with isomorphic second projection
\[
\overline{\text{pr}}_2 : G \to \text{pr}_2(S^{-1}HS)
\]
and first projection
\[
\overline{\text{pr}}_1 : G \to \overline{\text{pr}}_1(G) < Aut(C_1)
\]
with kernel \( \ker(\overline{\text{pr}}_1|G) \simeq \ker(\text{pr}_1|S^{-1}HS) \).

Proof. (i) It is well known that for any finite set \( \{L(h) \mid h \in H\} \) of commuting matrices, there exists \( S \in GL(2, \mathbb{C}) \), such that
\[
S^{-1}L(h)S = L(S^{-1}hS) = \begin{pmatrix}
\lambda_1L(h) & 0 \\
0 & \lambda_2L(h)
\end{pmatrix}
\]
are diagonal for all \( h \in H \). Namely, if there is \( h_o \in H \), whose linear part \( L(h_o) \) has two different eigenvalues \( \lambda_1L(h_o) \neq \lambda_2L(h_o) \), then one takes the \( j \)-th column of \( S \in \mathbb{Q}(\sqrt{-1})_{2 \times 2} \) to be an eigenvector, associated with \( \lambda_jL(h_o), 1 \leq j \leq 2 \). The conjugate \( S^{-1}L(h_o)S \) is a diagonal matrix. It suffices to show that \( v_j \) are eigenvectors of all \( L(h) \), in order to conclude that \( S^{-1}L(h)S \) are diagonal, as the matrices of \( L(h) \) with respect to the basis \( v_1, v_2 \) of \( \mathbb{C}^2 \). Indeed, for any \( h \in H \) the relation \( L(h)L(h_o) = L(h_o)L(h) \) implies that
\[
\lambda_jL(h_o)[L(h)v_j] = L(h)L(h_o)v_j = L(h_o)[L(h)v_j].
\]
Therefore \( L(h)v_j \) is an eigenvector of \( L(h_o) \) with associated eigenvalue \( \lambda_jL(h_o) \), so that \( L(h)v_j \) is proportional to \( v_j \), i.e., \( L(h)v_j = c_hv_j \) for some \( c_h \in \mathbb{C} \), which turns to be an eigenvalue \( c_h = \lambda_jL(h) \) of \( L(h) \). If \( \lambda_1L(h) = \lambda_2L(h) \) for \( \forall h \in H \) then all \( L(h) \) are scalar matrices. In particular, \( L(h) \) are diagonal.
(ii) Note that the direct product \( A = E \times E \) of elliptic curves coincides with their direct sum. If
\[
S^{-1}A := S^{-1}A/\pi_1(A) = \mathbb{C}^2/\pi_1(A),
\]
then \( S^{-1}A \to S^{-1}A \) is an isomorphism of abelian surfaces and
\[
S^{-1}(A) = S^{-1}(E \times E) = S^{-1}[(E \times \tilde{\phi}_E) \times (\tilde{\phi}_E \times E)] = S^{-1}(E \times \tilde{\phi}_E) \times S^{-1}(\tilde{\phi}_E \times E) = F_1 \times F_2.
\]
The canonical projections \( pr_k : S^{-1}HS \to Aut(F_k) \) are group homomorphisms, according to
\[
pr_k((\tau_{V_1}\lambda_1\mathcal{L}(g), \tau_{V_2}\lambda_2\mathcal{L}(g)))(\tau_{U_1}\lambda_1\mathcal{L}(h), \tau_{U_2}\lambda_2\mathcal{L}(h)) = pr_k(\tau_{V_1+\lambda_1\mathcal{L}(g)}V_1(\lambda\mathcal{L}(\lambda_1\mathcal{L}(h)), \tau_{V_2+\lambda_2\mathcal{L}(g)}U_2(\lambda_2\mathcal{L}(\lambda_2\mathcal{L}(h)))) = \tau_{V_2+hS}(\lambda_k\mathcal{L}(g), \lambda_k\mathcal{L}(h)) = pr_k(\tau_{V_2}\lambda_k\mathcal{L}(g), \tau_{U_2}\lambda_k\mathcal{L}(h)),\chi_k(\tau_{U_2}\lambda_k\mathcal{L}(h)) \text{ for } \forall g, h \in W \text{ with } S^{-1}gS = \tau_{V_1,V_2}(\mathcal{L}(S^{-1}gS), S^{-1}hS) = \tau_{U_1,U_2}(\mathcal{L}(S^{-1}hS)). \text{ The image } pr_k(S^{-1}HS) \text{ of } S^{-1}HS \text{ is a finite subgroup of } Aut(F_k) \text{ for } 1 \leq k \leq 2.
\]
(iii) If \( h_j = (\alpha_j, \beta_j) \in S^{-1}HS \) are liftings of the generators \( \beta_j \) of \( pr_2(S^{-1}HS) \), then \( \ker(pr_2)\langle h_1, \ldots, h_t \rangle \) is a subgroup of \( S^{-1}HS \), as far as \( \ker(pr_2) \) is a normal subgroup of \( S^{-1}HS \). For any \( pr_2(S^{-1}hS) = \beta_1^m \ldots \beta_t^m \) for some \( m_i \in \mathbb{Z} \), one has \( (S^{-1}HS)\langle h_1^m, \ldots, h_t^m \rangle \in \ker(pr_2) \), so that \( S^{-1}hS \in \ker(pr_2)\langle h_1, \ldots, h_t \rangle \) and \( S^{-1}HS = \ker(pr_2)\langle h_1, \ldots, h_t \rangle \).

(iv) The subgroup \( \ker(pr_2) \) of \( S^{-1}HS \) acts identically on \( F_2 \) and can be thought of as a subgroup of \( Aut(F_1) \), \( pr_1(\ker(pr_2)) \cong \ker(pr_2) \). Thus,
\[
S^{-1}A/\ker(pr_2) \cong [F_1/pr_1(\ker(pr_2))] \times F_2 = C_1 \times F_2
\]
with an elliptic curve \( C_1 \) exactly when \( pr_1(\ker(pr_2)) \) is a translation subgroup of \( Aut(F_1) \) or a rational curve \( C_1 \) for a non-translation subgroup \( pr_1(\ker(pr_2)) \) of the automorphism group \( Aut(F_1) \) of \( F_1 \).

(v) Since \( \ker(pr_2) \) is a normal subgroup of \( S^{-1}HS \) with quotient
\[
S^{-1}HS/\ker(pr_2) = [\ker(pr_2)\langle h_1, \ldots, h_t \rangle] / \ker(pr_2) = \langle h_1, \ldots, h_t \rangle / (\langle h_1, \ldots, h_t \rangle \cap \ker(pr_2)) = G,
\]
one has
\[
A/H \cong (S^{-1}A)/(S^{-1}HS) \cong [S^{-1}A/\ker(pr_2)]/[S^{-1}HS/\ker(pr_2)] = (C_1 \times F_2)/G.
\]
By the First Isomorphism Theorem, the epimorphism \( pr_2 : S^{-1}HS \to pr_2(S^{-1}HS) \) gives rise to an isomorphism
\[
\overline{pr}_2 : S^{-1}HS/\ker(pr_2) = G \longrightarrow pr_2(S^{-1}HS).
\]
The homomorphism $\text{pr}_1 : S^{-1}HS \to Aut(F_1)$ induces a homomorphism

$$\overline{\text{pr}} : S^{-1}HS/\ker(\text{pr}_2) = G \to Aut(F_1)/\text{pr}_1(\ker(\text{pr}_2)) \simeq Aut(C_1).$$

in the automorphism group of $C_1 = F_1/\text{pr}_1(\ker(\text{pr}_2))$. It suffices to show that the kernel

$$\ker(\overline{\text{pr}}) = \{S^{-1}hS \ker(\text{pr}_2) \mid \text{pr}_1(S^{-1}hS) \in \text{pr}_1 \ker(\text{pr}_2)\} = \ker(\text{pr}_2) \ker(\text{pr}_1)]/\ker(\text{pr}_2),$$

since

$$[\ker(\text{pr}_2) \ker(\text{pr}_1)]/\ker(\text{pr}_2) \simeq \ker(\text{pr}_1)/[\ker(\text{pr}_2) \cap \ker(\text{pr}_1)] = \ker(\text{pr}_1).$$

