МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2012
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2012
Proceedings of the Forty First Spring Conference
of the Union of Bulgarian Mathematicians
Borovetz, April 9–12, 2012

AN INEQUALITY FOR GENERALIZED CHROMATIC GRAPHS*

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Let G be a simple n-vertex graph with degree sequence $d_1, d_2, \ldots, d_n$ and vertex set $\mathrm{V}(G)$. The degree of $v \in \mathrm{V}(G)$ is denoted by $d(v)$. The smallest integer r for which $\mathrm{V}(G)$ has an r-partition

$$\mathrm{V}(G) = V_1 \cup V_2 \cup \cdots \cup V_r, \quad V_i \cap V_j = \emptyset, \quad , i \neq j$$

such that $d(v) \leq n - |V_i|$, $\forall v \in V_i$, $i = 1, 2, \ldots, r$ is denoted by $\varphi(G)$. In this note we prove the inequality

$$\varphi(G) \geq \frac{n}{n - \bar{\bar{d}}},$$

where $\bar{\bar{d}} = \sqrt{\dfrac{d_1^2 + d_2^2 + \cdots + d_n^2}{n}}$.

1. Introduction. We consider only finite, non-oriented graphs without loops and multiple edges. We use the following notations:

$\mathrm{V}(G)$ – the vertex set of G;
$e(G)$ – the number of edges of G;
$\mathrm{cl}(G)$ – the clique number of G;
$\chi(G)$ – the chromatic number of G;
$\mathrm{N}(v)$, $v \in \mathrm{V}(G)$ — the set of neighbours of a vertex v;
$\mathrm{N}(v_1, v_2, \ldots, v_k) = \bigcap_{i=1}^{k} \mathrm{N}(v_i)$;
$d(v)$ – the degree of a vertex v;
$\mathrm{G}[V]$, $V \subseteq \mathrm{V}(G)$ – induced subgraph by V.

Definition 1. *Let G be a graph, $|\mathrm{V}(G)| = n$ and $V \subseteq \mathrm{V}(G)$. Then, the set V is called a δ-set in G, if*

$$d(v) \leq n - |V| \text{ for all } v \in V.$$

Clearly, any independent set V of vertices of a graph G is a δ-set in G since $\mathrm{N}(v) \subseteq \mathrm{V}(G) \setminus V$ for all $v \in V$. It is obvious that if $V \subseteq \mathrm{V}(G)$ and $|V| \geq \max\{d(v) \mid v \in \mathrm{V}(G)\}$ then $\mathrm{V}(G) \setminus V$ is a δ-set in G (it is possible that $\mathrm{V}(G) \setminus V$ is not independent).

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*2000 Mathematics Subject Classification: Primary 05C35.
Key words: clique number,degree sequence.
This work was supported by the Scientific Research Fund of the St. Kliment Ohridski University of Sofia under contract No 187, 2011.

Definition 2. *A graph G is called a generalized r-partite graph if there is a r-partition*

$$\mathrm{V}(G) = V_1 \cup V_2 \cup \dots \cup V_r, \quad V_i \cap V_j = \emptyset, \quad , i \neq j$$

where the sets $V_1, V_2, \dots, V_r$ are δ-sets in G. The smallest integer r such that G is a generalized r-partite is denoted by $\varphi(G)$.

As any independent vertex set of G is a δ-set in G, we have $\varphi(G) \leq \chi(G)$. In fact, the following stronger inequality [10]

$$\varphi(G) \leq \mathrm{cl}(G) \tag{1}$$

holds.

Let $\mathrm{V}(G) = \{v_1, v_2, \dots, v_n\}$ and $\mathrm{cl}(G) = r$. Define

$$\bar{d} = \frac{d(v_1) + d(v_2) + \dots + d(v_n)}{n}, \qquad \bar{\bar{d}} = \sqrt{\frac{d^2(v_1) + d^2(v_2) + \dots + d^2(v_n)}{n}}.$$

By the classical Turan Theorem, [11] (see also [5]) we have

$$e(G) \leq \frac{n^2(r-1)}{2r}. \tag{2}$$

The equality in (2) holds if and only if $n \equiv 0 \pmod r$ and G is complete r-chromatic and regular.

It is proved in [6] that

$$e(G) \leq \frac{n^2(\varphi(G)-1)}{2\,\varphi(G)}. \tag{3}$$

According to (1) the inequality (3) is stronger than the inequality (2). But in case of equality in (3) the graph G is not unique as it is in the Turan theorem.