Indeed, if there exists $S^{-1}h_1S(\text{pr}_1(S^{-1}hS), Id_{F_2}) \in \ker(\text{pr}_2)$ then

$$S^{-1}(h_1^{-1}h)S = (Id_{F_1}, \text{pr}_2(S^{-1}hS)) \in S^{-1}HS \cap \ker(\text{pr}_1),$$

so that $S^{-1}hS \in S^{-1}h_1S \ker(\text{pr}_1) \subset \ker(\text{pr}_2) \ker(\text{pr}_1)$ for all $S^{-1}hS \ker(\text{pr}_2) \in \ker(\overline{\text{pr}})$. Conversely, any element of $[\ker(\text{pr}_2) \ker(\text{pr}_1)]/\ker(\text{pr}_2)$ is of the form

$$(g_1, Id_{F_2})(Id_{F_1}, g_2) \ker(\text{pr}_2) = (g_1, g_2) \ker(\text{pr}_2)$$

for some $(g_1, Id_{F_2}), (Id_{F_1}, g_2) \in S^{-1}HS \cap [Aut(F_1) \times Aut(F_2)]$, so that

$$\text{pr}_1(g_1, g_2) = g_1 = \text{pr}_1((g_1, Id_{F_2})) \in \text{pr}_1 \ker(\text{pr}_2)$$

reveals that $(g_1, g_2) \ker(\text{pr}_2) \in \ker(\overline{\text{pr}})$.

\[\square\]

According to Lemma 43, the finite automorphism groups of elliptic curves have at most three generators. Combining with Lemma 44(iii), one concludes that the finite subgroups $H$ of $Aut(E \times E)$ with abelian linear part $L(H)$ have at most six generators. Their linear parts $L(H)$ have at most two generators.

**Lemma 45.** Let $h = \tau_{[U,V]}L(h)$ be an automorphism of $A = E \times E$ and $w = (u,v) \in \mathbb{C}^2 = \mathbb{A}$ be a lifting of $(u,v) + \pi_1(A) = (U,V) \in A$. Then $h$ has no fixed points on $A$ if and only if for any $\mu = (\mu_1, \mu_2) \in \pi_1(A)$ the affine-linear transformation

$$\tilde{h}(w, \mu) = \tau_{w+\mu}L(h) \in Aff(\mathbb{C}^2, R) := (\mathbb{C}^2, +) \times GL(2, R)$$

has no fixed points on $\mathbb{C}^2$.

**Proof.** The statement of the lemma is equivalent to the fact that $Fix_A(h) \neq \emptyset$ exactly when $Fix_{\mathbb{C}^2}(\tilde{h}(w, \mu)) \neq \emptyset$ for some $\mu \in \pi_1(A)$. Indeed, if $(p, q) \in Fix_{\mathbb{C}^2}(\tilde{h}(w, \mu))$ then $(P, Q) = (p + \pi_1(E), q + \pi_1(E)) \in A$ is a fixed point of $h$, according to

$$h(P, Q) = L(h) \begin{pmatrix} P \\ Q \end{pmatrix} + \begin{pmatrix} U \\ V \end{pmatrix} = L(h) \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \pi_1(E) \\ \pi_1(E) \end{pmatrix} =$$

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\[
\begin{pmatrix}
    p \\
    q
\end{pmatrix}
+ \begin{pmatrix}
    \pi_1(E) \\
    \pi_1(E)
\end{pmatrix}
= \begin{pmatrix}
    P \\
    Q
\end{pmatrix}.
\]

Conversely, if
\[
\mathcal{L}(h) \begin{pmatrix}
    P \\
    Q
\end{pmatrix}
+ \begin{pmatrix}
    U \\
    V
\end{pmatrix}
= \begin{pmatrix}
    P \\
    Q
\end{pmatrix},
\]
then for any lifting \((p, q) \in \mathbb{C}^2\) of \((P, Q) = (p + \pi_1(E), q + \pi_1(E))\), one has
\[
\mathcal{L}(h) \begin{pmatrix}
    p \\
    q
\end{pmatrix}
+ \begin{pmatrix}
    U \\
    V
\end{pmatrix}
+ \begin{pmatrix}
    \pi_1(E) \\
    \pi_1(E)
\end{pmatrix}
= \begin{pmatrix}
    p \\
    q
\end{pmatrix}
+ \begin{pmatrix}
    \pi_1(E) \\
    \pi_1(E)
\end{pmatrix}.
\]

In other words,
\[
\mu = \begin{pmatrix}
    \mu_1 \\
    \mu_2
\end{pmatrix}
:= \mathcal{L}(h) \begin{pmatrix}
    p \\
    q
\end{pmatrix}
+ \begin{pmatrix}
    u \\
    v
\end{pmatrix}
- \begin{pmatrix}
    p \\
    q
\end{pmatrix}
\]
and \((p, q) \in \text{Fix}_{\mathbb{C}^2}(\widetilde{h}(w, -\mu))\).

\[\square\]

Now we are ready to characterize the automorphisms \(h \in \text{Aut}(A)\) without fixed points.

**Lemma 46.** An automorphism \(h = \tau_{(U, V)} \mathcal{L}(h) \in \text{Aut}(A) \setminus (\mathcal{T}_A, +)\) acts without fixed points on \(A = E \times E\) if and only if its linear part \(\mathcal{L}(h)\) has eigenvalues \(\lambda_1 \mathcal{L}(h) = 1\), \(\lambda_2 \mathcal{L}(h) \neq 1\) and
\[
\mathcal{L}(h) \begin{pmatrix}
    u \\
    v
\end{pmatrix}
\neq \lambda_2 \begin{pmatrix}
    u \\
    v
\end{pmatrix}
for any lifting \((u, v) \in \mathbb{C}^2\) of \((u + \pi_1(E), v + \pi_1(E)) = (U, V)\).

**Proof.** The fixed points \((P, Q) \in A\) of \(h = \tau_{(U, V)} \mathcal{L}(h)\) are described by the equality
\[
(\mathcal{L}(h) - I_2) \begin{pmatrix}
    P \\
    Q
\end{pmatrix}
= \begin{pmatrix}
    -U \\
    -V
\end{pmatrix}. \tag{19}
\]

If \(\det(\mathcal{L}(h) - I_2) \neq 0\) or \(1 \in \mathbb{C}\) is not an eigenvalues of \(\mathcal{L}(h)\), then consider the adjoint matrix
\[
(\mathcal{L}(h) - I_2)^* = \begin{pmatrix}
    d & -b \\
    -c & a
\end{pmatrix} \in \mathbb{R}_{2 \times 2}
\]
of
\[
\mathcal{L}(h) - I_2 = \begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix} \in \mathbb{R}_{2 \times 2}.
\]

According to \((\mathcal{L}(h) - I_2)^*(\mathcal{L}(h) - I_2) = \det(\mathcal{L}(h) - I_2)I_2 = (\mathcal{L}(h) - I_2)(\mathcal{L}(h) - I_2)^*\), one obtains
\[
\det(\mathcal{L}(h) - I_2) \begin{pmatrix}
    P \\
    Q
\end{pmatrix}
= (\mathcal{L}(h) - I_2)^*(\mathcal{L}(h) - I_2) \begin{pmatrix}
    u \\
    v
\end{pmatrix}
= -(\mathcal{L}(h) - I_2)^* \begin{pmatrix}
    U \\
    V
\end{pmatrix}. \tag{20}
\]
Then for an arbitrary lifting $(u_1, v_1) \in \mathbb{C}^2$ of
\[
\begin{pmatrix}
  u_1 + \pi_1(E) \\
v_1 + \pi_1(E)
\end{pmatrix} = \begin{pmatrix}
  U_1 \\
V_1
\end{pmatrix} := -(L(h) - I_2)^* \begin{pmatrix}
  U \\
V
\end{pmatrix},
\]
the point
\[
(p, q) = \left( \frac{u_1}{\det(L(h) - I_2)}, \frac{v_1}{\det(L(h) - I_2)} \right) \in \mathbb{C}^2
\]
descends to $(P, Q) = (p + \pi_1(E), q + \pi_1(E))$, subject to (20). As a result,
\[
(L(h) - I_2) \begin{pmatrix} P \\ Q \end{pmatrix} = \frac{1}{\det(L(h) - I_2)} (L(h) - I_2) \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} \pi_1(E) \\ \pi_1(E) \end{pmatrix} =
\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \pi_1(E) \\ \pi_1(E) \end{pmatrix}
\]
and $(P, Q) \in Fix_A(h)$.

From now on, let us suppose that the linear part $L(h) \in GL(2, R)$ of $h \in Aut(A) \setminus (\mathcal{T}_A, +)$ has eigenvalues $\lambda_1 L(h) = 1$ and $\lambda_2 L(h) = \det L(h) \in R^* \setminus \{1\}$. We claim that a lifting $(u, v) \in \mathbb{C}^2$ of $(u + \pi_1(E), v + \pi_1(E)) = (U, V) \in A$ satisfies
\[
L(h) \begin{pmatrix} u \\ v \end{pmatrix} = \lambda_2 L(h) \begin{pmatrix} u \\ v \end{pmatrix}
\]
if and only if there exists $(p, q) \in \mathbb{C}^2$ with
\[
(L(h) - I_2) \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -u \\ -v \end{pmatrix},
\]
which amounts to $(p, q) \in Fix_{\mathbb{C}^2}(\tau_{(u,v)}L(h))$. To this end, let us view $L(h) : \mathbb{C}^2 \to \mathbb{C}^2$ as a linear operator in $\mathbb{C}^2$ and reduce the claim to the equivalence of $(-u, -v) \in \ker(L(h) - \lambda_2 L(h)I_2)$ with $(-u, -v) \in Im(L(h) - I_2)$. In other word, the statement of the lemma reads as $\ker(L(h) - \lambda_2 L(h)I_2) = Im(L(h) - I_2)$ for the linear operators $L(h) - \lambda_2 L(h)I_2$ and $L(h) - I_2$ in $\mathbb{C}^2$. By Hamilton-Cayley Theorem, $L(h) \in \mathbb{C}_{2 \times 2}$ is a root of its characteristic polynomial
\[
\chi_{L(h)}(\lambda) = (\lambda - \lambda_1 L(h))(\lambda - 1).
\]
Thus,
\[
(L(h) - \lambda_2 L(h)I_2)Im(L(h) - I_2) = \{(0, 0)\}
\]
is the zero subspace of $\mathbb{C}^2$ and $Im(L(h) - I_2) \subseteq \ker(L(h) - \lambda_2 L(h)I_2)$. However, $\dim Im(L(h) - I_2) = rk(L(h) - I_2) = 1$ and
\[
\dim \ker(L(h) - \lambda_2 L(h)) = 2 - rk(L(h) - \lambda_2 L(h)I_2) = 2 - 1 = 1,
\]
so that $Im(L(h) - I_2) = \ker(L(h) - \lambda_2 L(h)I_2)$.

\[\square\]
Corollary 47. Let $H = \mathcal{T}(h)(h_o)$ be a finite subgroup of $\text{Aut}(A)$ for some $h_o \in H$ with
\[
\lambda_1 \mathcal{L}(h_o) = 1, \quad \lambda_2 \mathcal{L}(h_o) = e^{\frac{2\pi i}{s}}, \quad s \in \{2, 3, 4, 6\},
\]
$S \in \text{GL}(2, \mathbb{Q}(\sqrt{-d}))$ be a diagonalizing matrix for $h_o$ and
\[
S^{-1}h_oS = \left( \tau_W, e^{\frac{2\pi i}{s}} \right)
\]
after appropriate choice of an origin of $S^{-1}A = F_1 \times F_2$, $F_1 = S^{-1}(E \times \tilde{o}_E), F_2 = S^{-1}(\tilde{o}_E \times E)$. Then $A/H$ is a hyper-elliptic surface if and only if the kernel $\ker(\text{pr}_1)$ of the first canonical projection $\text{pr}_1 : S^{-1}HS \to \text{Aut}(F_1)$ is a translation subgroup of $\text{Aut}(F_2)$. If so, then
\[
S^{-1}A/[\ker(\text{pr}_2) \ker(\text{pr}_1)] \simeq C_1 \times C_2
\]
for some elliptic curves $C_1, C_2$ and
\[
A/H \simeq (C_1 \times C_2)/G,
\]
where the group $G$ is isomorphic to some of the groups
\[
G_{2}^{HE} = \langle (\tau_{U_1}, -1) \rangle \simeq \mathbb{C}_2
\]
with $U_1 \in C_{1}^{2-\text{tor}} \setminus \tilde{o}_{C_1}$,
\[
G_{2,2}^{HE} = \langle (\tau_{(P, Q_1)}) \rangle \times \langle (\tau_{U_1}, -1) \rangle \simeq \mathbb{C}_2 \times \mathbb{C}_2
\]
with $P, U_1 \in C_{1}^{2-\text{tor}} \setminus \tilde{o}_{C_1}$, $Q_1 \in C_{2}^{2-\text{tor}}$,
\[
G_{3}^{HE} = \langle (\tau_{U_1}, e^{\frac{2\pi i}{s}}) \rangle \simeq \mathbb{C}_3
\]
with $R = \mathcal{O}_{-3}, U_1 \in C_{1}^{3-\text{tor}} \setminus C_{2}^{2-\text{tor}}$,
\[
G_{3,3}^{HE} = \langle (\tau_{(P, Q_1)}) \rangle \times \langle (\tau_{U_1}, e^{\frac{2\pi i}{s}}) \rangle \simeq \mathbb{C}_3 \times \mathbb{C}_3
\]
with $R = \mathcal{O}_{-3}, P, U_1 \in C_{1}^{3-\text{tor}} \setminus C_{2}^{2-\text{tor}}, Q \in C_{2}^{3-\text{tor}} \setminus \tilde{o}_{C_2}$,
\[
G_{4}^{HE} = \langle (\tau_{U_1}, i) \rangle \simeq \mathbb{C}_4
\]
with $R = \mathbb{Z}[i], U_1 \in C_{1}^{4-\text{tor}} \setminus (C_{2}^{2-\text{tor}} \cup C_{1}^{3-\text{tor}})$,
\[
G_{4,4}^{HE} = \langle (\tau_{(P, Q_1)}) \rangle \times \langle (\tau_{U_1}, i) \rangle \simeq \mathbb{C}_2 \times \mathbb{C}_4
\]
with $R = \mathbb{Z}[i], P, U_1 \in C_{1}^{2-\text{tor}} \setminus \tilde{o}_{C_1}, Q_1 \in C_{2}^{(1)}(\text{tor}) \setminus \tilde{o}_{C_2}, U_1 \in C_{1}^{4-\text{tor}} \setminus (C_{2}^{2-\text{tor}} \cup C_{1}^{3-\text{tor}})$,
\[
G_{6}^{HE} = \langle (\tau_{U_1}, e^{\frac{2\pi i}{s}}) \rangle \simeq \mathbb{C}_6
\]
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with \( R = O_{-3}, \ U_1 \in C_1^{6-\text{tor}} \setminus (C_1^{3-\text{tor}} \cup C_1^{4-\text{tor}} \cup C_1^{5-\text{tor}}) \).

In the notations from Proposition 30, \( A/H \) is a hyper-elliptic surface exactly when \( H \cong S^{-1}HS \) is isomorphic to some of the groups:

\[
H_2^{HE}(m, n) = \langle (\tau_{M_j}, I_{F_2}), (Id_{F_1}, \tau_{N_k}), (\tau_W, -1) \mid 1 \leq j \leq m, \ 1 \leq k \leq n \rangle
\]

with \( W \notin \ker(pr_2), 2W \in \ker(pr_2), \mathcal{L}(H_2^{HE}(m, n)) \cong H_{C1}(1) \cong \mathbb{C}_2, \)

\[
H_2^{HE}(m, n) = \langle (\tau_{M_j}, I_{F_2}), (Id_{F_1}, \tau_{N_k}), \tau_{(X,Y)}, (\tau_W, -1) \mid 1 \leq j \leq m, \ 1 \leq k \leq n \rangle
\]

with \( 2X, 2W \in \ker(pr_2), X, W \notin \ker(pr_2), 2Y \in \ker(pr_1), Y \notin \ker(pr_1), \)
\[
\mathcal{L}(H_2^{HE}(m, n)) \cong H_{C1}(1) \cong \mathbb{C}_2
\]

\[
H_3^{HE}(m, n) = \langle (\tau_{M_j}, I_{F_2}), (Id_{F_1}, \tau_{N_k}), (\tau_W, e^{\frac{2\pi}{3}}) \mid 1 \leq j \leq m, \ 1 \leq k \leq n \rangle
\]