Since $\bar{d}(G) = \dfrac{2e(G)}{n}$, it follows from (3) that

$$\varphi(G) \geq \frac{n}{n - \bar{d}(G)}. \tag{4}$$

In this note we give the following improvement of the inequality (4).

Theorem 1. *Let G be a n-vertex graph. Then,*

$$\varphi(G) \geq \frac{n}{n - \bar{\bar{d}}(G)}. \tag{5}$$

The equality in (5) *holds if and only if* $n \equiv 0 \pmod{\varphi(G)}$ *and G is regular graph of degree* $\dfrac{n(\varphi(G)-1)}{\varphi(G)}$.

2. Auxiliary results. We denote the elementary symmetric polynomial of degree s by $\sigma_s(x_1, x_2, \dots, x_n)$, $1 \leq s \leq n$, i.e.

$$\sigma_s(x_1, x_2, \dots, x_n) = x_1 x_2 \dots x_s + \cdots .$$

Further, we use the following equalities:

$$x_1^2 + x_2^2 + \dots + x_n^2 = \sigma_1^2 - 2\sigma_2, \tag{6}$$

$$x_1^3 + x_2^3 + \dots + x_n^3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3, \tag{7}$$

where $\sigma_i = \sigma_i(x_1, x_2, \dots, x_n)$.

In order to prove Theorem 1 we use the following well-known inequality (particular case of the Maclaurin inequality, see [2], [3]).

Theorem 2. *Let $x_1, x_2, \ldots, x_n$ be non-negative reals and $\sigma_s(x_1, x_2, \ldots, x_n) = \sigma_s$. Then,*

$$\sqrt[s]{\frac{\sigma_s}{\binom{n}{s}}} \leq \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sigma_1}{n}, \quad 1 \leq s \leq n. \tag{8}$$

If $s \geq 2$, then the equality in (8) holds if and only if $x_1 = x_2 = \cdots = x_n$.

A straight and very short prove of Theorem 2 is given in [4].

3. Proof of Theorem 1. Let $\varphi(G) = r$, $\mathrm{V}(G) = \{v_1, v_2, \ldots, v_n\}$ and

$$\mathrm{V}(G) = V_1 \cup V_2 \cup \cdots \cup V_r, \quad V_i \cap V_j = \emptyset, \quad i \neq j, \tag{9}$$

where $V_1, V_2, \ldots, V_r$ are δ-sets in G, i.e. if $n_i = |V_i|$, $i = 1, 2, \ldots, r$, then

$$d(v) \leq n - n_i, \quad \forall v \in V_i. \tag{10}$$

It follows from (9) that

$$d^2(v_1) + d^2(v_2) + \cdots + d^2(v_n) = \sum_{i=1}^{r} \sum_{v \in V_i} d^2(v).$$

According to (10)

$$\sum_{v \in V_i} d^2(v) \leq n_i(n - n_i)^2.$$

Thus we have

$$d^2(v_1) + d^2(v_2) + \cdots + d^2(v_n) \leq \sum_{i=1}^{r} n_i(n - n_i)^2.$$

From (6) and (7) we see that

$$\sum_{i=1}^{r} n_i(n - n_i)^2 = n\sigma_2 + 3\sigma_3,$$

where $\sigma_2 = \sigma_2(n_1, n_2, \ldots, n_r)$, $\sigma_3 = \sigma_3(n_1, n_2, \ldots, n_r)$.

Thus we obtain the inequality

$$d^2(v_1) + d^2(v_2) + \cdots + d^2(v_n) \leq n\sigma_2 + 3\sigma_3. \tag{11}$$

Since $\sigma_1 = n$, Theorem 2 yields

$$\sigma_2 \leq \frac{n^2(r-1)}{2r} \quad \text{and} \quad \sigma_3 \leq \frac{n^3(r-1)(r-2)}{6r^2}. \tag{12}$$

Now, the inequality (5) follows from (11) and (12).

Obviously, if $n \equiv 0 \pmod{r}$ and $d(v_1) = d(v_2) = \cdots = d(v_r) = \dfrac{n(r-1)}{r}$, then we have equality in (5). Now, let us suppose that we have equality in inequality (5). Then, we have equality in (12) and (10) too. From $r = \varphi(G) = \dfrac{n}{n - \bar{\bar{d}}}$ it is clear that r divides n. By Theorem 2, we have

$$n_1 = n_2 = \cdots = n_r = \frac{n}{r}.$$

Because of the equality in (10), i.e. $d(v) = n - n_i$, $v \in V_i$, we have

$$d(v_1) = d(v_2) = \cdots = d(v_r) = \frac{n(r-1)}{r}.$$

Theorem 1 is proved.