with \( R = O_{3}, 3W \in \ker(pr_2), 2W \notin \ker(pr_2), \mathcal{L}(H_3^{HE}(m, n)) \cong H_{C1}(2) \cong \mathbb{C}_3, \)

\[
H_3^{HE}(m, n) = \langle (\tau_{M_j}, I_{F_2}), (Id_{F_1}, \tau_{N_k}), \tau_{(X,Y)}, (\tau_W, e^{\frac{2\pi}{3}}) \mid 1 \leq j \leq m, \ 1 \leq k \leq n \rangle
\]

with \( R = O_{3}, 3X, 3W \in \ker(pr_2), 2X, 2W \notin \ker(pr_2), 3Y \in \ker(pr_1), Y \notin \ker(pr_1), \)
\[
\mathcal{L}(H_3^{HE}(m, n)) \cong H_{C1}(2) \cong \mathbb{C}_3
\]

\[
H_4^{HE}(m, n) = \langle (\tau_{M_j}, I_{F_2}), (Id_{F_1}, \tau_{N_k}), (\tau_W, i) \mid 1 \leq j \leq m, \ 1 \leq k \leq n \rangle
\]

with \( R = \mathbb{Z}[i], 4W \in \ker(pr_2), 2W, 3W \notin \ker(pr_2), \mathcal{L}(H_4^{HE}(m, n)) \cong H_{C1}(i) \cong \mathbb{C}_4, \)

\[
H_4^{HE}(m, n) = \langle (\tau_{M_j}, I_{F_2}), (Id_{F_1}, \tau_{N_k}), \tau_{(X,Y)}, (\tau_W, i) \mid 1 \leq j \leq m, \ 1 \leq k \leq n \rangle
\]

with \( R = \mathbb{Z}[i], 2X \in \ker(pr_2), X \notin \ker(pr_2), (1+Y) \in \ker(pr_1), Y \notin \ker(pr_1), 4W \in \ker(pr_2), 2W, 3W \notin \ker(pr_2), \mathcal{L}(H_4^{HE}(m, n)) \cong H_{C1}(3) \cong \mathbb{C}_4, \)

\[
H_6^{HE}(m, n) = \langle (\tau_{M_j}, I_{F_2}), (Id_{F_1}, \tau_{N_k}), (\tau_W, e^{\frac{2\pi}{3}}) \mid 1 \leq j \leq m, \ 1 \leq k \leq n \rangle
\]

with \( R = O_{-3}, 6W \in \ker(pr_2), 3W, 4W, 5W \notin \ker(pr_2), \) where \( m, n \in \{0,1,2\}. \)

Proof. In the notations from Lemma 44, the kernel \( \ker(pr_2) \) of the second canonical projection \( pr_2 : S^{-1}HS \to \text{Aut}(F_2) \) is a translation group, so that

\[
S^{-1}A \to S^{-1}A/\ker(pr_2) = C_1 \times F_2
\]

is unramified and \( C_1 \) is an elliptic curve. Thus, the covering \( A \to A/H \) is unramified if and only if \( C_1 \times F_2 \to (C_1 \times F_2)/G \cong A/H \) is unramified. In other words, \( A/H \) is a hyper-elliptic surface exactly when the group \( G \) has no fixed point on \( C_1 \times F_2 \). For any \( g \in G \) with \( \mathcal{L}(g) \notin I_2 \) the second component \( pr_2^{-1}(g) = \tau_{r_2} e^{\frac{2\pi i}{s-j}} \) for some \( 1 \leq j \leq s-1 \), \( V_2 \in F_2 \) has a fixed point on \( F_2 \). Towards \( F_i x_{C_1 \times F_2}(g) = \emptyset \) one has to have
\( \overline{pr}_1(g) \neq Id_{C_1}, \) so that \( \ker(\overline{pr}_1) \subseteq T(G) = G \cap \ker(\mathcal{L}) \) and \( \ker(\overline{pr}_1) \subseteq H = H \cap \ker(\mathcal{L}) \) are translation groups. The covering \( C_1 \times F_2 \rightarrow (C_1 \times F_2)/\ker(\overline{pr}_1) = C_1 \times C_2 \) is unramified, \( C_2 \) is an elliptic curve and \( A_1/H \) is a hyper-elliptic surface exactly when \( G_o = G/\ker(\overline{pr}_1) \) has no fixed points on \( (C_1 \times F_2)/\ker(\overline{pr}_1) \). The canonical projections

\[
\overline{pr}_1 : G_o \rightarrow Aut(C_1) \text{ and } \overline{pr}_2 : G_o \rightarrow Aut(C_2)
\]

are injective. Since \( \overline{pr}_1(G_o) \) is a translation subgroup of \( Aut(C_1) \), the group \( G_o \simeq \overline{pr}_1 \) is abelian and has at most two generators. As a result, \( \overline{pr}_2(G_o) \simeq G_o \) is an abelian subgroup of \( Aut(C_2) \) with at most two generators and non-trivial linear part \( \mathcal{L}(\overline{pr}_2(G_o)) = \langle e^{2\pi i} \rangle \simeq \mathbb{C}_s \) for some \( s \in \{2, 3, 4, 6\} \). According to Lemma 43,

\[
\overline{pr}_2(G_o) \simeq \langle \tau_{Q_1} \rangle \times \langle e^{2\pi i} \rangle \simeq \mathbb{C}_m \times \mathbb{C}_s
\]

for some \( Q_1 \in C_2 \) with \( \tau_{Q_1} = \text{Ad}_{e^{2\pi i}}(\tau_{Q_1}) = \tau_{e^{2\pi i} Q_1} \). In other words, the point \( \tau_{Q_1} \in C_2^{(e^{2\pi i} - 1)_{\text{tor}}} \setminus \{\partial C_2\} \). If \( s = 2 \) then any \( Q_1 \in C_2^{2 \text{-tor}} \) works out and the order of \( Q_1 \in (\mathbb{C}_2, +) \) is \( m = 2 \).

For \( s = 3 \) note that the endomorphism ring of \( C_2 \) is \( \text{End}(C_2) = \mathcal{O}_3 \). Therefore the fundamental group \( \pi_1(C_2) = c(\mathbb{Z} + \tau \mathbb{Z}) \) for some \( \tau \in \mathbb{Q}(\sqrt{-3}) \) and \( c \in \mathbb{C}^* \). By \( c \in \pi_1(C_2) \) and \( e^{\pi i} \in \text{End}(C_2) \) one has \( e^{\pi i} c \in \pi_1(C_2) \). Due to the linear independence of \( c \) and \( e^{\pi i} \) over \( \mathbb{Z} \), one has \( \pi_1(C_2) = c(\mathbb{Z} + e^{\pi i} \mathbb{Z}) = c\mathcal{O}_3 \). For \( \alpha = e^{\pi i} - 1 = -\frac{3}{2} + \frac{\sqrt{3}}{2} i \) the equation

\[
\alpha \left( x + e^{\pi i} y \right) = \left( a + e^{\pi i} b \right) c \text{ for some } a, b \in \mathbb{Z}
\]

has a solution \( x = -\frac{a+b}{3}, y = -\frac{a-2b}{3} \). Note that \( x(\text{mod}\mathbb{Z}) \equiv y(\text{mod}\mathbb{Z}) \) and

\[
\left( x + e^{\pi i} y \right) c(\text{mod}\mathbb{Z} + e^{\pi i} \mathbb{Z}) = \left( x + e^{\pi i} \right)(\text{mod}\pi_1(C_2)) \in \left\{ \partial C_2, \pm \left( 1 + e^{\pi i} \right)(\text{mod}\pi_1(C_2)) \right\} = C_2^{3 \text{-tor}}
\]

whereas \( C_2^{2 \text{-tor}} = C_2^{3 \text{-tor}} \) and \( m = 3 \). Thus, \( Q_1 \in C_2^{3 \text{-tor}} \setminus \{\partial C_2\} \) in the case of \( s = 3 \).

If \( s = 4 \) then \( \text{End}(C_2) = \mathbb{Z}[i] \) and \( \pi_1(C_2) = c\mathbb{Z}[i] \) for some \( c \in \mathbb{C}^* \). The equation \( (i-1)(x + iy)c = (a + bi)c \) for some \( a, b \in \mathbb{Z} \) has a solution \( x = -\frac{a+b}{2}, y = -\frac{a-b}{2} \) with

\[
(x + iy)c(\text{mod}\mathbb{Z}[i]) = x + iy(\text{mod}\pi_1(C_2)) \in \left\{ \partial C_2, \left( \frac{1+i}{2} \right) c(\text{mod}\pi_1(C_2)) \right\} = C_2^{(i+1) \text{-tor}}
\]

so that \( m = 4 \) and \( Q_1 \in C_2^{(i+1) \text{-tor}} \setminus \{\partial C_2\} \).

For \( s = 6 \) one has \( e^{\pi i} - 1 = e^{2\pi i} \) and \( C_2^{2\pi i \text{-tor}} = \{\partial C_2\} \), Therefore \( \overline{pr}_2(G_o) = \langle e^{\pi i} \rangle \simeq \mathbb{C}_6 \) in this case.
The restrictions on $P_1, U_1 \in C_1$ arise from the isomorphism $G_o \simeq \overline{\text{pr}}_1(G_o) \simeq \overline{\text{pr}}_2(G_o)$. Namely, $(\tau_{U_1}, e^{\frac{2\pi i}{s}}) \in G_o$ with $\overline{\text{pr}}_2(\tau_{U_1}, e^{\frac{2\pi i}{s}}) = E^{\frac{2\pi i}{s}}$ of order $s \in \{2, 34, 6\}$ has to have $\tau_{U_1} = \overline{\text{pr}}_1(\tau_{U_1}, e^{\frac{2\pi i}{s}}) \in (C_1, +)$ of order $s$. That amounts to $U_1 \in C_1^{s-\text{tor}}$ and $U_1 \not\in C_1^{q-\text{tor}}$ for all $1 \leq t < s$. If $\overline{\text{pr}}_2(G_o) = \langle \tau Q_1 \rangle \times \langle e^{\frac{2\pi i}{s}} \rangle$ with $Q_1 \neq \delta C_2$ then the order $m$ of $Q_1 \in C_2$ has to coincide with the order of $P_1 \in C_1$.

In order to relate the classification $G^{HE}_o, G^{HE}_m$ of $G_o$ with the classification of the groups $H^{HE}_o(m, n), H^{HE}_o(m, n)$ of $H \simeq S^{-1}HS$, note that $P_1, U_1 \in C_1^{p-\text{tor}} \setminus C_1^{q-\text{tor}}$ for some natural numbers $p > q$ exactly when the corresponding liftings $X, W \in F_1$ are subject to $pX, pQ \in \ker(\text{pr}_2), qX, qW \not\in \ker(\text{pr}_2)$. Similarly, $Q_1 \in C_2^{p-\text{tor}} \setminus C_2^{q-\text{tor}}$ for $p, q \in \mathbb{N}, P > q$ if and only if an arbitrary lifting $Y \in F_2$ satisfies $pY \in \ker(\text{pr}_1), qY \not\in \ker(\text{pr}_1)$.

\[ \square \]

Bearing in mind that $A/H$ with $H = \mathcal{T}(H) \langle h_o \rangle$, $\gamma_1 \mathcal{L}(h_o) = 1, \gamma_2 \mathcal{L}(h_o) \in \mathbb{R}^* \setminus \{1\}$ is either hyper-elliptic or a ruled surface with an elliptic base, one obtains the following

**Corollary 48.** Let $H = \mathcal{T}(H) \langle h_o \rangle$ be a finite subgroup of $\text{Aut}(A)$ for some $h_o \in H$ with $\gamma_1 \mathcal{L}(h_o) = 1, \gamma_2 \mathcal{L}(h_o) = e^{\frac{2\pi i}{s}}, s \in \{2, 3, 4, 6\}, S \in \text{GL}(2, \mathbb{Q}(\sqrt{-d}))$ be a diagonalizing matrix for $h_o$ and

\[ S^{-1}h_oS = \left(\tau_{U_1}, e^{\frac{2\pi i}{s}}\right) \]

after an appropriate choice of an origin of $S^{-1}(A) = F_1 \times F_2, F_1 = S^{-1}(E \times \delta E), F_2 = S^{-1}(\delta E \times E)$. Then $A/H$ is a ruled surface with an elliptic base if and only if the kernel $\ker(\text{pr}_1)$ of the first canonical projection $\text{pr}_1 : S^{-1}HS \to \text{Aut}(F_1)$ contains a non-translation element $S^{-1}hS = \left(\text{Id}_{F_1}, \tau_{U_2}e^{\frac{2\pi i}{s}}\right)$ for some $1 \leq k \leq s - 1, V_2 \in F_2$.

In the notations from Lemma 44, the quotient $A/H \simeq (C_1 \times F_2)/G$ of the split abelian surface $C_1 \times F_2 = S^{-1}A/\ker(\text{pr}_2)$ by its finite automorphism group $G = S^{-1}HS/\ker(\text{pr}_2)$ is a ruled surface with an elliptic base exactly when $G$ is isomorphic to some of the groups

\[ G^{RE}_2(m, n) = \langle \tau(P_1, Q_1), \tau(P_2, Q_2), \gamma_1 \mathcal{L}(h_o) \rangle \times \langle (\tau_{U_1}, -1) \rangle \simeq (\mathbb{C}_m \times \mathbb{C}_n) \times_{(-1, -1)} \mathbb{C}_2 = \langle \{a, b, c | a^m = 1, b^n = 1, cac^{-1} = a^{-1}, bcbc^{-1} = b^{-1}\} \rangle \times \langle (\tau_{U_1}, e^{\frac{2\pi i}{s}}) \rangle \simeq \mathbb{C}_m \times_j \mathbb{C}_3 = \langle \{a, c | a^m = 1, c^3 = 1, cac^{-1} = a^{-1}\} \rangle \times \langle (\tau_{U_1}, e^{\frac{2\pi i}{3}}) \rangle \simeq \mathbb{C}_m \times_j \mathbb{C}_4 = \mathbb{C}_2 \]

with $R = \mathcal{O}_3, 2U_1 \in \langle (\tau P_1) \rangle \simeq \mathbb{C}_m$ for some $j \in \mathbb{Z}_m^*$ of order 1 or 3,
\[ = \langle a \rangle \times_j \langle c \rangle = \langle a, c \mid a^m = 1, \ c^4 = 1, \ cac^{-1} = a^j \rangle \]

with \( R = \mathbb{Z}[i] \) for some \( j \in \mathbb{Z}_m^* \) or order 1, 2 or 4,

\[ G_6^{RE}(m, j) = \langle \tau(P, Q_1) \rangle \times \left( \tau_{U_1}, e^{\frac{2\pi i}{3}} \right) \simeq \mathbb{C}_m \times_j \mathbb{C}_6 = \]

\[ = \langle a \rangle \times_j \langle c \rangle = \langle a, c \mid a^m = 1, \ c^6 = 1, \ cac^{-1} = a^j \rangle \]

with \( R = \mathcal{O}_{-3} \) and at least one of \( 3U_1, 4U_1 \) or \( 5U_1 \) from \((\tau_{P_1}, +)\) for some \( j \in \mathbb{Z}_m^* \) of order 1, 2, 3 or 6.

The classification of \( G \) is an immediate application of the group isomorphism \( \mathfrak{P}_2 : G \to \text{pr}_2(S^{-1}HS) \) from Lemma 44 (v) and the classification of \( \text{Aut}(F_2) \), given in Lemma 43.

**Lemma 49.** Let \( G \) be a finite subgroup of \( GL(2, R) \) with \( G \cap SL(2, R) \neq \{I_2\} \), such that any \( g \in G \) has an eigenvalue \( \lambda_1(g) = 1 \). Then:

(i) \( G = G_s = \langle g_s, g_o \rangle \) is generated by \( g_s \in SL(2, R) \) of order \( s \in \{2, 3, 4, 6\} \) and \( g_o \in GL(2, R) \) with \( \det(g_o) = -1, \ \text{tr}(g_o) = 0, \ \text{subject to} \ g_o g_s g_o^{-1} = g_s^{-1} \);

(ii) and \( g \in G \) has eigenvalues \( \lambda_1(g) = 1 \) and \( \lambda_2(g) = -1 \);

(iii) the group

\[ G_s = \langle g_s, g_o \mid g_s^2 = I_2, \ g_o^2 = I_2, \ g_o g_s g_o^{-1} = g_s^{-1} \rangle \simeq D_s \]

is dihedral of order 2s for \( s \in \{3, 4, 6\} \) or the Klein group \( G_2 \simeq \mathbb{C}_2 \times \mathbb{C}_2 \) for \( s = 2 \).