4. Some corollaries.

Definition 3 ([5]). *Let G be a graph and $v_1, v_2, \ldots, v_r \in \mathrm{V}(G)$. Then, the sequence $v_1, v_2, \ldots, v_r$ is called an α-sequence in G if the following conditions are satisfied:*

(*i*) $d(v_1) = \max\{d(v) \mid v \in \mid \mathrm{V}(G)\}$;

(*ii*) $v_i \in \mathrm{N}[v_1, v_2, \ldots, v_{i-1}]$ *and* v_i *has maximal degree in the induced subgraph* $\mathrm{G}[\mathrm{N}(v_1, v_2, \ldots, v_{i-1}]$, $2 \le i \le r$.

Definition 4. *Let G be a graph and $v_1, v_2, \ldots, v_r \in \mathrm{V}(G)$. Then, the sequence $v_1, v_2, \ldots, v_r$ is called a β-sequence in G if the following conditions are satisfied:*

(*i*) $d(v_1) = \max\{d(v) \mid v \in \mid \mathrm{V}(G)\}$;

(*ii*) $v_i \in \mathrm{N}(v_1, v_2, \ldots, v_{i-1})$ *and* $d(v_i) = \max\{d(v) \mid v \in \mathrm{N}(v_1, v_2, \ldots, v_{i-1})\}$, $2 \le i \le r$.

Corollary 1. *Let $v_1, v_2, \ldots, v_r$, $r \ge 2$ be an α- or a β-sequence in an n-vertex graph G such that $\mathrm{N}(v_1, v_2, \ldots, v_r)$ is a δ-set. Then,*

$$r \ge \frac{n}{n - \bar{\bar{d}}}. \tag{13}$$

Proof. Since $\mathrm{N}(v_1, v_2, \ldots, v_p)$ is a δ-set, G is a generalized r-partite graph, [9]. Thus, $r \ge \varphi(G)$ and (13) follows from Theorem 1. □

Corollary 2. *Let $v_1, v_2, \ldots, v_r$, $r \ge 2$, be a β-sequence in n-vertex graph G such that*

$$d(v_1) + d(v_2) + \cdots + d(v_r) \le (r-1)n. \tag{14}$$

Then, the inequality (13) *holds.*

Proof. From (14) it follows that G is a generalized r-partite graph ([7], [8]). □

The next corollary follows from (1) and Theorem 1.

Corollary 3 ([1]). *Let G be an n-vertex graph. Then,*

$$\mathrm{cl}(G) \ge \frac{n}{n - \bar{\bar{d}}}. \tag{15}$$

Remark 1. The prove of the inequality (15) given in [1] is incorrect, since the arguments on p. 53, rows 8 and 9 from the top, is not valid.

Corollary 4. *Let G be an n-vertex graph such that*

$$\mathrm{cl}(G) = \frac{n}{n - \bar{\bar{d}}}. \tag{16}$$

Then, G is regular and complete $\mathrm{cl}(G)$*-chromatic graph.*

Proof. Let $\varphi(G) = r$. Then, by (16), (1) and Theorem 1 we have

$$\mathrm{cl}(G) = \varphi(G) = r = \frac{n}{n - \bar{\bar{d}}}.$$

By Theorem 1, $n \equiv 0 \pmod r$ and G is a regular graph of degree $\dfrac{n(r-1)}{r}$. Thus

$$e(G) = \frac{n^2(r-1)}{2r} = \frac{n^2(\mathrm{cl}(G) - 1)}{2\,\mathrm{cl}(G)}.$$

According to Turan's Theorem, G is complete r-chromatic and regular. □

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ЕДНО НЕРАВЕНСТВО ЗА ОБОБЩЕНИ ХРОМАТИЧНИ ГРАФИ

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Нека G е n-върхов граф и редицата от степените на върховете му е $d_1, d_2, \ldots, d_n$, а $\mathrm{V}(G)$ е множеството от върховете на G. Степента на върха v бележим с $d(v)$. Най-малкото естествено число r, за което $\mathrm{V}(G)$ има r-разлагане

$$\mathrm{V}(G) = V_1 \cup V_2 \cup \cdots \cup V_r, \quad V_i \cap V_j = \emptyset, \quad , i \neq j$$

такова, че $d(v) \leq n - |V_i|$, $\forall v \in V_i$, $i = 1, 2, \ldots, r$ е означено с $\varphi(G)$. В тази работа доказваме неравенството

$$\varphi(G) \geq \frac{n}{n - \bar{\bar{d}}},$$

където $\bar{\bar{d}} = \sqrt{\dfrac{d_1^2 + d_2^2 + \cdots + d_n^2}{n}}$.