**Proof.** Note that \( g \in G \) has an eigenvalue 1 exactly when the characteristic polynomial \( X_g(\lambda) = \lambda^2 - \text{tr}(g)\lambda + \det(g) \in R[\lambda] \) of \( g \) vanishes at \( \lambda = 1 \). This is equivalent to

\[ \text{tr}(g) = \det(g) + 1. \]

If \(-I_2 \notin G\), then Proposition 24 specifies that \( G \cap SL(2, R) = \langle g_3 \rangle \simeq \mathbb{C}_3 \). In the notations from Proposition 35, all the finite subgroups \( H_{C3}(i) = [H_{C3}(i) \cap SL(2, R)] \langle g_o \rangle \) of \( GL(2, R) \) with \( H_{C3}(i) \cap SL(2, R) \simeq \mathbb{C}_3 \), such that \( g_o \) has an eigenvalue \( \lambda_1(g_o) = 1 \) are isomorphic to

\[ H_{C3}(4) = \langle g, \ g_o \ g^3 = g_o^3 = I_2, \ g_o g_s g_o^{-1} = g^{-1} \rangle \simeq S_3 \simeq D_3 \]

for some \( g \in SL(2, R) \) with \( \text{tr}(g) = -1 \) and \( \lambda_1(g_o) = 1, \ \lambda_2(g_o) = -1 \). Since \( g_o \) is of order 2, the complement

\[ H_{C3}(4) \setminus SL(2, R) = \langle g \rangle g_o = \{g^j g_o \mid 0 \leq j \leq 2\} \]

consists of matrices \( g^j g_o \) of determinant \( \det(g^j g_o) = \det(g_o) = -1 \) and \( g \in H_{C3}(4) \setminus SL(2, R) \) has as eigenvalue 1 exactly when \( \text{tr}(g^j g_o) = 0 \). Bearing in mind the invariance of the trace under conjugation, one can consider

\[ g = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix} \quad \text{and} \quad g_o = \begin{pmatrix} a_o & b_o \\ c_o & -a_o \end{pmatrix} \]
with $a_o^2 + b_o c_o = 1$. Then
\[
g_0 g_0^{-1} = g_0 g o = \begin{pmatrix}
e^{-\frac{2\pi i}{3}} + \sqrt{-3} a_o^2 & \sqrt{-3} a_o b_o \\
\sqrt{-3} a_o c_o & e^{\frac{2\pi i}{3}} + \sqrt{-3} a_o^2
\end{pmatrix} = \begin{pmatrix}
e^{-\frac{2\pi i}{3}} & 0 \\
0 & e^{\frac{2\pi i}{3}}
\end{pmatrix} = g^{-1}
\]
is equivalent to $a_o = 0$ and
\[
g^j g_o = \begin{pmatrix}
e^{\frac{2\pi i j}{3}} & 0 \\
0 & e^{-\frac{2\pi i j}{3}}
\end{pmatrix} \begin{pmatrix}
0 & b_o \\
\frac{1}{b_o} & 0
\end{pmatrix} = \begin{pmatrix}
0 & e^{\frac{2\pi i j}{3}} b_o \\
e^{-\frac{2\pi i j}{3}} b_o & 0
\end{pmatrix}
\]
have tr$(g^j g_o) = 0$ for all $0 \leq j \leq 2$. Thus, any $g \in H_{C3}(4) \setminus SL(2, R)$ has an eigenvalue $\lambda_1(g) = 1$.

If $-I_2 \in G$, then for any $g \in G \setminus SL(2, R)$ with $\lambda_1(g) = 1$, $\lambda_2(g) = \det(g) \in R^* \setminus \{1\}$, one has $-g \in G \setminus SL(2, R)$ with $\lambda_1(-g) = -1$, $\lambda_2(-g) = -\det(g)$. Thus, $-g$ has an eigenvalue 1 exactly when $\lambda_2(-g) = -\det(g) = 1$ or $\lambda_2(g) = \det(g) = -1$. In particular,
\[
G = [G \cap SL(2, R)](g_o)
\]
for some $g_o \in G$ with $\det(g_o) = -1$, tr$(g_o) = 0$ and $G \setminus SL(2, R) = [G \cap SL(2, R)]g_o$.

Thus, for any $g \in G \setminus SL(2, R)$ has $\det(g) = -1$ and $g$ has an eigenvalue $\lambda_1(g) = 1$ exactly when tr$(g) = 0$.

We claim that tr$(g_{1} g_o) = 0$ for all $g_1 \in G \cap SL(2, R)$ and some $g_o \in G$ with $\det(g_o) = -1$, tr$(g_o) = -1$ requires $G \cap SL(2, R)$ to be a cyclic group. Assume the opposite. Then by Proposition 24, either $G \cap SL(2, R)$ contains a subgroup $K_4 = \langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \ g_1 g_2 g_1^{-1} = g_2^{-1} \rangle \cong \mathbb{Q}_8$

isomorphic to the quaternion group $\mathbb{Q}_8$ of order 8, or
\[
G \cap SL(2, R) = K_7 = \langle g_1, g_4 \mid g_1^2 = g_4^3 = -I_2, \ g_1 g_4 g_1^{-1} = g_4^{-1} \rangle \cong \mathbb{Q}_{12}
\]
is isomorphic to the dicyclic group $\mathbb{Q}_{12}$ of order 12. In either case, one has $h_1, h_2 \in SL(2, R)$ with tr$(h_1) = 0$ and $h_2$ of order $s \in \{4, 6\}$, such that $h_1 h_2 h_1^{-1} = h_2^{-1}$. Let us consider
\[
D_1 = S^{-1} h_1 S = \begin{pmatrix} a_1 & b_1 \\
c_1 & -a_1 \end{pmatrix} \in SL\left(2, \mathbb{Q}\left(\sqrt{-d}, E\left(\frac{-2\pi i}{s}\right)\right)\right),
\]
\[
D_2 = S^{-1} h_2 S = \begin{pmatrix} e^{\frac{2\pi i}{s}} & 0 \\
0 & e^{-\frac{2\pi i}{s}} \end{pmatrix}
\]
and
\[
D_o = S^{-1} g_o S = \begin{pmatrix} a_o & b_o \\
c_o & -a_o \end{pmatrix} \in GL\left(2, \mathbb{Q}\left(\sqrt{-d}, e^{\frac{2\pi i}{s}}\right)\right)
\]

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with \(a_0^2 + b_0c_0 = 1\). The relation

\[
D_1D_2D_1^{-1} = -D_1D_2D_1 = \begin{pmatrix}
-2i\text{Im}(e^{2\pi i})a_1^2 & -2i\text{Im}(e^{2\pi i})a_1b_1 \\
-2i\text{Im}(e^{2\pi i})a_1c_1 & e^{2\pi i} + 2i\text{Im}(e^{2\pi i})a_1^2
\end{pmatrix} = \begin{pmatrix}
e^{-2\pi i} & 0 \\
0 & e^{2\pi i}
\end{pmatrix} = D_2^{-1}
\]

requires \(a_1 = 0\) and

\[
D_1 = \begin{pmatrix}
0 & b_1 \\
-\frac{1}{b_1} & 0
\end{pmatrix}
\]

for some \(b_1 \in \mathbb{Q}(\sqrt{-d}, e^{2\pi i})\).

Now,

\[
\text{tr}(D_2D_0) = \text{tr}\begin{pmatrix}
e^{2\pi i}a_o & e^{2\pi i}b_o \\
e^{-2\pi i}c_o & -e^{2\pi i}a_o
\end{pmatrix} = 2i\text{Im}(e^{2\pi i})a_o = 0
\]

specifies the vanishing of \(a_o\), whereas

\[
D_o = \begin{pmatrix}
0 & b_o \\
\frac{1}{b_o} & 0
\end{pmatrix}
\]

for some \(b_o \in \mathbb{Q}(\sqrt{-de^{2\pi i}})\).

The condition

\[
\text{tr}(D_1D_0) = \text{tr}\begin{pmatrix}
\frac{b_1}{b_o} & 0 \\
0 & -\frac{b_o}{b_1}
\end{pmatrix} = \frac{b_1}{b_o} - \frac{b_o}{b_1} = 0
\]

requires \(b_1 = \varepsilon b_o\) for some \(\varepsilon \in \{\pm\}\) and

\[
\text{tr}(D_1D_2D_o) = \text{tr}\begin{pmatrix}
\varepsilon e^{-2\pi i} & 0 \\
\text{mbox} & 0
\end{pmatrix} = -\varepsilon (e^{2\pi i} - e^{-2\pi i}) = -2i\text{Im}(e^{2\pi i})\varepsilon \neq 0
\]

contradicts the assumption. Therefore \(G \cap SL(2, R) = \langle g \rangle \simeq \mathbb{C}_s\) is cyclic group of order \(s \in \{2, 4, 6\}\). If \(G = [G \cap SL(2, R)]\langle g_o \rangle\) has a normal subgroup \(G \cap SL(2, R) = \langle g \rangle \simeq \mathbb{C}_s\) then \(g = -I_2\) and \(g_o(-I_2) = (-I_2)g_o\), as far as \(-I_2\) is a scalar matrix. As a result, \(G = \langle g \rangle \times \langle g_o \rangle \simeq \mathbb{C}_2 \times \mathbb{C}_2\). For \(G = [G \cap SL(2, R)]\langle g_o \rangle\) with a normal subgroup \(G \cap SL(2, R) = \langle g \rangle \simeq \mathbb{C}_s\) of order \(\{4, 6\}\) note that the element \(g_o^{-1}\) of \(\langle g \rangle\) is of order \(s\), so that either \(g_o^{-1} = g\) or \(g_o^{-1} = g^{-1}\), according to \(\mathbb{Z}_4^* = \{\pm 1(\text{mod}4)\}\), \(\mathbb{Z}_6^* = \{\pm 1(\text{mod}6)\}\). If \(g_o = g^-1\), then there exists a matrix \(S \in GL(2, \mathbb{Q}(\sqrt{-d}, e^{2\pi i}))\), such that

\[
D = S^{-1}gS = \begin{pmatrix} e^{2\pi i} & 0 \\
0 & e^{-2\pi i}
\end{pmatrix} \quad \text{and} \quad D_o = S^{-1}g_oS = \begin{pmatrix} 1 & 0 \\
0 & -1
\end{pmatrix}
\]

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are diagonal. Then $\text{tr}(gg_o) = \text{tr}(DD_o) = e^{2\pi i s} - e^{-2\pi i s} = 2i \text{Im} \left( e^{2\pi i s} \right) \neq 0$ and 1 is not an eigenvalue of $gg_o$. Therefore $g_0 gg_o^{-1} = g^{-1}$. If

$$D = S^{-1} gS = \begin{pmatrix} e^{\frac{2\pi i}{s}} & 0 \\ 0 & e^{-\frac{2\pi i}{s}} \end{pmatrix}$$

then the relation

$$D_o DD_o^{-1} = D_o DD_o = \begin{pmatrix} e^{-\frac{2\pi i}{s}} + 2i \text{Im} \left( e^{\frac{2\pi i}{s}} \right) a_o^2 & 2i \text{Im} \left( e^{\frac{2\pi i}{s}} \right) a_o b_o \\ 2i \text{Im} \left( e^{\frac{2\pi i}{s}} \right) a_o c_o & e^{\frac{2\pi i}{s}} - 2i \text{Im} \left( e^{\frac{2\pi i}{s}} \right) a_o^2 \end{pmatrix} = \begin{pmatrix} e^{-\frac{2\pi i}{s}} & 0 \\ 0 & e^{\frac{2\pi i}{s}} \end{pmatrix} = D^{-1}$$

specifies that $a_o = 0$ and

$$D_o = \begin{pmatrix} 0 & b_o \\ 1 & 0 \end{pmatrix}$$

for some $b_o \in \mathbb{Q} \left( e^{\frac{2\pi i}{s}} \right)$.

The non-trivial coset

$$S^{-1} GS \setminus SL \left( 2, \mathbb{Q} \left( \sqrt{-d}, e^{\frac{2\pi i}{s}} \right) \right) = \langle D \rangle D_o = \{ D^j D_o \mid 0 \leq j \leq s - 1 \}$$

consists of elements of trace

$$\text{tr}(D^j D_o) = \text{tr} \left( \begin{pmatrix} 0 & e^{\frac{2\pi i}{s} b_o} \\ e^{-\frac{2\pi i}{s} b_o} & 0 \end{pmatrix} \right) = 0,$$

so that any $\Delta \in S^{-1} GS \setminus SL \left( 2, \mathbb{Q} \left( \sqrt{-d}, e^{\frac{2\pi i}{s}} \right) \right)$ has an eigenvalue 1 and any $g = S \Delta S^{-1} \in G \setminus SL(2, R)$ has an eigenvalue 1.

**Proposition 50.** The quotient $A/H$ of $A = E \times E$ is an Enriques surface if and only if $H$ is generated by $h \in H$ of order $s \in \{2, 3, 4, 6\}$ with $L(h) \in SL(2, R)$ and $h_o \in H$ with $\lambda_1 L(h_o) = 1$, $\lambda_2 L(h_o) = -1$, $\tau(h_o) = h_o L(h_o)^{-1} = \tau(U_o V_o)$, subject to $h_o h h_o^{-1} = h_o h h_o = h^{-1}$ and

$$L(h_o) \begin{pmatrix} U_o \\ V_o \end{pmatrix} \neq - \begin{pmatrix} U_o \\ V_o \end{pmatrix}.$$  \hfill (21)
In particular, for $s = 2$ the group

$$H \simeq \mathcal{L}(H) \simeq \mathbb{C}_2 \times \mathbb{C}_2$$

is isomorphic to the Klein group of order 4, while for $s \in \{3, 4, 6\}$ one has a dihedral group

$$H \simeq \mathcal{L}(H) \simeq \mathcal{D}_s = \langle a, b \mid a^s = 1, b^2 = 1, bab^{-1} = a^{-1} \rangle$$
of order $2s$.

Proof. According to Lemmas 41 and 49, the finite subgroups $H$ of $Aut(E \times E)$ with Enriques quotient $A/H$ are of the form

$$H = \langle \tau_{(P_i,Q_i)}, h, h_o \mid 1 \leq i \leq m \rangle$$

with $0 \leq m \leq 3$ and

$$\mathcal{L}(H) = \langle \mathcal{L}(h), \mathcal{L}(h_o) \rangle \quad \mathcal{L}(h)^s = I_2, \quad \mathcal{L}(h_o)^2 = I_2, \quad \mathcal{L}(h_o)\mathcal{L}(h)\mathcal{L}(h_o)^{-1} = \mathcal{L}(h)^{-1} \simeq \mathcal{D}_s$$

for some $\mathcal{L}(h) \in SL(2, \mathbb{R}), \mathcal{L}(h_o) \in GL(2, \mathbb{R}), \lambda_1\mathcal{L}(h_o) = 1, \lambda_2\mathcal{L}(h_o) = -1$. Note that

$$K := \mathcal{L}^{-1}(\mathcal{L}(H) \cap SL(2, \mathbb{R})) = \langle \tau_{(P_i,Q_i)} \mid 1 \leq i \leq m \rangle \langle h \rangle$$
is a normal subgroup of $H$ with a single non-trivial coset

$$H \setminus K = Kh_o = \left\{ \tau_{h(z,j)} = \sum_{i=1}^{m} z_i(P_i,Q_i) h^i h_o \mid z_i \in \mathbb{Z}, \ 0 \leq j \leq s-1 \right\}.$$  

The automorphism $h$, whose linear part $\mathcal{L}(h)$ has eigenvalues $\lambda_1\mathcal{L}(h) = e^{\frac{2\pi i}{s}}$, $\lambda_2\mathcal{L}(h) = e^{-\frac{2\pi i}{s}}$, different from 1 has always a fixed point on $A$. Without loss of generality, one can assume that $h = \mathcal{L}(h) \in GL(2, \mathbb{R})$, after moving the origin of $A$ at a fixed point of $h$. If $h_o = \tau_{(U_o,V_o)}\mathcal{L}(h_o)$ for some $(U_o,V_o) \in A$ then the translation parts

$$\tau(h(z,j)) = h(z,j)\mathcal{L}(h(z,j))^{-1} = \tau_{\sum_{i=1}^{m} z_i(P_i,Q_i) + h^j(U_o,V_o)} \quad \forall z = (z_1, \ldots, z_m) \in \mathbb{Z}^m$$

and $0 \leq j \leq s - 1$. The linear parts $\mathcal{L}(h(z,j)) = \mathcal{L}(h^j h_o) = h^j \mathcal{L}(h_o)$ have eigenvalues $\lambda_1(h^j \mathcal{L}(h_o)) = 1, \lambda_2(h^j \mathcal{L}(h_o)) = -1$ for all $0 \leq j \leq s - 1$. Applying Lemma 46, one concludes that $\text{Fix}_A(h(z,j)) = \emptyset$ if and only if no one lifting $(x(z,j), y(z,j)) \in \mathbb{C}^2$ of $\tau(h(z,j))$ is in the kernel of the linear operator $\psi_j = h^j \mathcal{L}(h_o) + I_2 : \mathbb{C}^2 \to \mathbb{C}^2$. For any fixed $0 \leq j \leq s - 1$, note that $(x(z,j), y(z,j)) \notin \ker(\phi_j)$ for all $z = (z_1, \ldots, z_m) \in \mathbb{Z}^m$ implies that the lifting of the $\mathbb{R}$-span of $\langle \tau_{(P_i,Q_i)} \mid 1 \leq i \leq m \rangle$ to $\mathbb{C}^2$ is parallel to $\ker(\psi_j)$. It suffices to establish that $\ker(\psi_0) \cap \ker(\psi_1) = \{0, 0\}$, in order to conclude that $m = 0$ and $H = \langle h, h_o \rangle = \langle h_o, h \rangle$. Since the claim $\ker(\psi_0) \cap \ker(\psi_1) = \{0, 0\}$
is independent on the choice of a coordinate system on $C^2$, one can use Lemma 49 to assume that
\[
\mathcal{L}(h_o) = D_o = \left( \begin{array}{cc} 0 & b_o \\ \frac{1}{b_o} & 0 \end{array} \right) \quad \text{and} \quad h = \mathcal{L}(h) = \left( \begin{array}{cc} e^{2\pi i \alpha} & 0 \\ 0 & e^{-2\pi i \alpha} \end{array} \right)
\]
for some $s \in \{2, 3, 4, 6\}$. Then $\psi_0 = \mathcal{L}(h_o) + I_2$ has kernel $\ker(\psi_0) = \text{Span}_C(b_o, -1)$, while
\[
\psi_1 = h\mathcal{L}(h_o) + I_2 = \left( \begin{array}{cc} 1 & e^{2\pi i \alpha}b_o^{-1} \\ e^{-2\pi i \alpha}b_o & 1 \end{array} \right)
\]
has kernel $\ker(\psi_1) = \text{Span}_C\left(e^{2\pi i \alpha}b_o, -1\right)$. For $s \in \{2, 3, 4, 6\}$ the vectors $(b_o, -1)$ and $(e^{2\pi i \alpha}b_o, -1)$ are linearly independent over $C$, so that $\ker(\psi_0) \cap \ker(\psi_1) = \{(0, 0)\}$. Now, $\mathcal{L}(h^j h_o) = h^j \mathcal{L}(h_o) \neq I_2$ for any $0 \leq j \leq s - 1$, as far as $\mathcal{L}(h_o) \not\in \langle h \rangle < SL(2, R)$. On the other hand, the subgroup $\langle h = \mathcal{L}(h) \rangle$ of $H$ is contained in $SL(2, R)$, so that the translation part $T(H) = \ker(\mathcal{L}|_H) = Id_A$ is trivial. As a result, $\mathcal{L} : H \rightarrow \mathcal{L}(H)$ is a group isomorphism and the relation $\mathcal{L}(h_o) h \mathcal{L}(h_o)^{-1} = h^{-1}$ implies that
\[
h_o h_o^{-1} = (\tau_{(U_o, V_o)}\mathcal{L}(h_o)) h (\tau_{\mathcal{L}(h_o)^{-1}(U_o, V_o)}\mathcal{L}(h_o)^{-1}) =
\]
\[
= \tau_{(U_o, V_o) - \mathcal{L}(h_o) h \mathcal{L}(h_o)^{-1}(U_o, V_o)} \mathcal{L}(h_o) h \mathcal{L}(h_o)^{-1} = \tau_{(U_o, V_o) - h^{-1}(U_o, V_o) h^{-1}} = h^{-1}.
\]
After acting by $h$ on $(U_o, V_o) = h^{-1}(U_o, V_o)$, one obtains that $h(U_o, V_o) = (U_o, V_o)$, or $(U_o, V_o) \in A$ is a fixed point of $h$. Bearing in mind that $K = \langle h \rangle \simeq \langle \mathcal{L}(h) \rangle = \mathcal{L}(H) \cap SL(2, R)$ is a normal subgroup of $H \simeq \mathcal{L}(H) = \mathcal{L}(H) \cap SL(2, R)\langle \mathcal{L}(h_o)\rangle$, let us represent the complement $H \setminus K$ as the set of the entries of the left coset
\[
H \setminus K = h_o K = \{h_o h^j \mid 0 \leq j \leq s - 1\}.
\]
Then $h_o h^j = \tau_{(U_o, V_o)}(\mathcal{L}(h_o) h^j)$ have translation parts
\[
\tau(h_o h^j) = h_o h^j \mathcal{L}(h_o h^j)^{-1} = h_o \mathcal{L}(h_o)^{-1} = \tau(h_o) = \tau_{(U_o, V_o)}
\]
and linear parts $\mathcal{L}(h_o) h^j$ with eigenvalues $\lambda_1(\mathcal{L}(h_o) h^j) = 1$, $\lambda_2(\mathcal{L}(h_o) h^j) = -1$. According to Lemma 46, the automorphism $h_o h^j \in \text{Aut}(A)$ has no fixed point on $A$ if and only if no one lifting $(u_o, v_o) \in C^2$ of $(u_o + \pi_1(E), v_o + \pi_1(E)) = (U_o, V_o)$ is in the kernel of $\varphi_j = \mathcal{L}(h_o) h^j + I_2$. We claim that if
\[
h \left( \begin{array}{c} u_o \\ v_o \end{array} \right) = \left( \begin{array}{c} u_o \\ v_o \end{array} \right) + \left( \begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right) \quad \text{for some} \quad (\mu_1, \mu_2) \in \pi_1(A),
\]
then $\varphi_j(u_o, v_o) - \varphi_0(u_o, v_o) \in \pi_1(A)$. Indeed, by an induction on $j$, one has
\[
h^j \left( \begin{array}{c} u_o \\ v_o \end{array} \right) - \left( \begin{array}{c} u_o \\ v_o \end{array} \right) \in \pi_1(A),
\]
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whereas
\[
\varphi_j(u_o, v_o) - \varphi_0(u_o, v_o) = \mathcal{L}(h_o)h_j \begin{pmatrix} u_o \\ v_o \end{pmatrix} - \mathcal{L}(h_o) \begin{pmatrix} u_o \\ v_o \end{pmatrix} \in \pi_1(A).
\]

Thus, the assumption \((u_o, v_o) \in \ker(\varphi_j)\) implies that
\[
\varphi_0(u_o, v_o) = \mathcal{L}(h_o)(u_o, v_o) + (u_o, v_o) = (\mu_1', \mu_2') \in \pi_1(A),
\]
whereas
\[
\mathcal{L}(h_o) \begin{pmatrix} U_o \\ V_o \end{pmatrix} = - \begin{pmatrix} U_o \\ V_o \end{pmatrix},
\]
contrary to the assumption (21). Note that (21) is equivalent to \(\varphi_0(u_o, v_o) \not\in \pi_1(A)\) for all liftings \((u_o, v_o) \in \mathbb{C}^2\) of \((u_o + \pi_1(E), v_o + \pi_1(E)) = (U_o, V_o)\) and is slightly stronger than \(Fix_A(h_o) = \emptyset\), which amounts to \(\varphi_0(u_o, v_o) \neq 0\) for \(\forall (u_o, v_o) \in \mathbb{C}^2\) with \((u_o + \pi_1(E), v_o + \pi_1(E)) = (U_o, V_o)\).
References


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